Complexity, Combinatorial Positivity, and Newton Polytopes

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Computational Complexity Theory

Poorly understood issue: Why are do some decision problems have fast algorithms and others seem to need costly search?

Some complexity classes:

- NP: LP ($\exists x \geq 0$, Ax=b?)
- coNP: Primes
- P: LP and Primes!
- NP-complete: Graph coloring

Famous theoretical computer science problems:

- P $\stackrel{?}{=}$ NP
- NP $\stackrel{?}{=}$ coNP
- NP $\cap$ coNP $\stackrel{?}{=}$ P

Polynomials

In algebraic combinatorics and combinatorial representation theory we often study:

$$F_\diamond = \sum_{\alpha} c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \mathrm{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n]$$

Example 1: $\diamond = \lambda \implies F_\diamond = s_\lambda$ (Schur), $c_{\alpha,G} =$ Kostka coeff.

Example 2: $\diamond = G = (V, E) \implies F_\diamond = \chi_G$ (Stanley's chromatic symmetric polynomial), $c_{\alpha,G} = \#$proper colorings of G with α_i-many colors i

Example 3: $\diamond = w \in S_\infty \implies F_\diamond = \mathfrak{S}_w$ (Schubert polynomial). More later.

Nonvanishing

Nonvanishing: What is the complexity of deciding $\underline{c_{\alpha,\diamond} \neq 0}$ as measured in the length of the input $(\alpha, \diamond)$ assuming arithmetic takes constant time?

- In general undecidable: Gödel incompleteness '31, Turing's halting problem '36.
- Our cases of interest have combinatorial positivity: $\exists$ rule for $c_{\alpha,\diamond} \in \mathbb{Z}_{\geq 0} \implies \text{Nonvanishing}(F_\diamond) \in \text{NP}$.

Newton polytopes

Evidently, nonvanishing concerns the *Newton polytope*,

$$\mathrm{Newton}(F_\diamond) = \mathrm{conv}\{\alpha : c_{\alpha,\diamond} \neq 0\} \subseteq \mathbb{R}^n.$$

- Monical-Tokcan-Y. '17: $F_\diamond$ has *saturated Newton polytope* (SNP) if $\beta \in \mathrm{Newton}(F_\diamond) \iff c_{\beta,\diamond} \neq 0$
- Many polynomials have this property.

Importance of SNP property:

Observation 1: SNP $\Rightarrow$ nonvanishing$(F_\diamond)$ is equivalent to checking membership of a lattice point in Newton$(F_\diamond)$.

Observation 1': SNP + "efficient" halfspace description of Newton$(F_\diamond) \implies$ nonvanishing$(F_\diamond) \in$ coNP.

$\therefore$ in many cases nonvanishing$(F_\diamond) \in \mathrm{NP} \cap \mathrm{coNP}$.

Nonvanishing and NP

Example 1': s_λ has SNP. Newton$(s_\lambda) = \mathcal{P}_\lambda$ (the permutahedron). Nonvanishing$(s_\lambda) \in$ P by dominance order (Rado's theorem).

Example 2': χ_G does not have SNP.

coloring $\in$ NP$-$complete $\implies$ Nonvanishing$(\chi_G) \in$ NP$-$complete.

$\therefore$ nonvanishing hits the extremes of NP.

Question: What about the nonextremes?

- Many problems *suspected* of being NP-intermediate: e.g., graph isomorphism, factorization
- Ladner's theorem: $\mathrm{P} \neq \mathrm{NP} \implies \mathrm{NP} - \mathrm{intermediate} \neq \emptyset$
- NP $\cap$ coNP is important to this discussion:

$$\mathrm{coNP} \cap \mathrm{NP} - \mathrm{complete} \neq \emptyset \implies \mathrm{NP} = \mathrm{coNP}!$$

- This is why factorization is not expected to be NP-complete.
- Most public key cryptography relies on NP $\cap$ coNP $\neq$ P.

Possible application of algebraic combinatorics to TCS?

Conjecture 1: [Stanley '95] If G is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then χ_G is Schur positive.

Conjecture 2: [C. Monical '18] If χ_G is Schur positive, then it is SNP.

Conjecture 1+2: If G is claw-free then χ_G is SNP.

Theorem: (Holyer '81) Coloring of claw-free G is NP-complete.

Corollary: $\text{nonvanishing}(\chi_{\text{claw-free}\,G}) \in$ NP-complete.

$\therefore$ Conjecture 1+2 and a halfspace description of $\text{Newton}(\chi_{\text{clawfree}\,G}) \implies \text{NP} = \text{coNP}$

Suggests a new complexity-theoretic rationale for the study of χ_G.

An algebraic combinatorics paradigm for complexity

In many cases of algebraic combinatorics, $\{F_\diamond\}$ has combinatorial positivity and SNP. If one also has an efficient halfspace description of Newton$(F_\diamond)$, then nonvanishing$(F_\diamond) \in \mathsf{NP} \cap \mathsf{coNP}$.

Four possible outcomes of such a study:

(I) **Unknown**: it is an open problem to find additional problems that are in NP ∩ coNP that are not *known* to be in P.

(II) **P**: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.

(III) **NP-complete**: proof solves NP $\stackrel{?}{=}$ coNP with "$=$".

(IV) **NP-intermediate**: proof solves NP-intermediate$\stackrel{?}{=} \emptyset$ with "$\neq$", i.e., P $\neq$ NP.

Next: do this for Schubert polynomials (outcomes (I) and (II)).

Schubert polynomials

B acts on GL_n/B with *finitely many orbits*, the Schubert cells, whose closures X_w, $w \in S_n$ are the **Schubert varieties**.

Lascoux and Schützenberger's (1982) main idea in type A (after Bernstein-Gelfand-Gelfand):

- Pick $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ as an especially nice representative of the class of a point
- Apply *Newton's divided difference operator*

$$\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}},$$

to recursively define all other $\mathfrak{S}_w$ using weak Bruhat order.

This starts the theory of *Schubert polynomials*.

Complexity results

There are many combinatorial rules that establish that $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$.

However, none of these prove nonvanishing$(\mathfrak{S}_w) \in \mathrm{P}$ since they involve exponential search.

Theorem A: (Adve-Robichaux-Y. '18) $c_{\alpha,w}$ is #P-complete.

$\therefore$ no polynomial time algorithm to compute $c_{\alpha,w}$ exists unless $\mathrm{P} = \mathrm{NP}$.

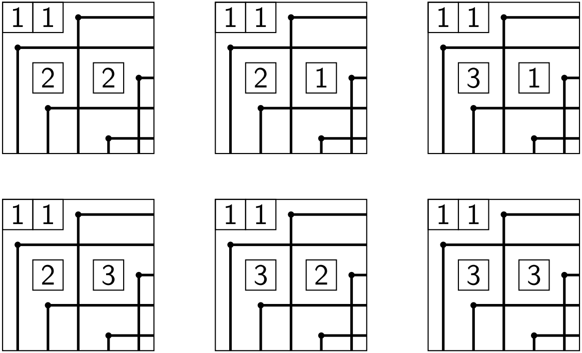
Counting is hard, nonvanishing is easy:

Theorem B: (Adve-Robichaux-Y. '18) nonvanishing$(\mathfrak{S}_w) \in \mathrm{P}$

Analogy: Computing the permanent of a $0, 1$-matrix is #P-complete but nonzeroness is easy (Edmonds-Karp matching algorithm).

A tableau rule for nonvanishing

Fillings of the Rothe diagram of 31524:



Theorem C: (Adve-Robichaux-Y. '18)
$c_{\alpha,w} \neq 0 \iff \mathrm{Tab}(w,\alpha) \neq \emptyset.$

Proofs

- The *Schubitope* $\mathcal{S}_D$ was introduced by Monical-Tokcan-Y. '17 for any $D \subseteq [n]^2$.
- We give a generalization of tableau of Theorem C to any D.
- Then introduce a new polytope $\mathcal{T}_D$ whose integer points biject with tableaux.
- Integer linear programming is hard but $\mathcal{T}_D$ is totally unimodular. Now use LPfeasibility $\in$ P.
- Link to Schubert polynomials: For $D = D(w)$, Monical-Tokcan-Y. '17 conjectured $\mathcal{S}_D = \text{Newton}(\mathfrak{S}_w)$. Proved by Fink-Mészáros-St. Dizier '18.
- First proved that nonvanishing$(\mathfrak{S}_w) \in \text{NP} \cap \text{coNP}$ hinting $\in$ P.
- NP and $\#P$ proof via transition.

Conclusions and summary

- In this talk we described an algebraic combinatorics paradigm for complexity on theoretical computer *science*.
- Conversely, complexity gives some new perspectives on algebraic combinatorics.
- In our main example, we obtain new results about Schubert polynomials and the Schubitope.