The point processes at turning points of large lozenge tilings

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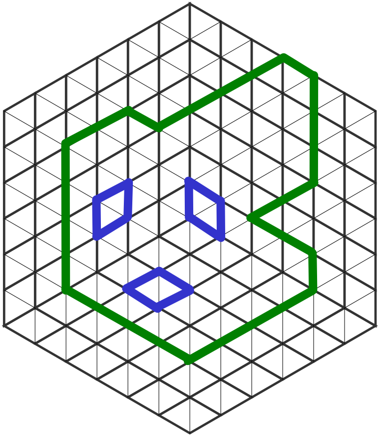
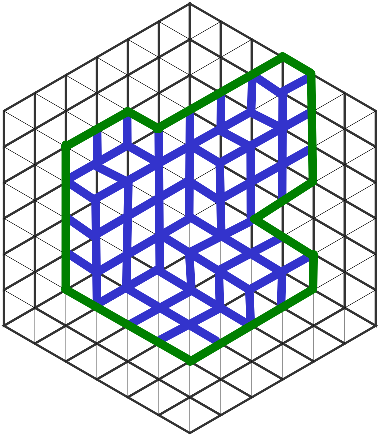
Lozenge tilings

Take a portion of the triangular lattice which is tilable by lozenges.

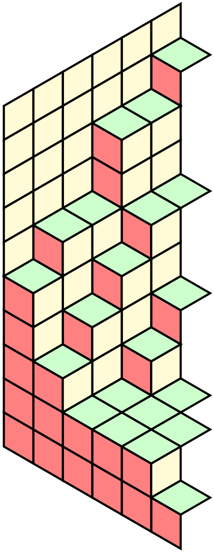


Lozenge tilings

There are many different tilings.
The lozenge tiling model $\leftrightarrow$ study random tilings of a region.

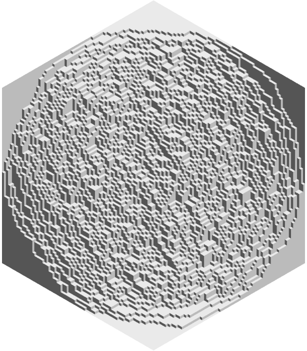


Stacks of boxes



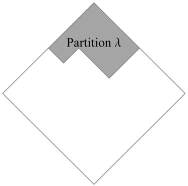
Limit shapes

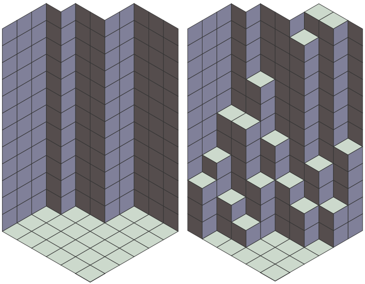
Take a region, (in this picture a hexagon of size $n \times n \times n$), and consider uniformly random tilings of it by lozenges in the limit when $n \to \infty$.



- Limit shape results via a variational principle - Cohn Larsen Propp 1998, Cohn Kenyon Propp 2001
- Frozen boundaries for polygonal regions and limiting processes - Kenyon Okounkov 2006, Kenyon Okounkov Sheffield 2006

Infinite regions - volume measure





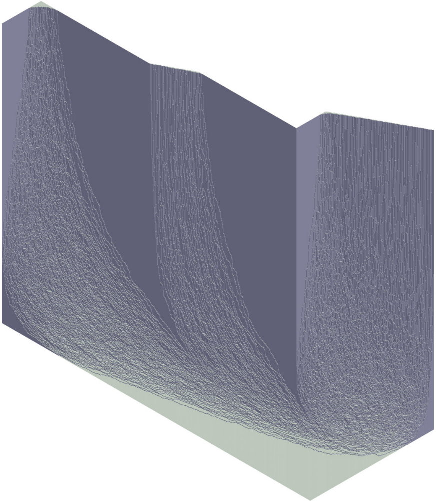
Consider the stacks of boxes with boundary λ, confined to the $aN \times bN$ box, with the distribution

$$Prob(\pi) \propto q^{|\pi|} = q^{volume},$$

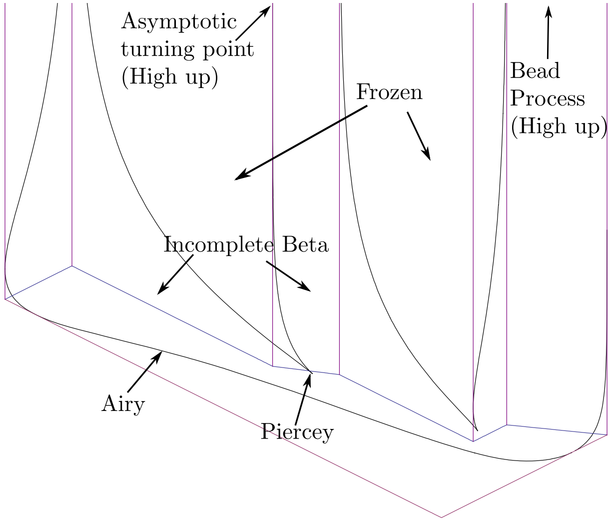
for some $q \in (0,1)$, where $|\pi|$ is the total volume (number of boxes).

Limit shapes - infinite regions

Slopes ± 1 - Okounkov Reshetikhin 2003, 2007,
Rational slopes $(-1,1)$ - Boutillier, M., Reshetikhin, Tingley 2012,
Arbitrary slopes $[-1,1]$ - M. 2011.

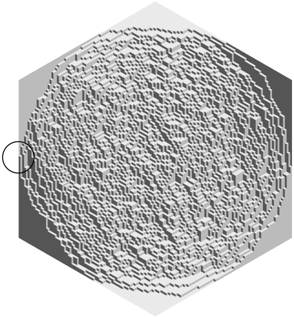


Scaling limit - limit shape



The birth of a random matrix

The birth of a random matrix



Conjecture (Okounkov-Reshetikhin 2006)

The point process at turning points converges to the GUE corners process.

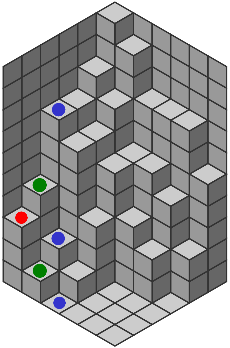
Turning points

Consider the process near turning points:

- On a slice at distance k from the edge we have k horizontal lozenges. Let the heights be

$$x_1^k \leq x_2^k \leq \cdots \leq x_k^k.$$

- Slices interlace:



$$\begin{array}{ccccc}
 & & & & x_3^3 \\
 & & & \leq & \\
 & & x_2^2 & & \\
 & \leq & & \geq & \\
x_1^1 & & & & x_2^3 \\
 & \geq & & \leq & \\
 & & x_1^2 & & \\
 & & & \geq & \\
 & & & & x_1^3
\end{array}$$

The GUE

The Gaussian Unitary Ensemble is an ensemble of $N \times N$ hermitian random matrices H with independent Gaussian entries:

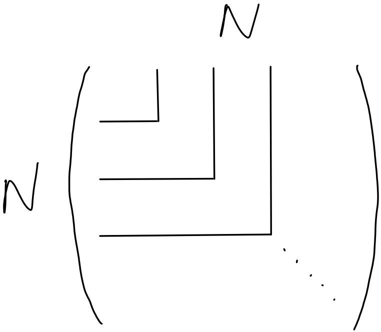
$$H_{i,i} \sim N(0,1)$$
$$\Re H_{i,j}, \Im H_{i,j} \sim N(0,1/2)$$
$$H_{j,i} = \overline{H_{i,j}}.$$

A different way to describe the law is to say H is an $N \times N$ hermitian random matrix with density

$$\frac{1}{Z_N} e^{-tr(H^2)/2},$$

where Z_N is a normalization constant.

The GUE corners process



Let $\lambda_1^k \le \lambda_2^k \le \cdots \le \lambda_k^k$ be the eigenvalues of the $k \times k$ corner of H.
Two neighboring rows interlace:

$$
\begin{array}{ccccccccc}
\lambda_1^k & & \lambda_2^k & & \cdots & & \lambda_{k-1}^k & & \lambda_k^k. \\
& \rotatebox{-45}{$\le$} \quad \rotatebox{45}{$\ge$} & & \rotatebox{-45}{$\le$} \quad \rotatebox{45}{$\ge$} & & & & \rotatebox{-45}{$\le$} \quad \rotatebox{45}{$\ge$} & \\
& \lambda_1^{k-1} & & \lambda_2^{k-1} & \cdots & & & \lambda_{k-1}^{k-1}. &
\end{array}
$$

The GUE corners process

- The joint distribution of

$$\begin{array}{c} \lambda_1^k, \lambda_2^k, \quad \dots, \quad \lambda_{k-1}^k, \lambda_k^k \\ \lambda_1^{k-1}, \lambda_2^{k-1}, \dots, \lambda_{k-1}^{k-1} \\ \dots \\ \lambda_1^3, \lambda_2^3, \lambda_3^3 \\ \lambda_1^2, \lambda_2^2 \\ \lambda_1^1 \end{array}$$

  is called the GUE corners process. Denote by $\mathbb{GUE}_k$.
- Conditioned on the top row and the interlacing condition, the lower entries are unifromly distributed.

The Okounkov-Reshetikhin conjecture

Conjecture (Okounkov-Reshetikhin 2006)

After appropriate centering and scaling the joint law of the heights x_i^k of the horizontal lozenges in the first k slices converges to the joint law of the eigenvalues λ_i^k of the first k corners of a GUE random matrix.

The Okounkov-Reshetikhin conjecture

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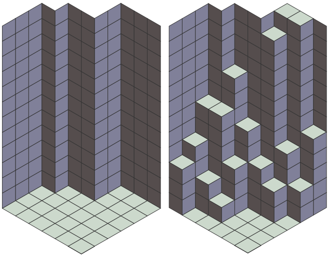


$$Prob(\pi) \propto q^{|\pi|} = q^{volume}$$

$$q \in (0, 1)$$

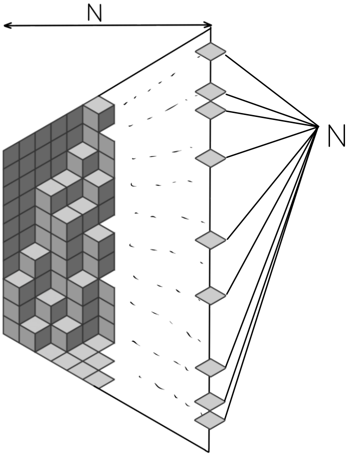
Let $x_i^k, i \leq k$ be the heights of the first k slices as defined above.

Theorem (Okounkov-Reshetikhin 2006)

Let $q = e^{-1/N}$. There exist constants C_0 and C_1 such that in the limit $q \to 1$ we have $\frac{x_i^k - NC_0}{\sqrt{NC_1}} \to \mathbb{GUE}_k$ in distribution.

CLT for the turning point

Let $x(1), x(2), \dots$ be a sequence of signatures, where $x(N)$ has N parts. Consider the q^{volume} distribution on lozenge tilings whose N'th slice is $x(N)$. Let $x^k(N)$ be the k'th slice.



CLT for the turning point

Theorem (M.,Petrov 2017)

Suppose there exists a nonconstant weakly decreasing function $f(t)$ such that $x_i(N)/N$ converges pointwise and uniformly to f. E.g.

$$\left|\frac{x_i(N)}{N} - f(i/N)\right| = o(1/\sqrt{N})$$

as $N \to \infty$ will suffice. Then for every k, as $N \to \infty$ and $q \to 1$ as $q = e^{-\gamma/N}$ for some constant $\gamma \geq 0$, we have

$$\frac{x^k(N) - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k,$$

in the sense of weak convergence, for some explicit constants $E(f)$ and $S(f)$.

- $\gamma = 0$ corresponds to the uniform measure. Obtained earlier by Gorin,Panova.
- If we let f to be piecewise constant, we get the result for certain polygonal regions, incuding the hexagon. The case of the hexagon with the uniform measure was done first by Johansson, Nordenstam.

Breaking away from the GUE corners process

Given $\{q_i\}_{i\in\mathbb{Z}}$, $q_i > 0$, consider plane partitions with the distribution

$$Prob(\pi) \propto \prod_{i\in\mathbb{Z}} q_i^{|\pi^i|},$$

where $|\pi^i|$ is the total volume of the i-th slice of π.

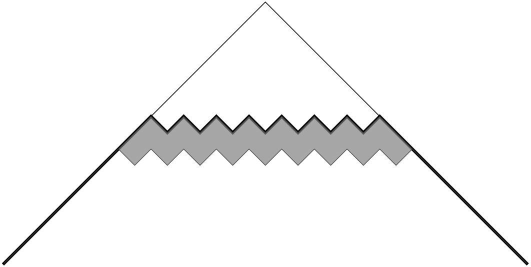
Periodic weights

- Consider weights with

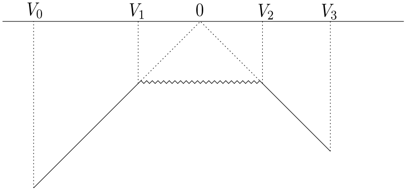
$$
\begin{aligned}
q_0 &= q_{\pm k} = q_{\pm 2k} = \dots \\
q_1 &= q_{\pm k+1} = q_{\pm 2k+1} = \dots \\
q_2 &= q_{\pm k+2} = q_{\pm 2k+2} = \dots \\
&\dots \\
q_{k-1} &= q_{\pm 2k-1} = q_{\pm 3k-1} = \dots
\end{aligned}
$$

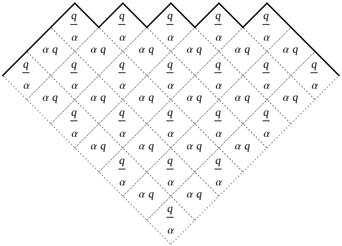
- What scaling limit should we study?
- For simplicity, set $k = 2$ for now.
- Nothing new, if you take $q_0 \to 1^-$ and $q_1 \to 1^-$.
- More interesting: $\alpha \geq 1$, $q_0 = \alpha q$, $q_1 = \alpha^{-1} q$ and $q \to 1^-$.

Periodic weights

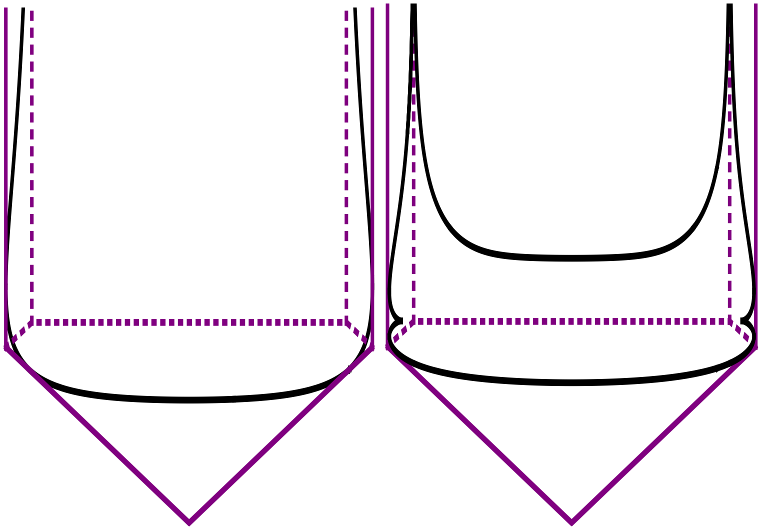


Periodic weights

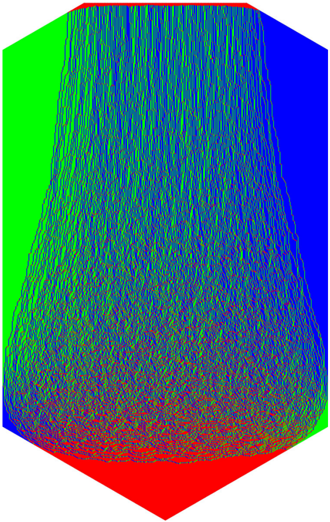


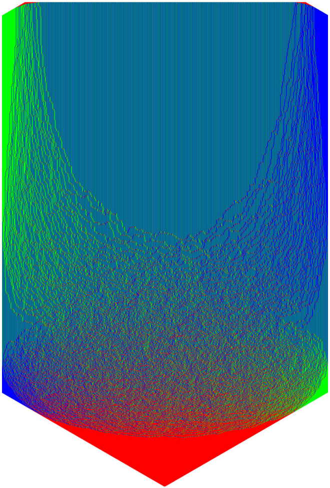


Periodic weights



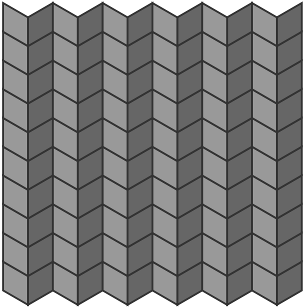
Bounded floor: A sample



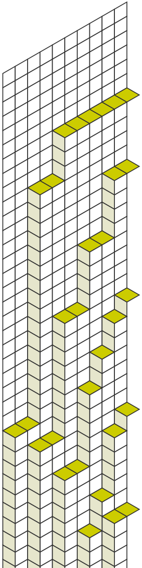


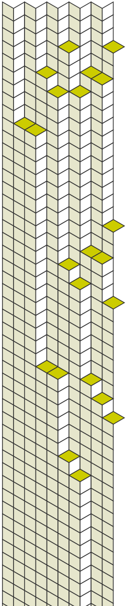
Turning points

- There are two turning points near each vertical boundary section.
- The fact that there are two turning points implies that locally you do not have the interlacing property from slice to slice.
- The distance between the turning points converges to zero when α converges to 1.
- Turning points are separated by a deterministic region of two types of tiles:



Turning points and the GUE Corners process





Turning point correlations

Theorem (M.)

Let χ be the expected hight of a turning point and let $h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$, where $q = e^{-r}$. The correlation functions near a turning point of the system with periodic weights are given by

$$\lim_{r \to 0} r^{-\frac{1}{2}} K_{\lambda, \bar{q}}((t_1, h_1), (t_2, h_2)) = \frac{1}{(2\pi\mathfrak{i})^2} \iint e^{\frac{\sigma^2}{2}(\zeta^2 - \omega^2)} \frac{e^{\tilde{h}_2 \omega}}{e^{\tilde{h}_1 \zeta}} \frac{\omega^{\lfloor \frac{t_2+e}{2} \rfloor}}{\zeta^{\lfloor \frac{t_1+e}{2} \rfloor}} \frac{d\zeta \; d\omega}{\zeta - \omega},$$

where e is 1 or 2 depending on whether $\chi = \chi_{bottom}$ or $\chi = \chi_{top}$.

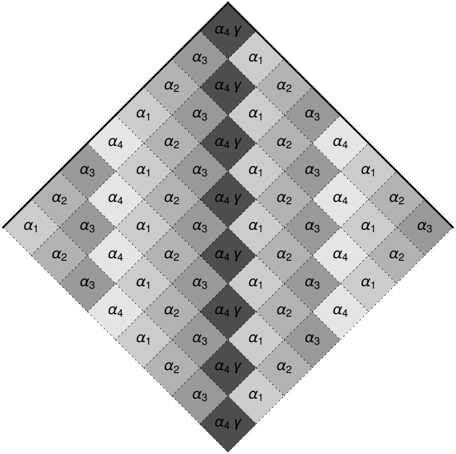
Remark: Not surprising that we don't get the GUE-corners process. Conditioned on the k'th slice, the previous slices are not uniform.
Remark: If we restrict the process to horizontal lozenges of only even or only odd distances from the edge, then the correlation kernel coincides with the correlation kernel of the GUE-corners process, so we have two GUE-corners processes non-trivially correlated.
Remark: Interlacing is not a geometric constraint anymore.

General k

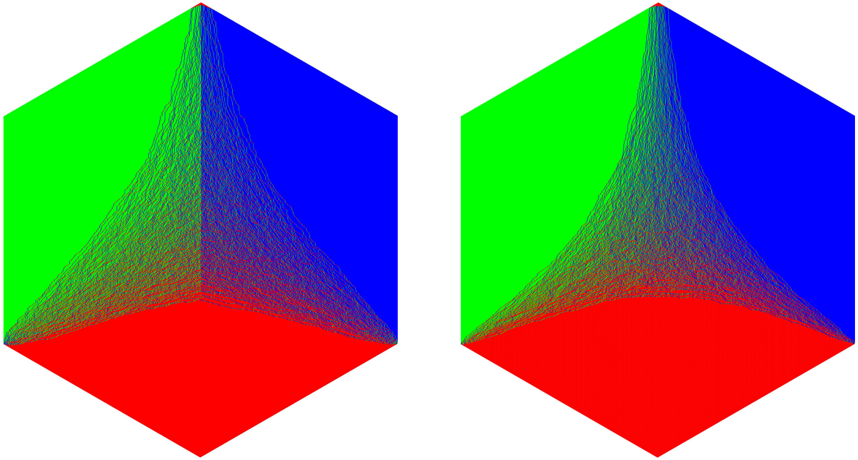
- What happens for arbitrary finite period k for the weights?
- Consider weights
$$q_i = \alpha_i q,$$
$i = 1, \ldots, k$, with $q_j = q_{j+k}, \forall j$. Let
$$\prod_{i=1}^{k} \alpha_i = 1$$
and consider the limit $q \to 1^-$.
- If we have $\alpha_i > 1$ for some i, we run into the same issues with the measure being infinite as before.
- Modifying the boundary as in the case $k = 2$ does not work anymore.

Periodic weights with one defect

- Alternate solution: periodic weights with one "defect".
- Define
$$\gamma = \prod_{\alpha_i < 1} \alpha_i .$$
- Consider plane partitions with weights
$$q_i = \begin{cases} \alpha_{i \bmod k} q, & i \neq 0 \\ \gamma \alpha_0 q, & i = 0. \end{cases} .$$
- This way, for all $m < n$ we have
$$\prod_{i=m}^{n} q_i < 1,$$
so the partition function doesn't blow up.



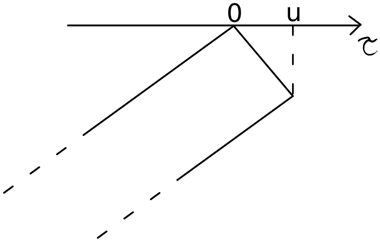
First order phase transition



(These two figures not completely accurate)

Turning points with k-periodic weights

- For simplicity consider a semi-infinite floor.



- Let $\beta_i = \prod_{j=1}^{i} \alpha_j$ for $i = 1, \dots, k$.
- Let

$$\tilde{\beta}_1 < \cdots < \tilde{\beta}_m$$

  be distinct such that

$$\{\beta_1, \dots, \beta_k\} = \{\tilde{\beta}_1, \dots, \tilde{\beta}_m\}.$$

Turning points for k-periodic weights

Theorem (M.)

- *The system developes m turning points near the boundary* $\tau = u$. *Let* $\chi_1 < \cdots < \chi_m$ *be the vertical coordinates of the turning points.*
- *The correlation functions near the turning point* $(\chi_j, \tau = u)$ *of the system with k-periodic weights are given by*

$$\lim_{r \to 0} r^{-\frac{1}{2}} K_{\lambda,\bar{q}}((t_1, h_1), (t_2, h_2)) = \frac{1}{(2\pi i)^2} \iint e^{\frac{\sigma^2}{2}(\zeta^2 - \omega^2)} \frac{e^{\tilde{h}_2 \omega}}{e^{\tilde{h}_1 \zeta}} \frac{\omega^{c_j(t_2)}}{\zeta^{c_j(t_1)}} \frac{d\zeta \; d\omega}{\zeta - \omega},$$

where $h_i = \lfloor \frac{\chi_j}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$, $q = e^{-r}$, *and* $c_j(t) = \#\{t \leq m < u : \beta_m = \tilde{\beta}_j\}$.

GUE corners, Semi-frozen regions

Corollary

- *The number of horizontal lozenges on slice t near turning point (χ_j, u) is $c_j(t)$.*
- *For any sequence of slices $t_1 > t_2 > \dots$ such that $c_j(t_i) = i$ (i.e. the i'th chosen slice has i horizontal lozenges), then the point process of the horizontal lozenges on these slices is the GUE corner process.*

Corollary

- *If $m = k$, i.e. if all β's are distinct, then at the turning point (χ_j, u) two semi-frozen regions meet: they both have k-periodic profiles of left and right lozenges, with one having $j-1$ left and $k-(j-1)$ right lozenges, and the other having j left and $k-j$ right lozenges.*
- *For any semi-frozen region of left and right lozenges with period k there exist weights $\alpha_1, \dots, \alpha_k$ such that the system with those weights developes such a semi-frozen region.*

Intermediate regime

- Consider two-periodic weights αq, $\frac{1}{\alpha}q$ again.
- Question: What happens when $\alpha \to 1$?
- Consider two-periodic weights q_t given by

$$q_t = \begin{cases} e^{-r+\gamma r^{1/2}}, & t \text{ is even} \\ e^{-r-\gamma r^{1/2}}, & t \text{ is odd} \end{cases}, \tag{1}$$

  where $\gamma > 0$ is an arbitrary constant. This is an intermediate regime between the homogeneous weights and the inhomogeneous weights considered earlier.
- The macroscopic limit shape and correlations in the bulk are the same as in the homogeneous case.
- Periodicity disappears in the limit and we have a $\mathbb{Z} \times \mathbb{Z}$ translation invariant ergodic Gibbs measure in the bulk. However, the local point process at turning points is different from the homogeneous one. In particular, while we only have one turning point near each edge, we still do not have the GUE corners process, but rather a one-parameter deformation of it.

Turning point correlations in the intermediate regime

Theorem (M.)

Let χ *be the expected hight of a turning point and let* $h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$, *where* $q = e^{-r}$. *The correlation functions near a turning point of the system with periodic weights*(1) *are given by*

$$\lim_{r\to 0} r^{-\frac{1}{2}} K_{\lambda,\bar{q}}((t_1,h_1),(t_2,h_2))$$
$$= \frac{1}{(2\pi \mathrm{i})^2} \iint e^{C_{cr}(\zeta^2-\omega^2)} \frac{e^{\tilde{h}_2\omega}}{e^{\tilde{h}_1\zeta}} \frac{\omega^{\lfloor \frac{t_2+1}{2} \rfloor}}{\zeta^{\lfloor \frac{t_1+1}{2} \rfloor}} \frac{(\omega-\gamma)^{\lfloor \frac{t_2+2}{2} \rfloor}}{(\zeta-\gamma)^{\lfloor \frac{t_1+2}{2} \rfloor}} \frac{d\zeta \, d\omega}{\zeta-\omega}.$$