ASYMPTOTIC REGIME CHANGE FOR MULTIVARIATE GENERATING FUNCTIONS

Stephen Melczer
University of Pennsylvania

Generating Functions

Given a sequence $(c_n) = c_0, c_1, c_2, c_3, c_4, \ldots$ we can form its **generating function**

$$C(z) := \sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

Combinatorial definitions often **automatically** translate to generating function specifications.

Generating Functions

Given a sequence $(c_n) = c_0, c_1, c_2, c_3, c_4, \ldots$ we can form its **generating function**

$$C(z) := \sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

Combinatorial definitions often **automatically** translate to generating function specifications.

> Firſt, *Let the Relation of each Term to the two preceding ones be expreſſed in this manner,* viz. *Let* C *be* $= \mathrm{m}\,\mathrm{B}\,\mathrm{r} - \mathrm{n}\,\mathrm{A}\,\mathrm{rr}$; *and let* D *likewiſe be* $= \mathrm{m}\,\mathrm{C}\,\mathrm{r} - \mathrm{n}\,\mathrm{B}\,\mathrm{rr}$, *and ſo on: Then will the ſum of that Infinite Series be equal to* $\frac{\mathrm{A} + \mathrm{B} - \mathrm{mr}\,\mathrm{A}}{1 - \mathrm{mr} + \mathrm{nrr}}$.

A. de Moivre, The Doctrine of Chances or a Method of Calculating the Probabilities of Events in Play, 1718

Generating Functions

Given a sequence $(c_n) = c_0, c_1, c_2, c_3, c_4, \ldots$ we can form its **generating function**

$$C(z) := \sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

Combinatorial definitions often **automatically** translate to generating function specifications.

Class	Example	Encoding
Rational	Regular Languages	Numerator + Denominator
Algebraic	Types of trees	Min. Poly + Initial Terms
D-Finite	Linear Recurrences with Polynomial Coefficients	Differential Eq. + Initial Terms

Ge

Given
gene

Comb
gene

A *D-Finite* function is one which satisfies a linear differential equation with polynomial coefficients.

Example: The transcendental generating function

$$F(z) = \sum_{n \geq 0} \binom{2n}{n}^2 z^n$$

satisfies

$$(z - 16z^2)F''(z) + (1 - 32z)F'(z) - 4F(z) = 0$$

Cla	Example	Encoding
Rati nal	Regular Languages	Numerator + Denominator
Alge raic	Types of trees	Min. Poly + Initial Terms
D-Finite	Linear Recurrences with Polynomial Coefficients	Differential Eq. + Initial Terms

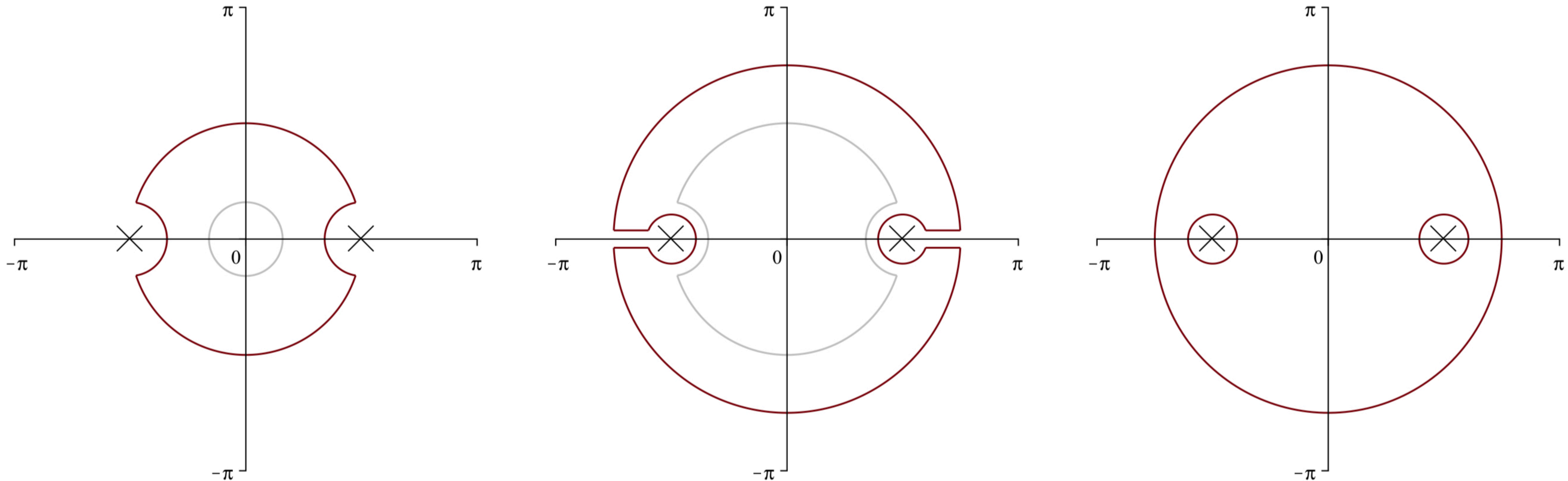
Basics of Analytic Combinatorics

There are deep links between **analytic properties** of a generating function and **asymptotics** of its coefficients.

If $F(z) = \sum_{n \geq 0} f_n z^n$ is analytic at the origin, then CIF implies

$$f_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{n+1}} dz$$

where C is a sufficiently small circle around the origin



D-Finite Functions

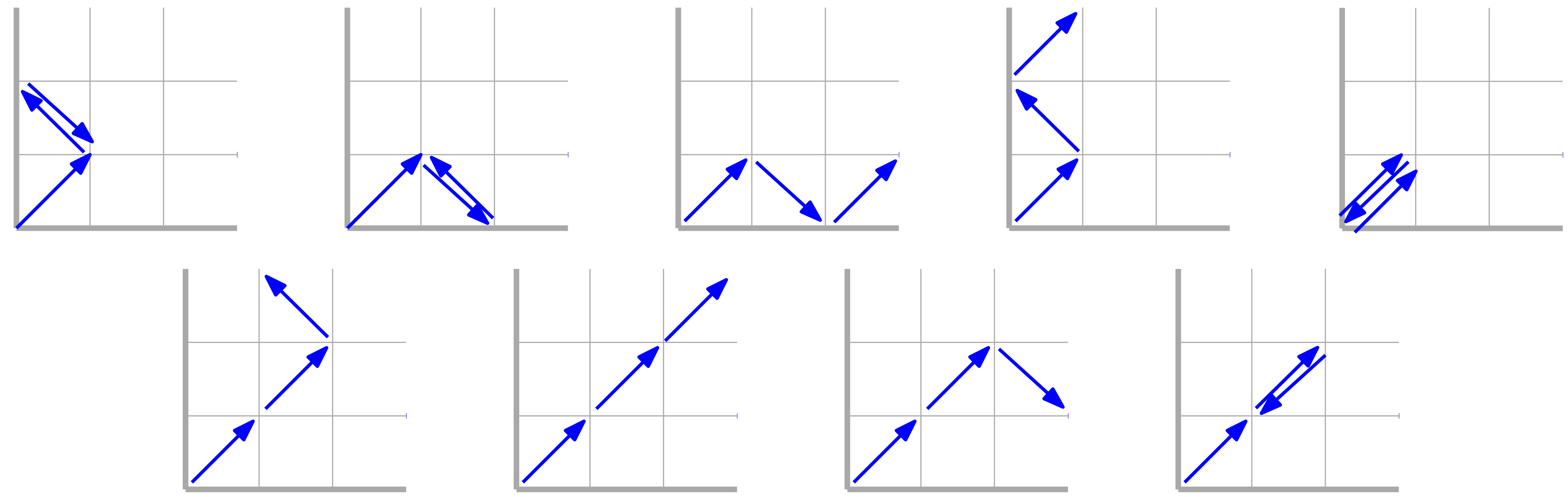
Example

The number of walks from the origin taking steps $\{NE, NW, SE, SW\}$ and staying in the first quadrant has GF satisfying

$$\mathcal{L} \cdot Q(z) = 1$$

where

$$\mathcal{L} = (32z^5 - 2z^3)\frac{d^3}{dz^3} + (240z^4 + 8z^3 - 9z^2)\frac{d^2}{dz^2} + (368z^3 + 24z^2 - 5z)\frac{d}{dz} + (80z^2 + 4z + 1)$$



D-Finite Functions

This is equivalent to the coefficient sequence (q_n) satisfying a linear recurrence relation with polynomial coefficients.

For the last example,

$$(2n+3)(n+3)^2 q_{n+2} = (8n^2+32n+28)q_{n+1} + 16(2n+5)(n+1)^2 q_n$$

There are methods to find an **asymptotic basis** of solutions of a linear recurrence, but one must write the sequence of interest as a linear combination of the basis elements (*connection problem*).

Here a basis has leading terms $\left\{\frac{4^n}{n}, \frac{(-4)^n}{n^2}\right\}$, so

$$q_n = \frac{4^n}{n}\left(C + O\left(\frac{1}{n}\right)\right)$$

D-Finite Functions

This is equivalent to the coefficient sequence (q_n) satisfying a linear recurrence relation with polynomial coefficients.

For the last example,

$$(2n+3)(n+3)^2 q_{n+2} = (8n^2+32n+28)q_{n+1} + 16(2n+5)(n+1)^2 q_n$$

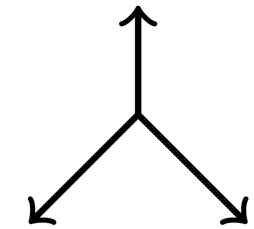
There are methods to find an **asymptotic basis** of solutions of a linear recurrence, but one must write the sequence of interest as a linear combination of the basis elements (*connection problem*).

Here a basis has leading terms $\left\{ \frac{4^n}{n}, \frac{(-4)^n}{n^2} \right\}$, so

$C = 0.6366\ldots$
$(= 2/\pi)$

$$q_n = \frac{4^n}{n}\left(C + O\left(\frac{1}{n}\right)\right)$$

Lattice Path Example



Consider walks on the steps {N, SE, SW}, restricted to the non-negative quadrant. The number of walks satisfies an order 15 linear recurrence with poly coefficients.

There is an asymptotic basis consisting of

$$3^n\, n^{-1/2} \left(1 - \frac{33}{16}\, n^{-1} + \cdots \right)$$

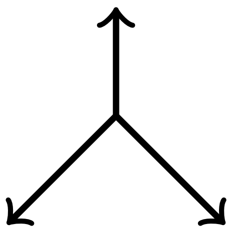
$$\left(2\sqrt{2}\right)^n\, n^{-2} \left(1 - \frac{32\sqrt{2}+57}{4} n^{-1} + \cdots \right)$$

$$\left(-2\sqrt{2}\right)^n\, n^{-2} \left(1 + \frac{32\sqrt{2}-57}{4} n^{-1} + \cdots \right)$$

with other elements $O\left(\frac{(2\sqrt{2})^n}{n^3}\right)$

Bostan, Chyzak, van Hoeij, Kauers, and Pech 2017

Lattice Path Example

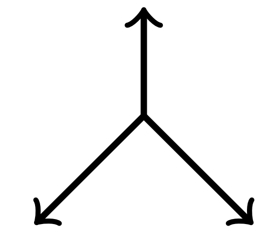


One can write

$$q_n = C_1 \frac{3^n}{\sqrt{n}} \left(1 - \frac{33}{16n} + \cdots\right) \quad + \quad C_2 \frac{(2\sqrt{2})^n}{n^2} \left(1 - \frac{32\sqrt{2}+57}{4n} + \cdots\right)$$

$$+ C_3 \frac{(-2\sqrt{2})^n}{n^2} \left(1 - \frac{32\sqrt{2}-57}{4n} + \cdots\right) \quad + \quad O\left(\frac{(2\sqrt{2})^n}{n^3}\right)$$

Lattice Path Example



One can write

$$q_n = C_1 \frac{3^n}{\sqrt{n}} \left(1 - \frac{33}{16n} + \cdots \right) \quad + \quad C_2 \frac{(2\sqrt{2})^n}{n^2} \left(1 - \frac{32\sqrt{2}+57}{4n} + \cdots \right)$$

$$+ C_3 \frac{(-2\sqrt{2})^n}{n^2} \left(1 - \frac{32\sqrt{2}-57}{4n} + \cdots \right) \quad + \quad O\left(\frac{(2\sqrt{2})^n}{n^3}\right)$$

Bostan, Chyzak, van Hoeij, Kauers, and Pech 2017

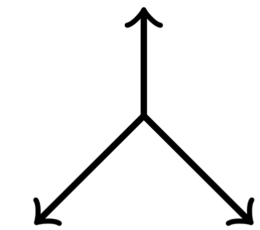
Let

$$\phi(t) = 2\frac{(1-6t^2-8t^3)\,{}_2F_1\left(\begin{matrix}1/4\ 3/4\\ 1\end{matrix}\,\middle|\, 64t^4\right) + 4t^3(1-7t+4t^2)\,{}_2F_1\left(\begin{matrix}3/4\ 5/4\\ 2\end{matrix}\,\middle|\, 64t^4\right)}{(1-2t)^2(1+t)^{3/2}}$$

Then $C_1 = 0$ if and only if

$$\int_0^{1/3} \frac{\phi(t)}{\sqrt{1-3t}}\,dt = 1$$

Lattice Path Example



One can write

$$q_n = {\color{red}C_1}\frac{3^n}{\sqrt{n}}\left(1-\frac{33}{16n}+\cdots\right) \quad + \quad {\color{red}C_2}\frac{(2\sqrt{2})^n}{n^2}\left(1-\frac{32\sqrt{2}+57}{4n}+\cdots\right)$$

$$+\,{\color{red}C_3}\frac{(-2\sqrt{2})^n}{n^2}\left(1-\frac{32\sqrt{2}-57}{4n}+\cdots\right) \quad + \quad O\left(\frac{(2\sqrt{2})^n}{n^3}\right)$$

Bostan, Chyzak, van Hoeij, Kauers, and Pech 2017

Let

$$\phi(t) = 2\frac{(1-6t^2-8t^3)\,{}_2F_1\left(\begin{matrix}1/4\;3/4\\1\end{matrix}\middle|\,64t^4\right)+4t^3(1-7t+4t^2)\,{}_2F_1\left(\begin{matrix}3/4\;5/4\\2\end{matrix}\middle|\,64t^4\right)}{(1-2t)^2(1+t)^{3/2}}$$

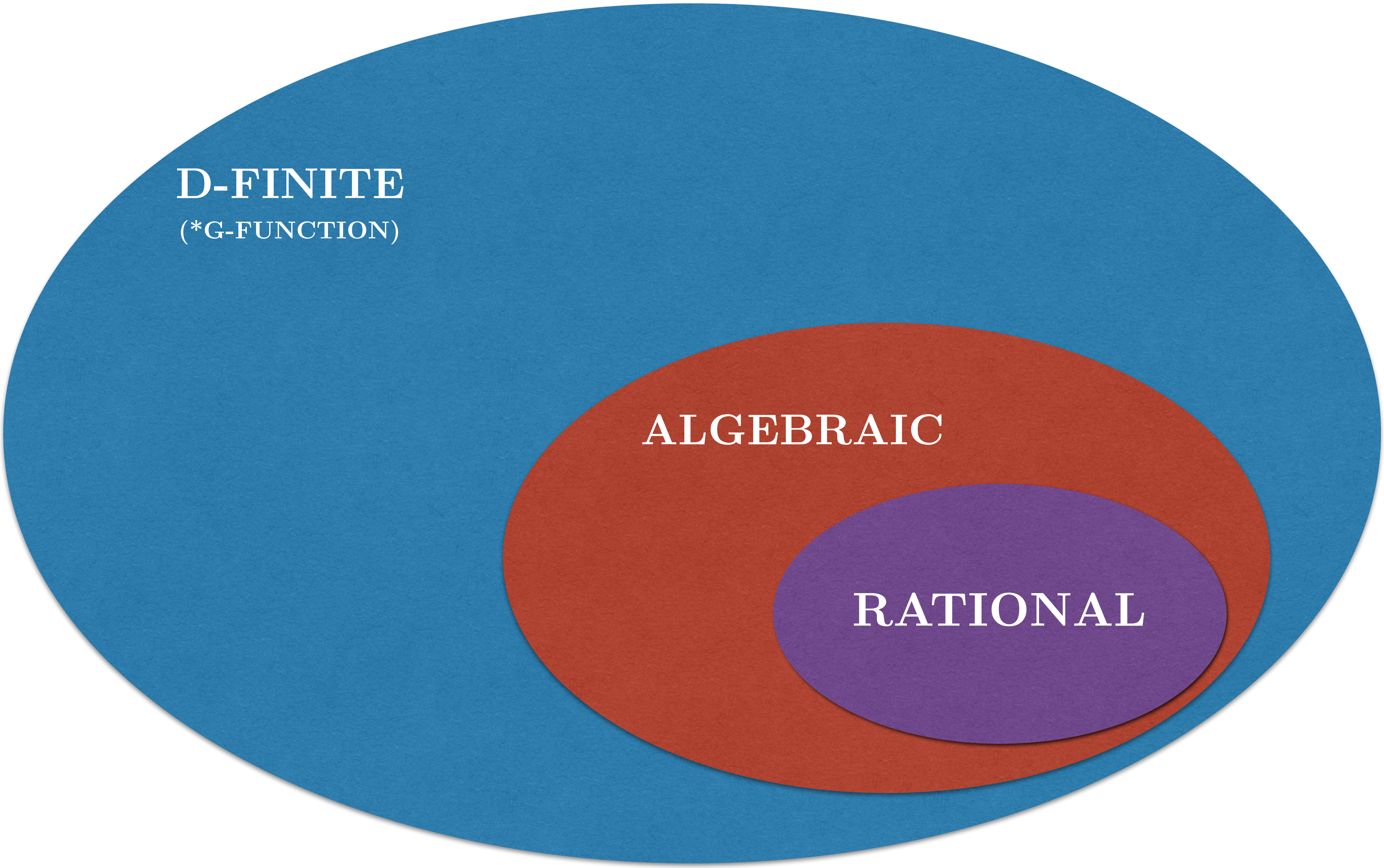
Then $C_1 = 0$ if and only if

$$\int_0^{1/3}\frac{\phi(t)}{\sqrt{1-3t}}dt = 1$$

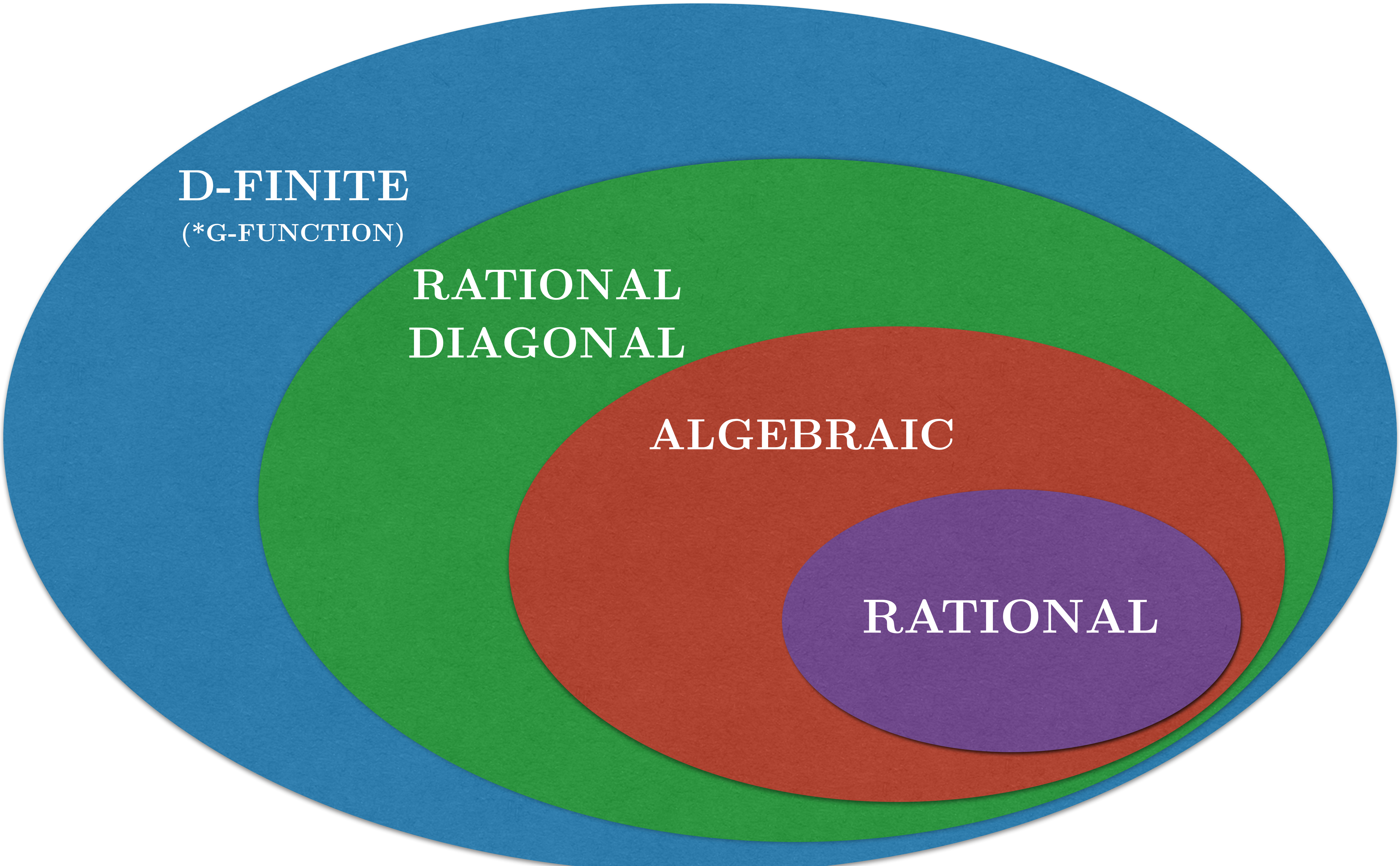
M. and Wilson 2016/18

$C_1 = 0$, values for other constants, and results for similar lattice path models

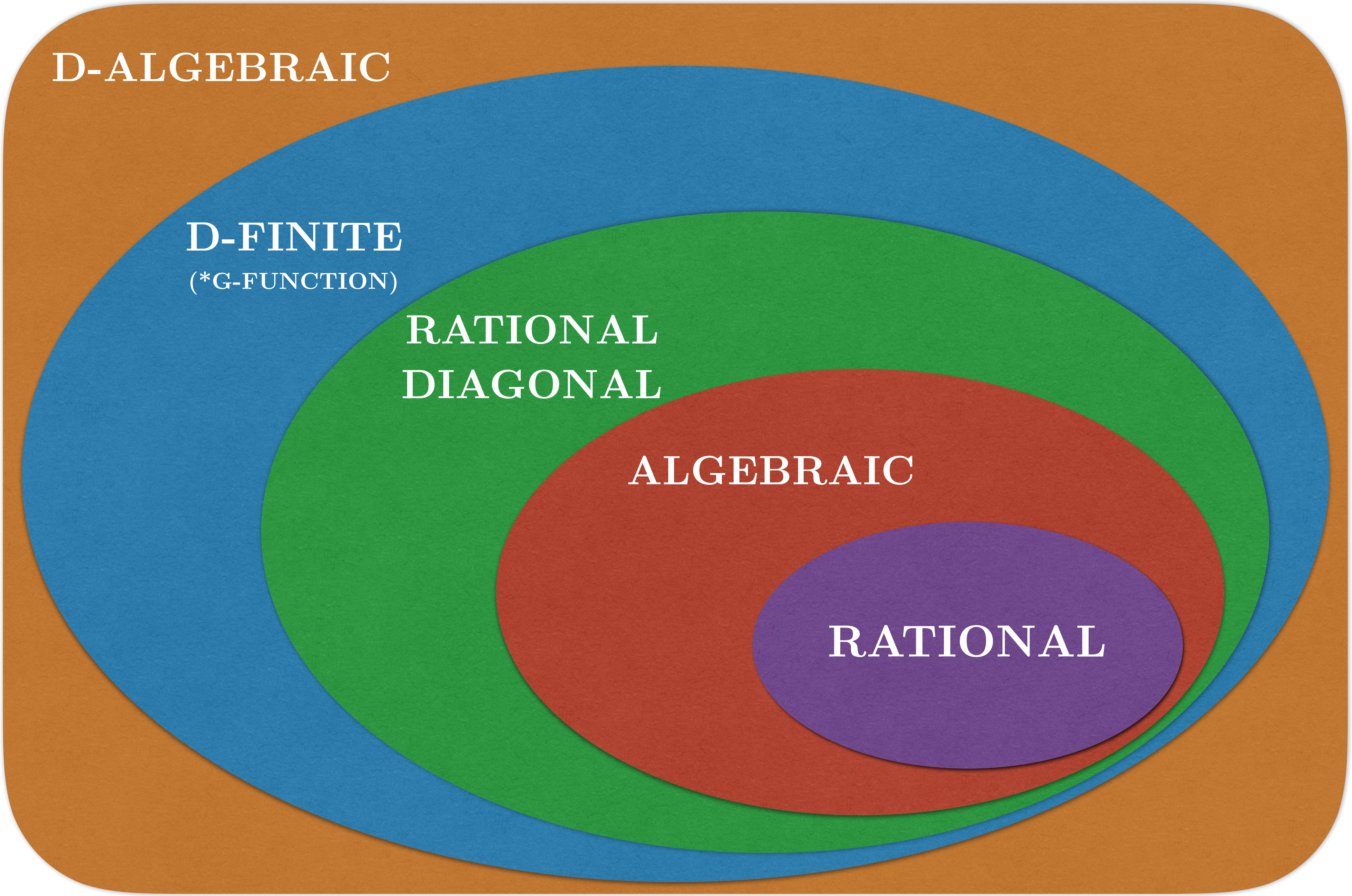
Generating Function Classes



Generating Function Classes



Generating Function Classes



Multivariate Rational Diagonals

Idea: Use a multivariate rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ to encode sequences

$$F(\mathbf{z}) = \sum_{(i_1,\dots,i_d)\in\mathbb{N}^d} f_{i_1,\dots,i_d} z_1^{i_1} \cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

Example (Main Diagonal)

The main diagonal sequence consists of the terms $f_{n,n,\dots,n}$

$$F(x,y) = \frac{1}{1-x-y}$$

$$= 1 + x + y + 2xy + x^2 + y^2 + x^3 + 3x^2y + 3xy^2 + y^3 + 6x^2y^2 + \cdots$$

Multivariate Rational Diagonals

Idea: Use a multivariate rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ to encode sequences

$$F(\mathbf{z}) = \sum_{(i_1,\dots,i_d)\in\mathbb{N}^d} f_{i_1,\dots,i_d} z_1^{i_1} \cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

Example (Apéry)

$$F(w,x,y,z) = \frac{1}{1 - z(1+w)(1+x)(1+y)(wxy + xy + x + y + 1)}$$

Here $(f_{n,n,n,n})_{n\geq 0}$ determines Apéry's sequence, related to his celebrated proof of the irrationality of $\zeta(3)$.

Multivariate Rational Diagonals

Idea: Use a multivariate rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ to encode sequences

$$F(\mathbf{z}) = \sum_{(i_1,\ldots,i_d)\in\mathbb{N}^d} f_{i_1,\ldots,i_d} z_1^{i_1} \cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

Exercise

Be the first in your block to prove by a 2-line argument that $\zeta(3)$ is irrational.[7]

(6) Given the definitions of (5) show that $a_n b_{n-1} - a_{n-1} b_n = b_n^{-3}$ and $b_n = O(\alpha^n)$ with $\alpha = (1+\sqrt{2})^4$. Conclude that $\zeta(3)$ is irrational because $\log \alpha > 3$.

A Proof that Euler Missed, Alfred van der Poorten

Multivariate Rational Diagonals

Idea: Use a multivariate rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ to encode sequences

$$F(\mathbf{z}) = \sum_{(i_1,\ldots,i_d)\in\mathbb{N}^d} f_{i_1,\ldots,i_d} z_1^{i_1}\cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$$

Example (Lattice Paths)

The number of walks on  which start at the origin and stay in the first quadrant form the diagonal of

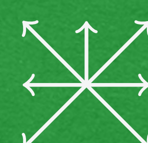
$$\frac{(1+x)(1+y)}{1-txy(x+y+1/x+1/y)}$$

Multivariate Rational Diagonals

Idea: Use a multivariate rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ to encode sequences

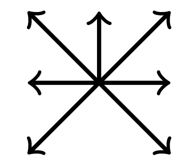
$$F(\mathbf{z}) = \sum_{(i_1,\dots,i_d)\in\mathbb{N}^d} f_{i_1,\dots,i_d} z_1^{i_1}\cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$$

Example (Lattice Paths)

The number of walks on  which start at the origin and stay in the first quadrant form the diagonal of

$$\frac{(1+x)(1-x^2y^2+x^2-y^2+x)}{(x^2+x+1)(1-y)(1-txy(xy+x+x/y+y/x+1/y+1/x+1/xy))}$$

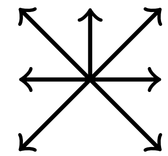
Lattice Path Example

The minimal order linear differential annihilator for  :

$$t^2 (7t-1)(2t+1)(5t+1)\left(4t^2-4t-1\right)\left(20t^2+4t-1\right)\left(t^2+t+1\right)$$
$$\left(337920\,t^{13}+1373184\,t^{12}-4304640\,t^{11}-6344576\,t^{10}-444096\,t^9+2010720\,t^8+901808\,t^7+180552\,t^6+55164\,t^5+31010\,t^4+11106\,t^3+1914\,t^2+106\,t-3\right)\partial_t{}^5$$
$$+t\left(47308800000\,t^{22}+227888332800\,t^{21}-542727905280\,t^{20}-1484019662848\,t^{19}-767620100096\,t^{18}+1085290090496\,t^{17}+1896743070208\,t^{16}+918495748096\,t^{15}\right.$$
$$-215512785664\,t^{14}-427218085376\,t^{13}-200103936864\,t^{12}-53308965120\,t^{11}-16198105488\,t^{10}-7684582384\,t^9-2788409498\,t^8-526917856\,t^7-11674372\,t^6$$
$$\left.+14725960\,t^5+2406665\,t^4+42072\,t^3-17460\,t^2-836\,t+27\right)\partial_t{}^4+2\left(189235200000\,t^{22}+910845542400\,t^{21}-2482106941440\,t^{20}-6004739067904\,t^{19}\right.$$
$$-2190950518784\,t^{18}+4468038376448\,t^{17}+5923732034560\,t^{16}+2057998031360\,t^{15}-1109933285888\,t^{14}-1316341967488\,t^{13}-560905256000\,t^{12}-165928469200\,t^{11}$$
$$\left.-64781167760\,t^{10}-30898350868\,t^9-10648213196\,t^8-2104167976\,t^7-150023840\,t^6+22705940\,t^5+5264545\,t^4+267944\,t^3-9053\,t^2-510\,t+24\right)\partial_t{}^3$$
$$+6\left(189235200000\,t^{21}+910137753600\,t^{20}-2797638696960\,t^{19}-6009599143936\,t^{18}-1214520197120\,t^{17}+4569763273728\,t^{16}+4392743400448\,t^{15}\right.$$
$$+735231523328\,t^{14}-1250713939968\,t^{13}-987314157184\,t^{12}-367899527360\,t^{11}-119740279344\,t^{10}-58557054080\,t^9-28856070484\,t^8-9660129468\,t^7$$
$$\left.-1939533508\,t^6-193545296\,t^5-497736\,t^4+1672921\,t^3+118532\,t^2+2559\,t+132\right)\partial_t{}^2+24\left(47308800000\,t^{20}+227357491200\,t^{19}-779376721920\,t^{18}\right.$$
$$-1487700047872\,t^{17}-35249020928\,t^{16}+1159020984320\,t^{15}+740359199744\,t^{14}-80316882176\,t^{13}-317740267264\,t^{12}-173358054912\,t^{11}-53419838208\,t^{10}$$
$$\left.-20200372344\,t^9-12618507248\,t^8-6380918656\,t^7-2053685840\,t^6-402111758\,t^5-44894842\,t^4-2517458\,t^3-78126\,t^2-6615\,t-384\right)\partial_t-22464-451296\,t$$
$$-26356354176\,t^6-7768879584\,t^5-1370419584\,t^4-140485008\,t^3-8567520\,t^2-64845759168\,t^8-495281498112\,t^{10}-109934770176\,t^9-53271954240\,t^7$$
$$-6994180227072\,t^{16}+1235817283584\,t^{15}+5584717234176\,t^{14}+1907260735488\,t^{13}-1376741382144\,t^{12}-1399425761280\,t^{11}+227082240000\,t^{19}$$
$$+1090466611200\,t^{18}-4130053816320\,t^{17}$$

Bostan, Chyzak, van Hoeij, Kauers, and Pech 2017

Lattice Path Example

An "explicit" expression for  :

$$\frac{1}{t(2t+1)}\int\left(1+\int\frac{(2t+1)(5t+1)}{\left(-35t^2-2t+1\right)^{5/2}}\left(10+\int\frac{12\left(-35\,t^2-2t+1\right)^{3/2}}{(5t+1)(12t^2+1)^{9/2}(2t+1)^2}\left((12t^2+1)\right.\right.\right.$$

$$(736t^5+2208t^4+1096t^3-44t^2+44t+1)\,{}_2\mathrm{F}_1\left(7/4,9/4;\;2;\;64\,\frac{\left(t^2+t+1\right)t^2}{\left(12\,t^2+1\right)^2}\right)$$

$$\left.\left.\left.-7\,t\left(1824\,t^6+2496\,t^5+1288\,t^4+452\,t^3+420\,t^2+53\,t+10\right){}_2\mathrm{F}_1\left(9/4,11/4;\;3;\;64\,\frac{\left(t^2+t+1\right)t^2}{\left(12\,t^2+1\right)^2}\right)\right)\right)\,\mathrm{d}t\right)\,\mathrm{d}t$$

Bostan, Chyzak, van Hoeij, Kauers, and Pech 2017

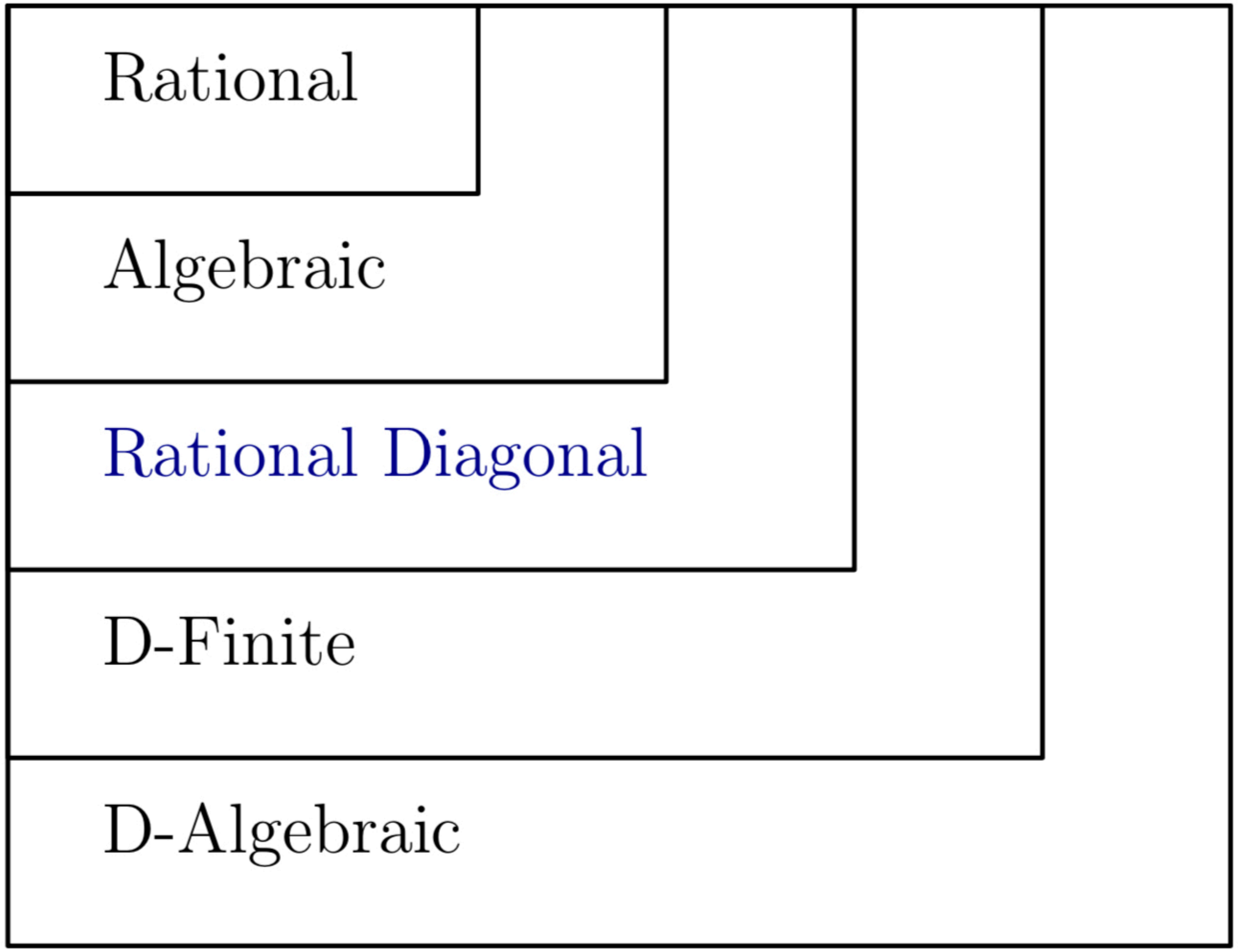
Multivariate Rational Diagonals

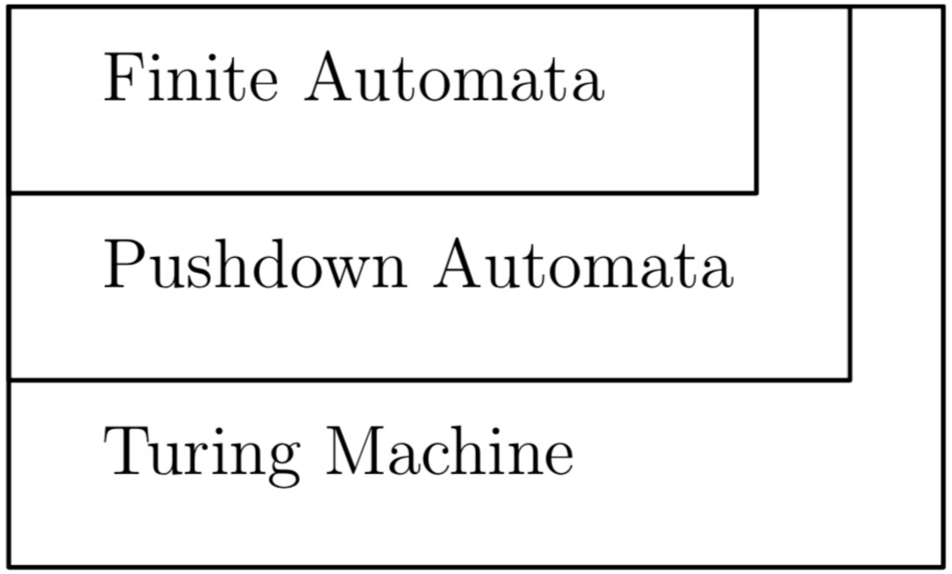
Many problems in

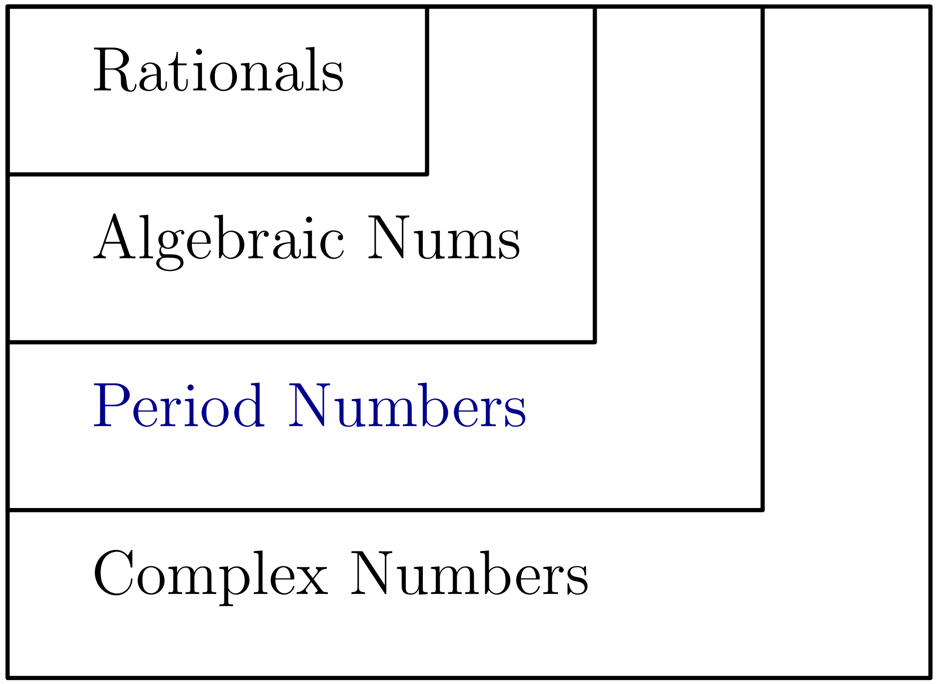
- combinatorics (lattice path enumeration, tilings, strings)
- probability theory (random walk models)
- number theory (binomial sums like Apéry's sequence)
- physics (the Ising model, rational period integrals)
- representation theory (Kronecker coefficients)
- computer science (automatic sequences, Kronecker coefficients)

and more appear naturally as questions about rational diagonals, which are compact **encodings**

Goal: Automatic asymptotics of rational diagonal sequences.


Rational
Algebraic
Rational Diagonal
D-Finite
D-Algebraic


Finite Automata
Pushdown Automata
Turing Machine


Rationals
Algebraic Nums
Period Numbers
Complex Numbers

Diagonal Asymptotics

Assume

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$$

is analytic at the origin, with open domain of convergence $\mathcal{D}$.

The singularities of $F(\mathbf{z})$ are given by $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$.
Points in $\partial\mathcal{D} \cap \mathcal{V}$ are called **minimal points**.

Diagonal Asymptotics

Assume

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

is analytic at the origin, with open domain of convergence $\mathcal{D}$.

The singularities of $F(\mathbf{z})$ are given by $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$.
Points in $\partial\mathcal{D} \cap \mathcal{V}$ are called **minimal points**.

Equivalently, no other singularities have smaller coordinate-wise *modulus*

Diagonal Asymptotics

Assume

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

is analytic at the origin, with open domain of convergence $\mathcal{D}$.

The singularities of $F(\mathbf{z})$ are given by $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$.
Points in $\partial\mathcal{D} \cap \mathcal{V}$ are called **minimal points**.

The Cauchy integral formula has a higher-dim generalization

$$f_{n,\ldots,n} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} \frac{F(\mathbf{z})}{(z_1 \cdots z_d)^{n+1}} d\mathbf{z}$$

The field of *analytic combinatorics in several variables (ACSV)* uses singularity analysis to determine asymptotics

Critical Points

We understand asymptotics of Gaussian integrals very well

$$\int_{-c}^{c} A(\mathbf{x}) e^{-n\left(\mathbf{x}^T \mathcal{H} \mathbf{x}\right)/2} d\mathbf{x} = A(\mathbf{0}) \sqrt{\frac{(2\pi)^d}{n^d \det \mathcal{H}}} + O\left(n^{-d/2-1}\right)$$

The Cauchy integral has the Fourier-Laplace form

$$\int A(\mathbf{z}) e^{-n\phi(\mathbf{z})} d\mathbf{z}$$

with

$$\phi(\mathbf{z}) = \log(z_1) + \cdots + \log(z_d)$$

Critical Points

ACSV uses complex residues to rewrite the Cauchy integral as a local integral restricted to part of $\mathcal{V}$

Thus, one decomposes $\mathcal{V}$ into a union of smooth manifolds and finds **critical points** of $\phi(\mathbf{z})$ on each *strata*

Critical points are defined by vanishing of matrix minors.
Simplest case: $\mathcal{V}$ is a manifold and critical points defined by

$$z_1 H_{z_1} = \cdots = z_d H_{z_d}, \qquad H = 0$$

Minimal points are those that the Cauchy integral can be deformed close to, critical points are those where saddle-point approximations can be made

Complexity Results for ACSV

Suppose that $G(\mathbf{z})$ and $H(\mathbf{z})$ have coefficients $\leq 2^h$ and degree $\boldsymbol{q}$

Suppose also that the power series of $F(\mathbf{z})$ has non-negative coefficients

Theorem (M. and Salvy, 2016)

Under generic and verifiable assumptions one can **find all minimal critical points** in $\tilde{O}(hq^{4d+5})$ bit operations

For any $M \in \mathbb{N}$ one can compute algebraic constants such that

$$f_{n,\dots,n} = \rho^n \, n^{(1-d)/2} \cdot \pi^{(1-d)/2} \left(\sum_{j=0}^{M} C_j^{(n)} n^{-j} + O(n^{-M-1}) \right)$$

C_0 is explicit and can be determined to $2^{-\kappa}$ in $\tilde{O}(\kappa q^{d+1} + hq^{3d+3})$ bit ops

Complexity Results for ACSV

Can remove non-negativity assumption, with increased complexity.

Theorem (M. and Salvy, 2018)

Under verifiable assumptions, there exists a probabilistic algorithm which finds minimal critical points in $\tilde{O}(hq^{9d+4}2^{3d})$ bit ops.

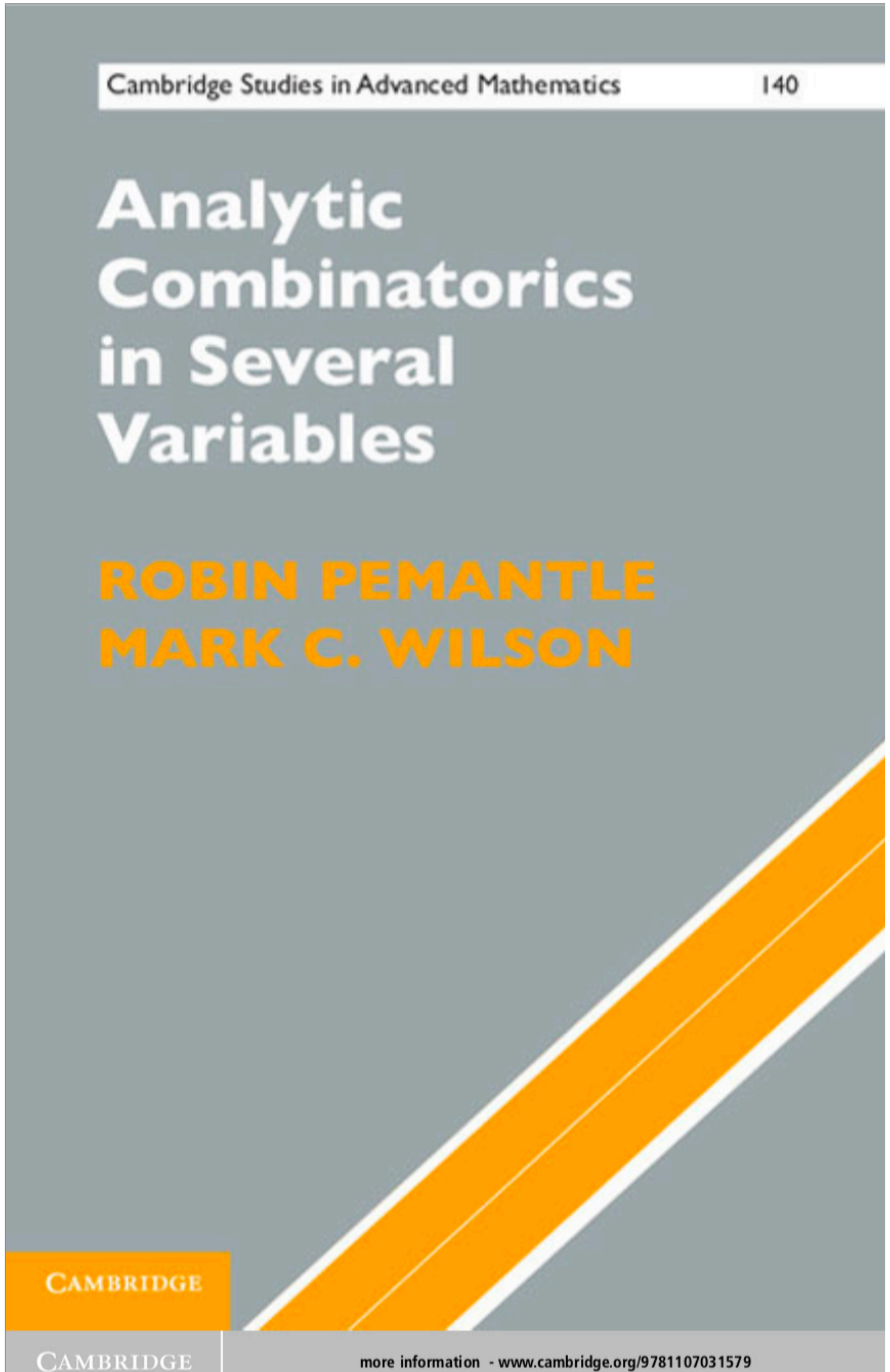
Example (Apéry)

$$F(w,x,y,z) = \frac{1}{1-z(1+w)(1+x)(1+y)(wxy+xy+x+y+1)}$$

```
> A, U, PRINT := DiagonalAsymptotics(numer(F),denom(F),[a,b,c,z],u,k, useFGb):
A, U;
```

$$\frac{1}{4}\,\frac{\left(\frac{2\,u-366}{34\,u+1458}\right)^{k}\sqrt{2}\,\sqrt{\frac{2\,u-366}{-96\,u-4192}}}{k^{3/2}\,\pi^{3/2}},\ \left[\mathit{RootOf}\left(\_Z^2-366\,\_Z-17711,\ -43.27416997969\right.\right.$$

Analytic Combinatorics in Several Variables



arXiv.org > math > arXiv:1709.05051

Mathematics > Combinatorics

Analytic Combinatorics in Several Variables: Effective Asymptotics and Lattice Path Enumeration

Stephen Melczer

Comments: PhD thesis, University of Waterloo and ENS Lyon – 259 pages
Subjects: **Combinatorics (math.CO)**; Symbolic Computation (cs.SC)
Cite as: arXiv:1709.05051 **[math.CO]**

Theory developing rapidly
(textbook based on thesis coming soon)

Diagonals in General Directions

In general, the ***r**-diagonal* of F forms the coefficient sequence of

$$(\Delta_{\mathbf{r}} F)(t) = \sum_{n \geq 0} f_{nr_1,\ldots,nr_d} z_1^{nr_1} \cdots z_d^{nr_d} = \sum_{n \geq 0} f_{n\mathbf{r}} \mathbf{z}^{n\mathbf{r}}$$

A priori, the coefficient $f_{n\mathbf{r}}$ is only nonzero if $n\mathbf{r} \in \mathbb{N}^d$

In particular, this sequence is only non-trivial when $\mathbf{r} \in \mathbb{Q}^d_{\geq 0}$

Diagonals in General Directions

In general, the ***r**-diagonal* of F forms the coefficient sequence of

$$(\Delta_{\mathbf{r}}F)(t) = \sum_{n\geq 0} f_{nr_1,\ldots,nr_d} z_1^{nr_1} \cdots z_d^{nr_d} = \sum_{n\geq 0} f_{n\mathbf{r}} \mathbf{z}^{n\mathbf{r}}$$

A priori, the coefficient $f_{n\mathbf{r}}$ is only nonzero if $n\mathbf{r} \in \mathbb{N}^d$
In particular, this sequence is only non-trivial when $\mathbf{r} \in \mathbb{Q}^d_{\geq 0}$
Again we can write

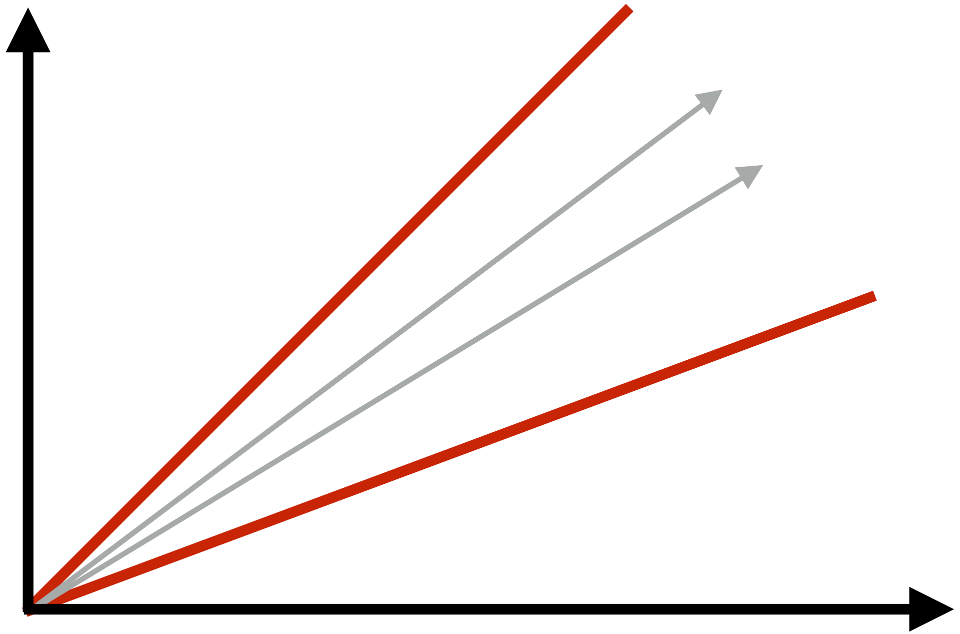
$$f_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}}$$

Generic Asymptotics

For "generic" directions $\mathbf{r}$ asymptotics have a uniform expression varying smoothly with $\mathbf{r}$ staying in fixed cones of $\mathbb{R}^d_{\geq 0}$

Thus, one can define asymptotics for any (generic) direction $\mathbf{r} \in \mathbb{R}^d_{\geq 0}$ as a limit!

$$f_{n\mathbf{r}} \to \lim_{\substack{\mathbf{s}\to\mathbf{r} \\ \mathbf{s}\in\mathbb{Q}^d}} \left(\lim_{n\to\infty} f_{n\mathbf{s}} \right)$$



Example

Consider

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

Then

$$[x^{an}y^{bn}]F(x,y) = \frac{\sqrt{1/a+1/b}}{\sqrt{2\pi\, n}} \left(\frac{a+b}{a}\right)^{an} \left(\frac{a+b}{b}\right)^{bn} \left(1+O\left(\frac{1}{n}\right)\right)$$

Example

Consider

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

Then

$$[x^{an}y^{bn}]F(x,y) = \frac{\sqrt{1/a+1/b}}{\sqrt{2\pi\, n}} \left(\frac{a+b}{a}\right)^{an} \left(\frac{a+b}{b}\right)^{bn} \left(1 + O\left(\frac{1}{n}\right)\right)$$

Interpreting as the limit gives asymptotics for

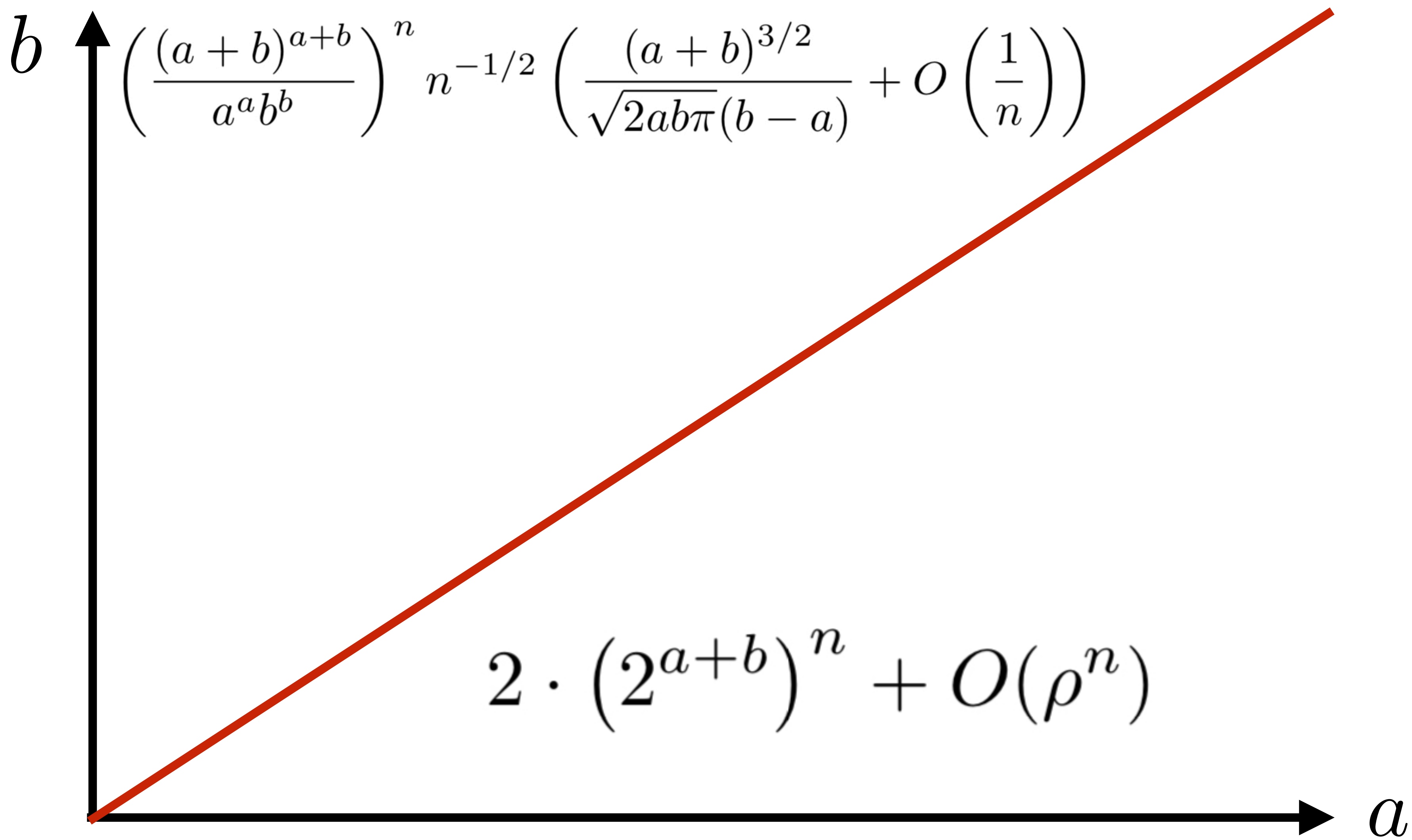
$$\binom{an+bn}{an} \approx \frac{(an+bn)!}{(an!)(bn)!} \approx \frac{\Gamma\left(an+bn+1\right)}{\Gamma\left(an+1\right)\,\Gamma\left(bn+1\right)}$$

Example #2

Let

$$F(x,y) = \frac{1}{(1-x-y)(1-2x)}$$

Then $[x^{an}y^{bn}]F(x,y)$ satisfies



Asymptotics in Generic Directions

After introducing negligible error terms, some residue computations reduce dominant asymptotics to finding asymptotics of a *Fourier-Laplace* integral

$$\int_{\mathbb{R}^r} \boldsymbol{\theta}^{\mathbf{m}} e^{-n\left(\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}\right)} \, d\boldsymbol{\theta} \qquad\qquad (r < d)$$

where $\mathbf{m} \in \mathbb{N}^r$ and $\mathcal{H}$ is a symmetric positive definite matrix

Terms in such an asymptotic expansion are known **explicitly**.

Asymptotics in Non-Generic Directions

In “non-generic” directions, one is not allowed to do all the necessary residue computations needed to reduce to a Fourier-Laplace integral, while still having acceptable error bounds

One ultimately obtains a modified expression of the form

$$\int_{\mathbb{R}^r + i(\epsilon,\ldots,\epsilon)} \boldsymbol{\theta}^{\mathbf{m}} e^{-n\left(\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}\right)} \, d\boldsymbol{\theta} \qquad (r < d)$$

where $\mathbf{m} \in \mathbb{Z}^r$.

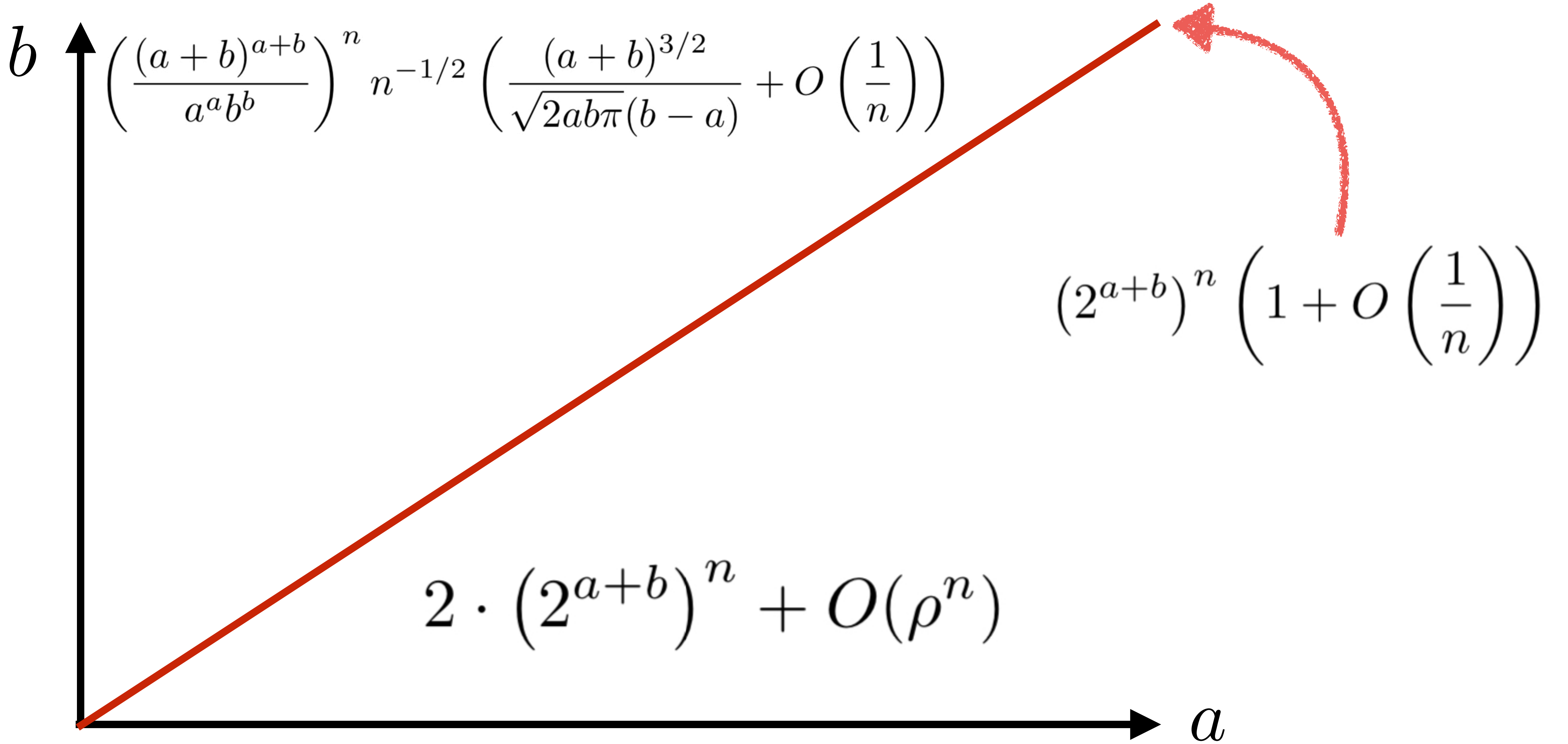
These “negative Gaussian moments” seem to be much less studied (one dimension is easy, otherwise ad hoc using e.g. int. by parts)

Another Example

Let

$$F(x,y) = \frac{1}{(1-x-y)(1-2x)}$$

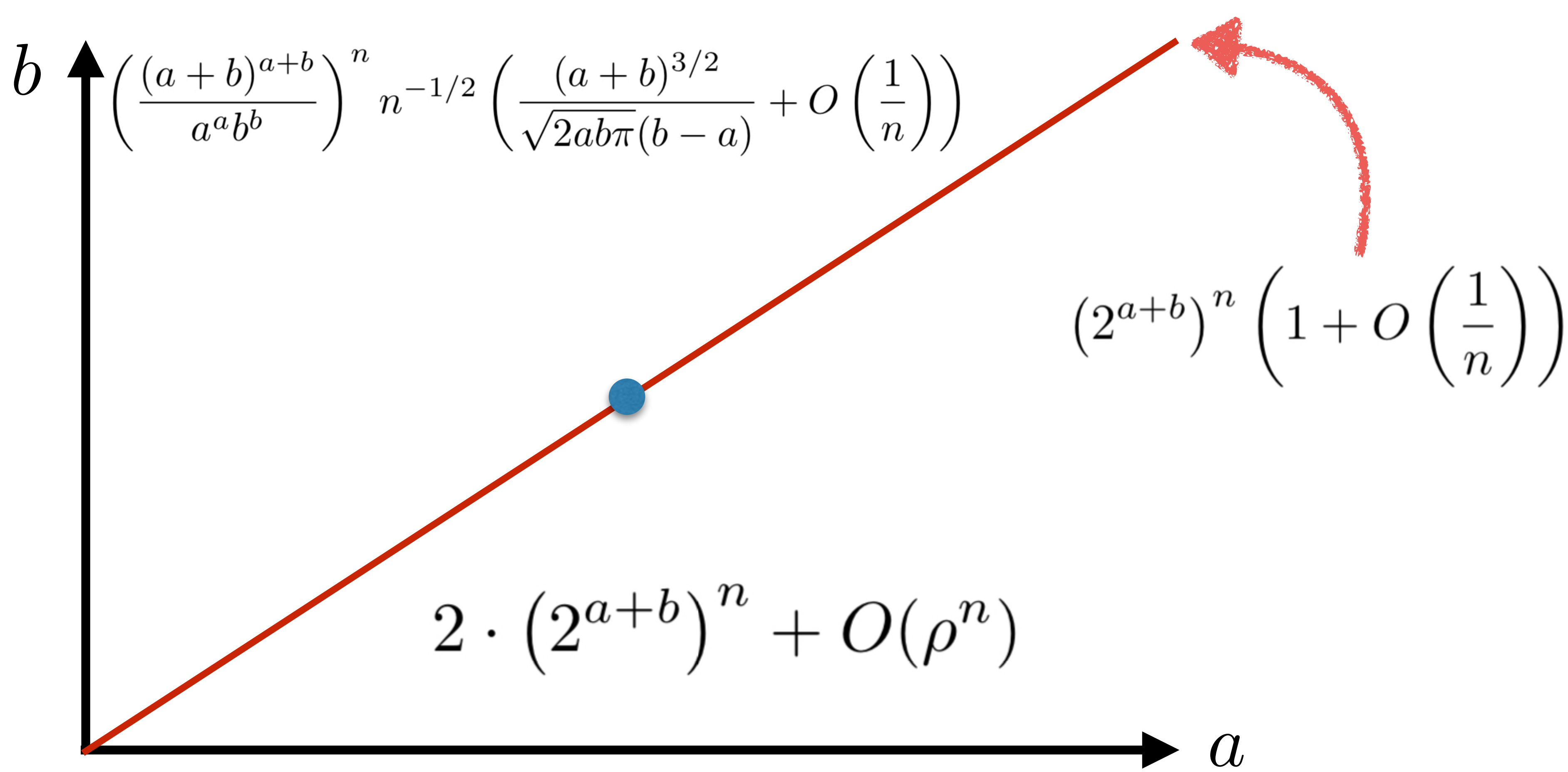
Then $[x^{an}y^{bn}]F(x,y)$ satisfies



Asymptotic Regime Change

The exponential growth of $[x^{an}y^{bn}]F(x,y)$ varies smoothly with (a,b), so scale by the exponential growth.

For our example, around $\mathbf{r} = (1,1)$ the remaining terms go from decaying as $n^{-1/2}$ to being the constant 2.



Asymptotic Regime Change

The exponential growth of $[x^{an}y^{bn}]F(x,y)$ varies smoothly with (a,b), so scale by the exponential growth.

For our example, around $\mathbf{r} = (1,1)$ the remaining terms go from decaying as $n^{-1/2}$ to being the constant 2.

How does this transition occur?
It makes sense to look at the transition on the square-root scale

$$[x^{n+t\sqrt{n}}y^n]F(x,y) \quad \text{for} \quad t = O(n^c) \text{ with } 0 < c < 1/2$$

Asymptotic Regime Change

The exponential growth of $[x^{an}y^{bn}]F(x,y)$ varies smoothly with (a,b), so scale by the exponential growth.

For our example, around $\mathbf{r} = (1,1)$ the remaining terms go from decaying as $n^{-1/2}$ to being the constant 2.

How does this transition occur?
It makes sense to look at the transition on the square-root scale

$$[x^{n+t\sqrt{n}}y^n]F(x,y) \quad \text{for} \quad t = O(n^c) \text{ with } 0 < c < 1/2$$

First step: Get data for our example!

Experimental Data

How do we usually generate $f_{n\mathbf{r}}$ for large n?

Theorem (Christol, Lipshitz): The sequence $f_{n\mathbf{r}}$ satisfies a linear recurrence relation with polynomial coefficients.

There are good algorithms (Lairez / Bostan, Lairez, Salvy) for determining such a recurrence and practical implementations (Best: Lairez's MAGMA package, Also Good: Koutschan's Mathematica package)

Experimental Data

How do we usually generate $f_{n\mathbf{r}}$ for large n?

Theorem (Christol, Lipshitz): The sequence $f_{n\mathbf{r}}$ satisfies a linear recurrence relation with polynomial coefficients.

There are good algorithms (Lairez / Bostan, Lairez, Salvy) for determining such a recurrence and practical implementations (Best: Lairez's MAGMA package, Also Good: Koutschan's Mathematica package)

Problem #1: Singly exponential complexity which increases with the numer/denom of $\mathbf{r}$'s coordinates

Problem #2: We need truly multidimensional data

Computing Coefficients

With Kevin Hyun and Éric Schost:
Efficient algorithm for generating terms of multivariate rational function (right now only in *bivariate case*)

Idea: Each *section* $\alpha_j(x) = \sum_{n \geq 0} f_{n,j} x^n$ is a rational function $\frac{P_j(x)}{H(x,0)^j}$

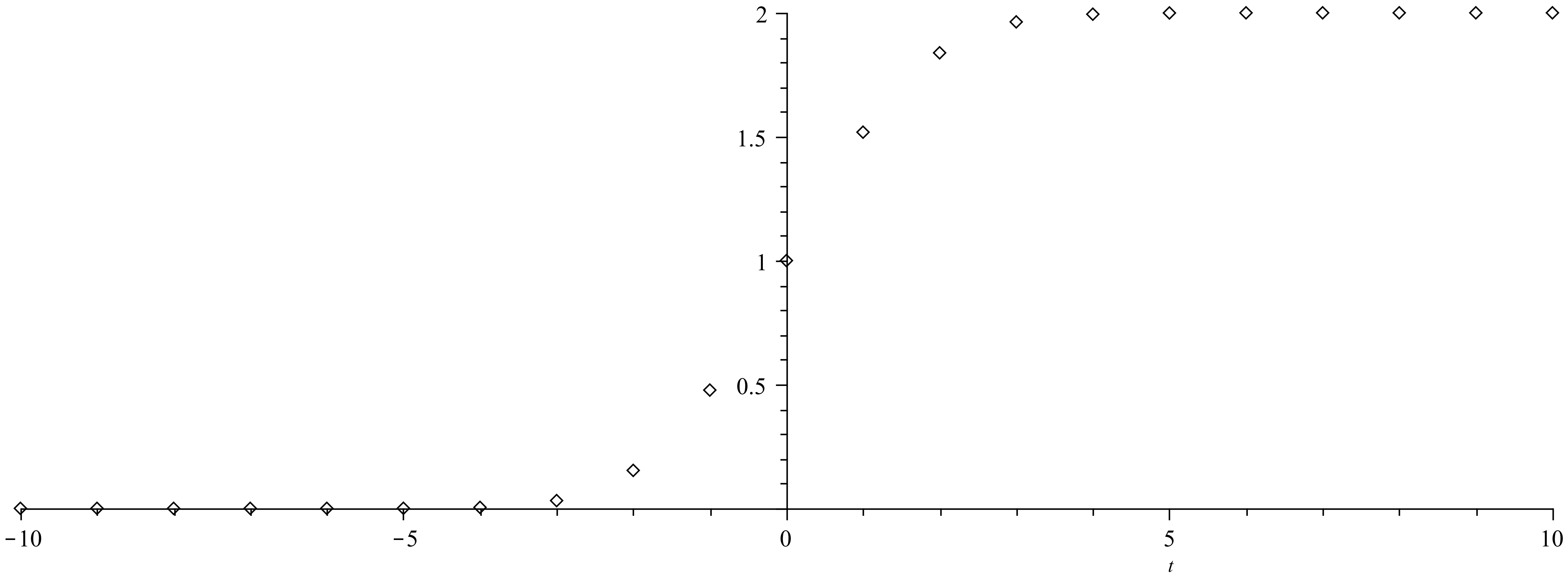
Can find P_j using fast interpolation procedures

Since denominator is a power of a fixed polynomial, can find terms in good complexity using work of Hyun, M., Schost, and St-Pierre

Very efficient implementation in C++ using Shoup's NTL library

```
void bivariate_lin_seq::find_row_geometric(zz_pX &num, zz_pX &den, const long &D){
    long degree = (D+1) * d1;
    zz_pX x;
    SetCoeff(x,1,1);

    zz_p x_0;
    random(x_0);
    zz_pX_Multipoint_Geometric eval(x_0, x_0, degree);

    Vec<zz_p> pointsX, pointsY;
    pointsX.SetLength(degree);
    pointsY.SetLength(degree);
    eval.evaluate(pointsX, x); // grabs all the points used for evaluation

    Vec<zz_pX> polX_num, polX_den;
    create_poly(polX_num, num_coeffs);
    create_poly(polX_den, den_coeffs);

    for (long i = 0; i < degree; i++){
        zz_pX eval_num, eval_den;
        eval_x(eval_num, pointsX[i], polX_num);
        eval_x(eval_den, pointsX[i], polX_den);

        Vec<zz_p> init = get_init(d2, eval_num, eval_den);
        auto rp = get_elem(D,reverse(eval_den), init);
        auto p_pow = power(ConstTerm(eval_den), D+1);

        pointsY[i] = (rp*p_pow);
    }
    eval.interpolate(num, pointsY);
    power(den, polX_den[0], D+1);
}

void bivariate_lin_seq::get_entry_sq_ZZ
(Vec<ZZ> &entries_num,
 Vec<ZZ> &entries_den,
```

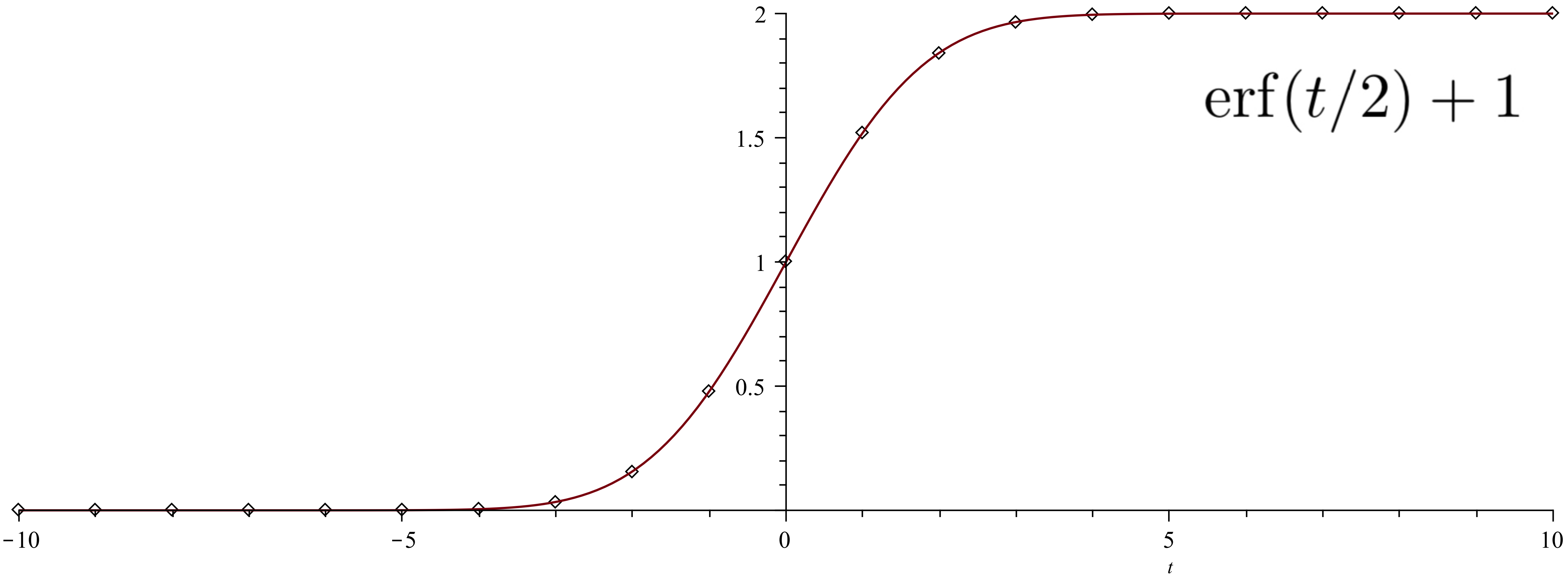
Asymptotic Transition For Our Example



$$4^{-2\cdot 50^2 - t50} \cdot [x^{50^2+t50} y^{50^2}] F(x,y) \text{ for } t = -10 \ldots 10$$

A Gaussian error curve!

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-y^2} dy$$



$$4^{-2\cdot 50^2 - t50} \cdot [x^{50^2+t50} y^{50^2}] F(x, y) \ \text{ for } \ t \ = \ -10 \ldots 10$$

Final term calculated (5501 bits)

9247112633865973228836926990252927536356128705864994391723960554842197828011919474188031
1840050067111278111780191338963196100213646176384616576895324325774311651633061291743511
1528172307641969079370616908774932526257748200792620808754002776970859314141249780545077
8103255913168249620154652817830950635794229671872993810041692625728133745324643626841293
0259564647442319740147252362804562844434857835125458940592134491474970770607230655221867
5366230681922963259368342680997668526477479402147170142640019971630836873779496410564406
5906486259309487970100334323892438718399499179010927377682177528243724037074218571133372
5542774057540268752388779449398881580396831894698931952530172625133010565323295147885324
9981002946718644699833713280981651736705195798719880743558954453380941098600643926040411
4496539256860182158422734589455124276305689168482910661467600355604435267838066675355087
9311733057968744439375914536704720736701280856507092158687171417876146691374315589264408
9749686947951486155583039909969190414112626413695581796272088309197088870117259664085189
7628170182782844835742032533698459985431963124199119073986596954833469830341670440503081
4142884824014900626562588911196406528928198509499728155987916438342256979170118456640402
7939362451483545842365315802379461162277246402661979338172430393316433538350972283167985
5945250295071620153743584846519241968287635621625773570912765784809250497309984552598716
2260107070515687329791339969156814011616512253084076327937423777720247529424544504161301
8998699781303328086317552377901540356213863616459034770127913986510273876354130346015132
1300228752061945518359933288559372125414239085199824338629314562144453776

Transition in this Example

Integral manipulations show

$$2^{-2n-t\sqrt{n}} \cdot [x^{n+t\sqrt{n}}y^n]F(x,y) \sim I(t) = \frac{1}{\pi i}\int_{\mathbb{R}-i\epsilon} \frac{e^{-4nz^2+2i\sqrt{n}tz}}{z}dz$$

Transition in this Example

Integral manipulations show

$$2^{-2n-t\sqrt{n}} \cdot [x^{n+t\sqrt{n}}y^n]F(x,y) \sim I(t) = \frac{1}{\pi i}\int_{\mathbb{R}-i\epsilon} \frac{e^{-4nz^2+2i\sqrt{n}tz}}{z}dz$$

$$(\partial I/\partial t)(t) = \frac{2\sqrt{n}}{\pi}\int_{\mathbb{R}-i\epsilon} e^{-4nz^2+2i\sqrt{n}tz}dt \quad = \frac{e^{-t^2/4}}{\sqrt{\pi}}$$

$$I(0) = \frac{1}{\pi i}\int_{\mathbb{R}-i\epsilon} \frac{e^{-nz^2}}{z}dz = 1$$

General (Linear) 2D Transition

Theorem (Baryshnikov, M., Pemantle): This error function appears more generally. For instance, suppose

$$F(x,y) = \frac{G(x,y)}{\ell_1(x,y)\ell_2(x,y)}$$

For "non-generic" directions where asymptotics are determined by a singularity $\boldsymbol{\sigma}$ there exist explicit constants $A, B \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$ such that

$$\boldsymbol{\sigma}^{n\mathbf{r}+t\sqrt{n}\mathbf{v}} \cdot \left[\mathbf{z}^{n\mathbf{r}+t\sqrt{n}\mathbf{v}}\right] F(\mathbf{z}) \sim A \cdot \operatorname{erf}(Bt) + A$$

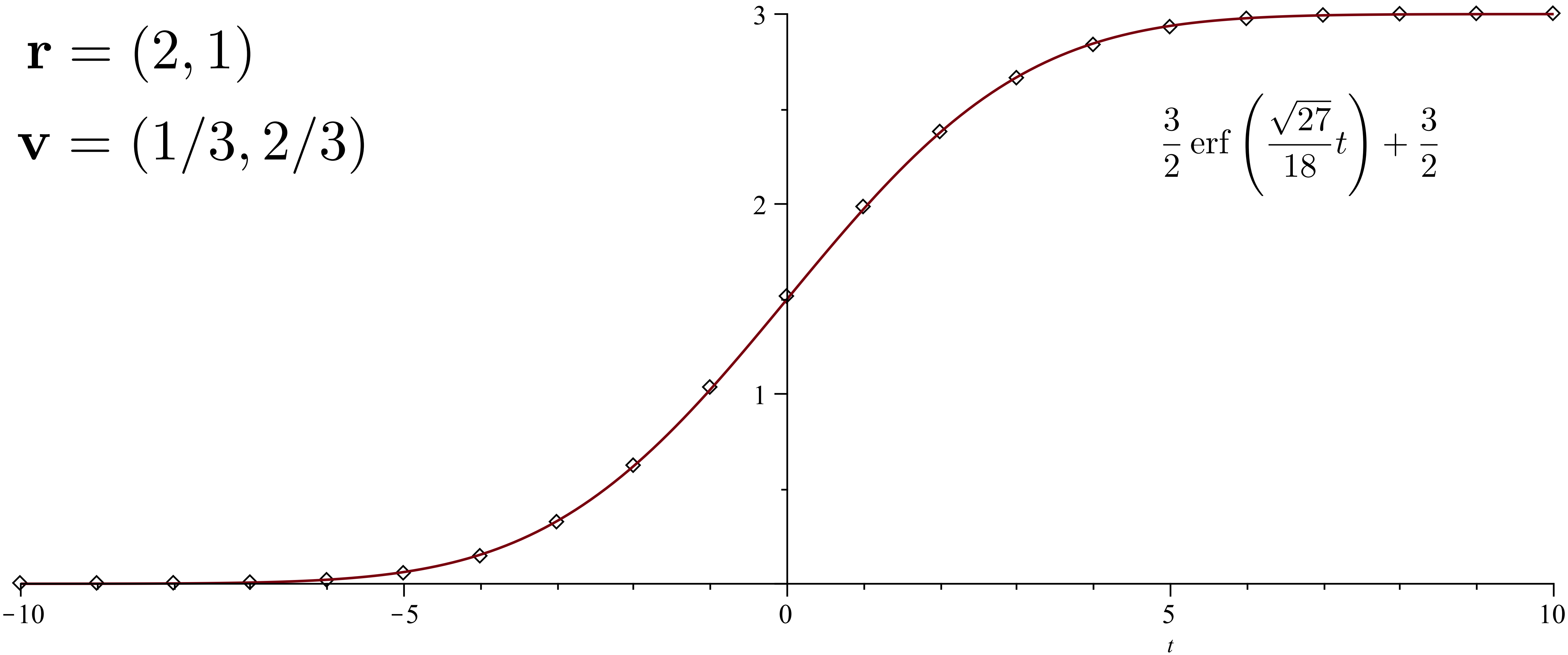
Similar results in more variables when denominator product of linear functions

Example #3

$$F(x,y) = \frac{1}{(1-2x-y)(1-x-2y)}$$

$\mathbf{r} = (2,1)$

$\mathbf{v} = (1/3, 2/3)$



$$9^{-3\cdot 30^2-30t}\left[x^{2\cdot 30^2+10t}y^{30^2+20t}\right]F(x,y)$$

CONCLUSION

Conclusion

- ACSV developing rapidly, including increasingly powerful algorithms

- Diagonals are data structures for univariate sequences, but ACSV also allows for treatment of truly multivariate questions

- Now that "generic" behaviour is starting to be figured out, time to branch out to more pathological cases (using Morse theory, algebraic geometry, ...)

- Perhaps most interesting, we can examine how behaviour transitions between different uniform regimes

- **Still many ways to generalize, and lots more to come!**

THANK YOU!

Stephen Melczer
www.math.upenn.edu/~smelczer

(Those interested in examining draft manuscript
in late summer 2019, please contact me!)