

Arctic Curves for Bounded Lecture Hall Tableaux

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BIRS Asymptotic Algebraic Combinatorics

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Outline

- 1 Bounded Lecture Hall Tableaux
- 2 Tangent Method
- 3 Examples
- 4 Further Questions

Bounded LHT

Fix positive integers n, t . Given $\lambda = (\lambda_1, \dots, \lambda_n)$ (some λ_i possibly zero). Consider tableaux T of shape λ satisfying

$$\frac{T_{ij}}{n-i+j} \geq \frac{T_{ij+1}}{n-i+(j+1)}$$
$$\frac{T_{ij}}{n-i+j} > \frac{T_{i+1j}}{n-(i+1)+j}$$
$$\frac{T_{ij}}{n-i+j} < t.$$

Call Z_λ^t the number of such tableaux.

Bounded LHT

$$\lambda = (5, 4, 3, 2, 1), \quad n = t = 5$$

24	23	25	24	25
16	18	17	18	
11	11	2		
6	2			
0				

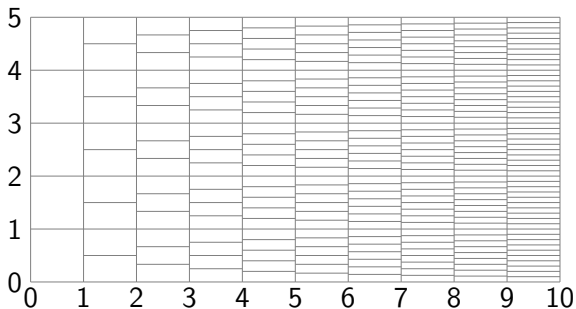
$$T_{ij}$$

$\frac{24}{5}$	$\frac{23}{6}$	$\frac{25}{7}$	$\frac{24}{8}$	$\frac{25}{9}$
$\frac{16}{4}$	$\frac{18}{5}$	$\frac{17}{6}$	$\frac{18}{7}$	
$\frac{11}{3}$	$\frac{11}{4}$	$\frac{2}{5}$		
$\frac{6}{2}$	$\frac{2}{3}$			
$\frac{0}{1}$				

$$\frac{T_{ij}}{n-i+j}$$

LHT as Nonintersecting Paths

Consider the graph below ($t = 5$)



LHT as Nonintersecting Paths

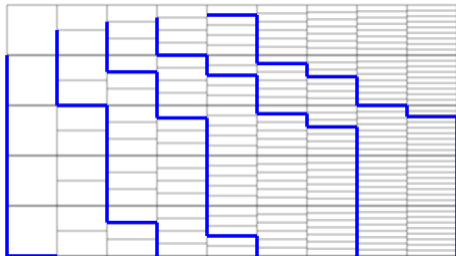
(Here $n = 5, t = 5$)

Starting points: $v_i = (n - i, t - \frac{1}{n-i+1})$.

Ending points: $u_j = (n + \lambda_j - j, 0)$.

Row i of $T \leftrightarrow$ path from v_i to u_i .

24	23	25	24	25
16	18	17	18	
11	11	2		
6	2			
0				



Proposition (Corteel, Kim, Savage 18)

Fix n, t, λ . The number of LHT with $\frac{T_{ij}}{n-i+j} < t$ is given by

$$Z_{\lambda}^t = t^{|\lambda|} s_{\lambda}(\underbrace{1, \dots, 1}_{n \text{ times}}) = t^{|\lambda|} \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Proposition (Corteel, Kim, Savage 18)

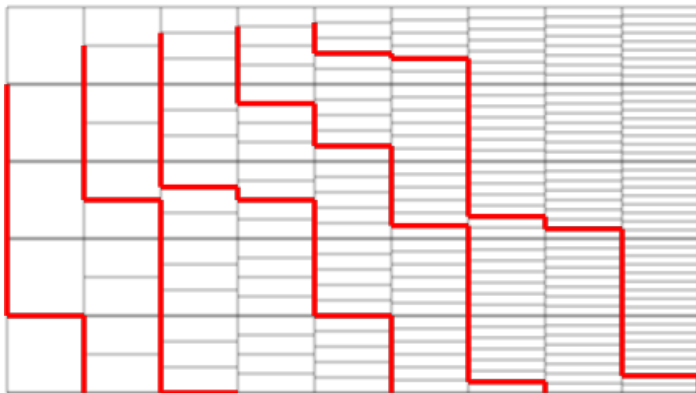
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$$Z_{\lambda}^t = t^{|\lambda|} s_{\lambda}(\underbrace{1, \dots, 1}_{n \text{ times}}) = t^{|\lambda|} \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

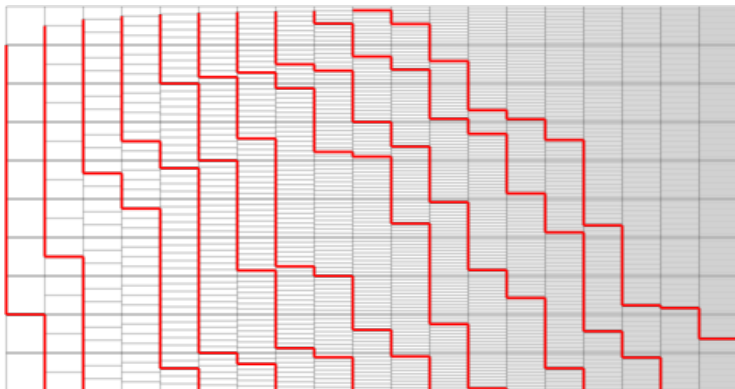
Proof.

Easy determinant evaluation. □

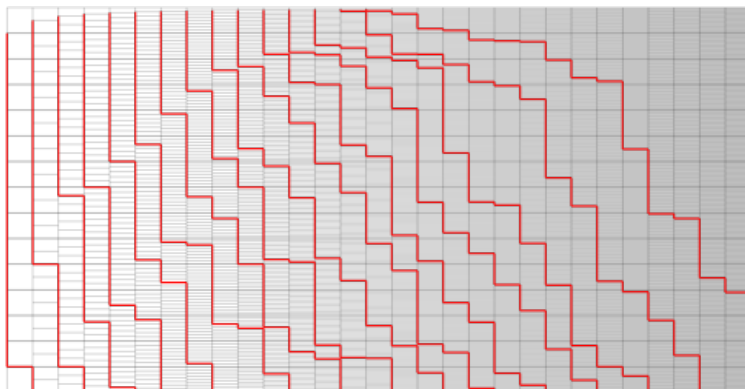
$$\lambda = (n, n - 1, \dots, 1)$$



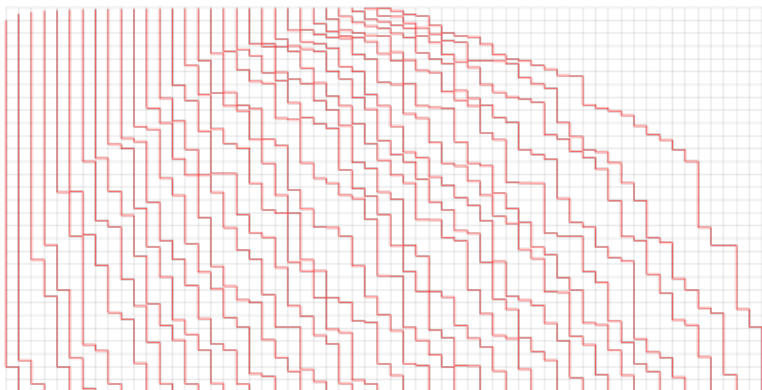
$$n = t = 5$$



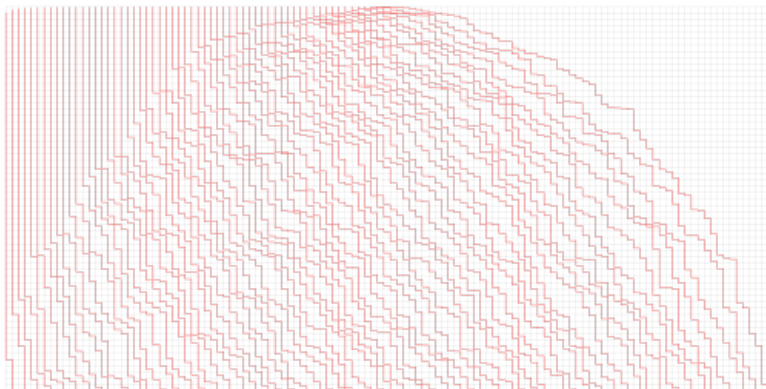
$$n = t = 10$$



$$n = t = 15$$



$$n = t = 30$$

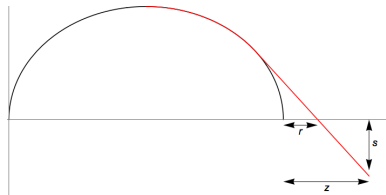


$$n = t = 60$$

Tangent Method

Developed by Colomo and Sportiello (2016). Idea:

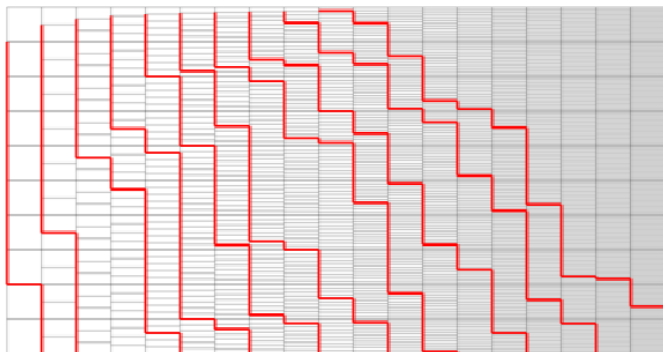
- In the thermodynamic limit, the outermost path follows the arctic curve.
- Extend outermost path by z, s as shown.



- Assumption: The path will follow the arctic curve until it can move in a straight line to its endpoint. The line is tangent to the arctic curve.
- Compute most probable r . The points $(n + \lambda_1 - 1 + r, 0)$ and $(n + \lambda_1 - 1 + z, -s)$ define the tangent line.
- Varying z gives a family of lines tangent to the arctic curve.

An Easy Example

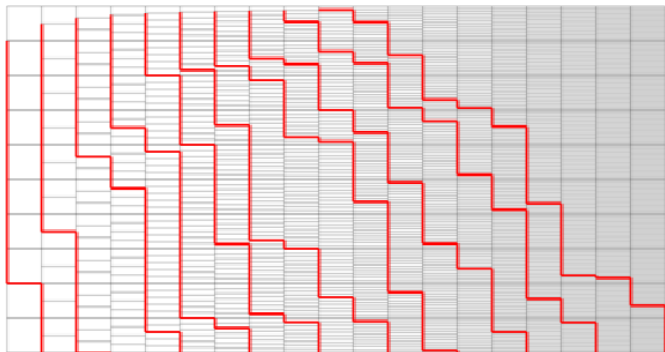
$$\lambda = (n, n-1, \dots, 1)$$



$$n = 10, t = 10$$

$$\lambda = (n, n-1, \dots, 1)$$

$$Z_{\lambda}^t = t^{|\lambda|} s_{\lambda}(1, \dots, 1) = (2t)^{\binom{n+1}{2}}$$

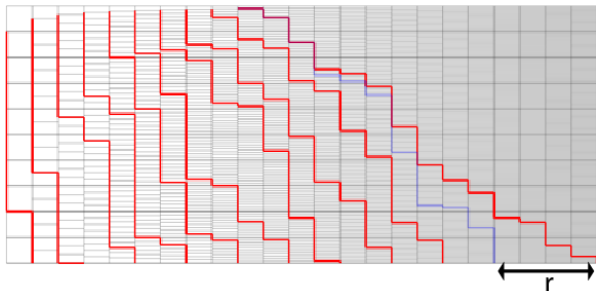


$$n = 10, t = 10$$

$$\lambda = (n, n-1, \dots, 1)$$

$$\frac{Z_{\lambda,r}^t}{Z_{\lambda}^t} = t^r \prod_{j=2}^n \frac{\lambda_1 + r - \lambda_j + j - 1}{\lambda_1 - \lambda_j + j - 1} = t^r \binom{n+k-1}{k}$$

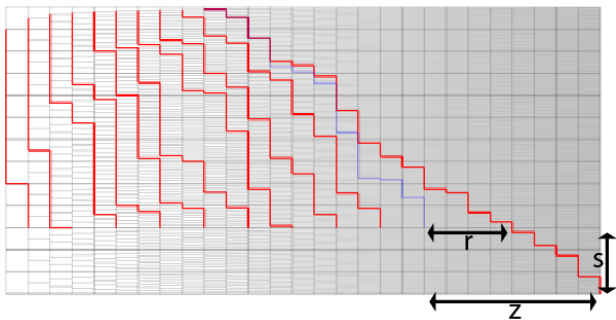
(for $r = 2k$)



$$n = 10, t = 10, r = 4$$

$$\lambda = (n, n-1, \dots, 1)$$

$$\frac{Z_{\lambda, z, s}^t}{Z_{\lambda}^t} = \sum_{r=0}^z \frac{Z_{\lambda, r}^t}{Z_{\lambda}^t} s^{z-r} \binom{2n+z-1}{z-r}$$



$$n = 10, t = 10, z = 8, s = 3$$

$$\lambda = (n, n-1, \dots, 1)$$

We take the limit $r = n\rho$, $z = n\zeta$, $t = n\tau$, $s = n\sigma$, and $n \rightarrow \infty$.

$$\frac{Z_{\lambda, z, s}^t}{Z_{\lambda}^t} \approx \frac{1}{2\pi} e^{\zeta n \ln(n)} \int_0^{\zeta} \sqrt{\frac{1+\zeta}{\rho(\zeta-\rho)}} e^{nS(\rho)} d\rho$$

$$S(\rho) = \rho \ln(\tau) + (\zeta - \rho) \ln(\sigma) - \frac{1}{2}(2+\rho) \ln(2+\rho) - \frac{1}{2}\rho \ln(\rho) - (\zeta - \rho) \ln(\zeta - \rho)$$

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Most probable $\rho \rightarrow$ maximum of $S(\rho)$

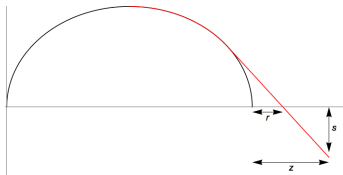
$$\lambda = (n, n - 1, \dots, 1)$$

$$S'(\rho) = 0 \implies \zeta - \rho = \frac{\sigma}{\tau}(2 + \rho)\sqrt{\frac{\rho}{2 + \rho}}$$

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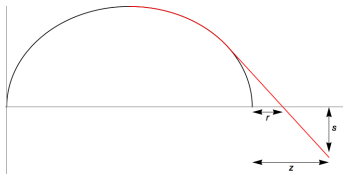
Pair of points: $(2 + \rho, 0)$ and $(2 + \zeta, -\sigma)$.



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$$S'(\rho) = 0 \implies \zeta - \rho = \frac{\sigma}{\tau}(2 + \rho)\sqrt{\frac{\rho}{2 + \rho}}$$

Pair of points: $(2 + \rho, 0)$ and $(2 + \zeta, -\sigma)$.



Family of tangent lines

$$Y = -\frac{\tau}{x}\sqrt{\frac{x}{x-2}}(X-x)$$

parametrized by $x = 2 + \rho$, $x \in [2, \infty)$.

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Parametrization:

$$X(x) = \frac{x}{x-1}$$

$$Y(x) = \tau \frac{x}{x-1} \sqrt{\frac{x-2}{x}}$$

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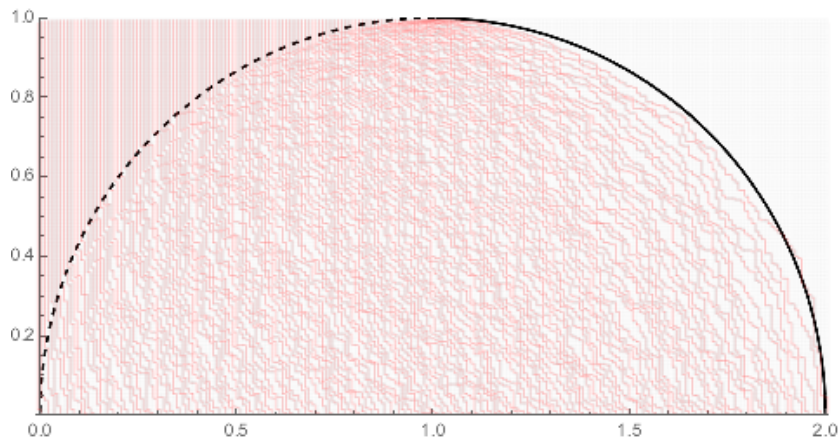
Parametrization:

$$X(x) = \frac{x}{x-1}$$

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Curve: $Y = \tau \sqrt{2X - X^2}$

$$\lambda = (n, n - 1, \dots, 1)$$

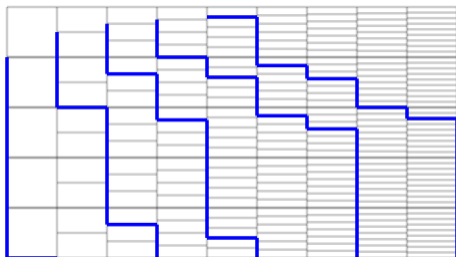


Other Portions of the Arctic Curve

Dual paths: Read tableaux by column rather than by row.

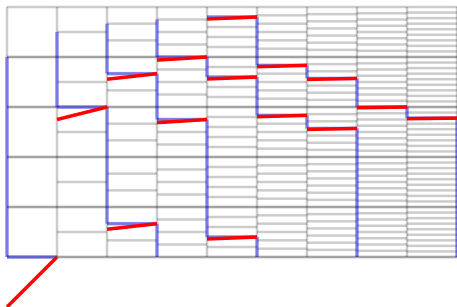
Other Portions of the Arctic Curve

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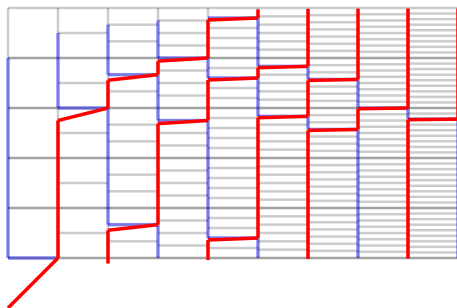
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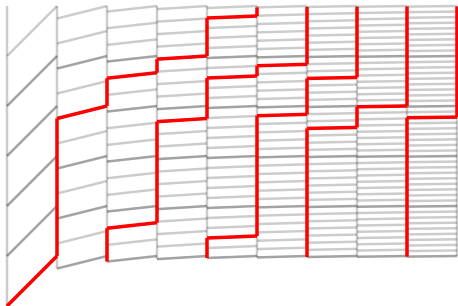
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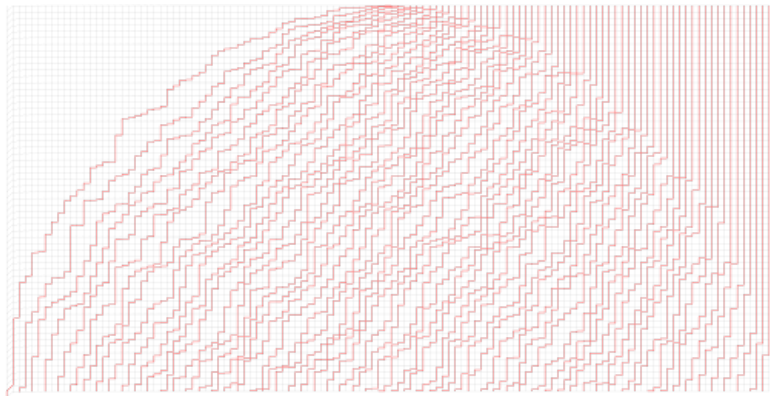


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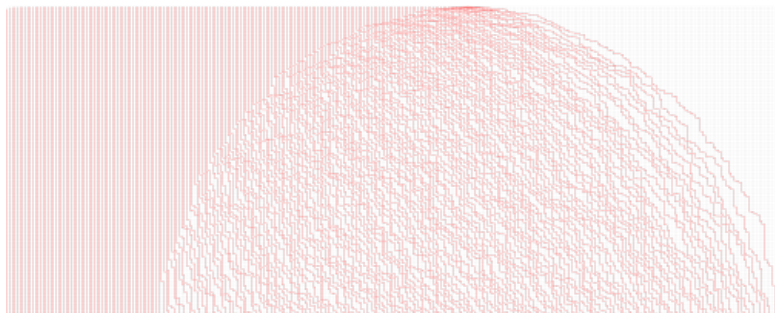
Other Portions of the Arctic Curve



$$n = t = 60$$

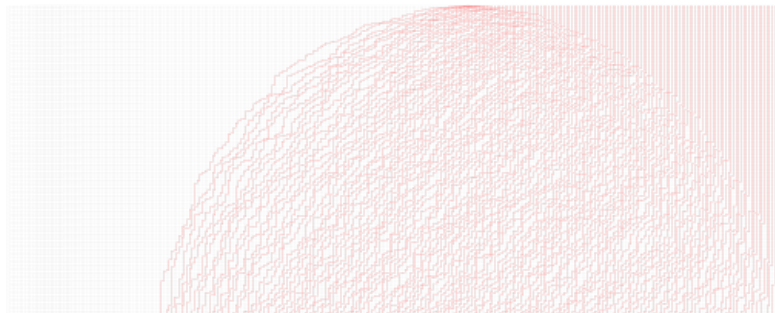
$$\lambda = (n, n - 1, \dots, 1)$$

Extend $\lambda = (n, n - 1, \dots, 1, 0, \dots, 0)$.



$$\lambda = (n, n - 1, \dots, 1)$$

Extend $\lambda = (n, n - 1, \dots, 1, 0, \dots, 0)$.

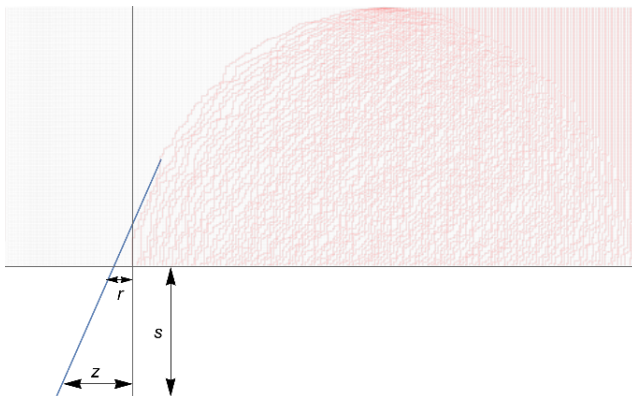


$$\lambda = (n, n - 1, \dots, 1)$$

Fix z, s .

Shifting dual path by r corresponds to

$$\lambda = (\lambda_1, \dots, \lambda_n, 1, \dots, 1, 0, \dots, 0).$$



$$\lambda = (n, n - 1, \dots, 1)$$

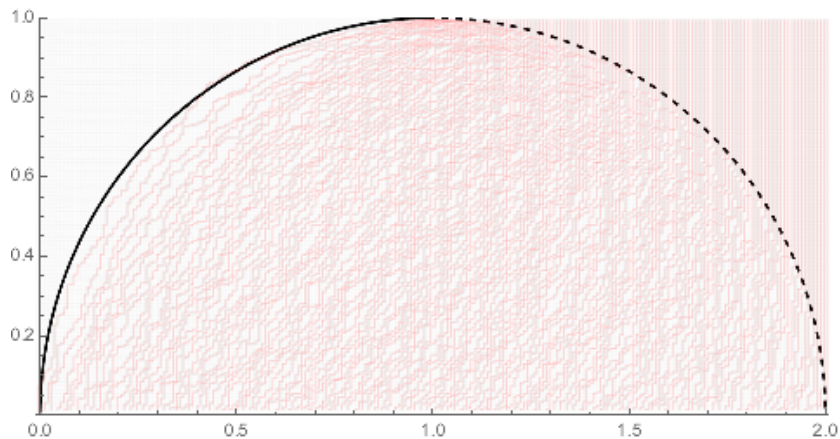
Parametrization:

$$X(x) = \frac{x}{x - 1}$$
$$Y(x) = \tau \frac{x}{x - 1} \sqrt{\frac{x - 2}{x}}$$

for $x \in (-\infty, 0]$.

Same parametrization as before!

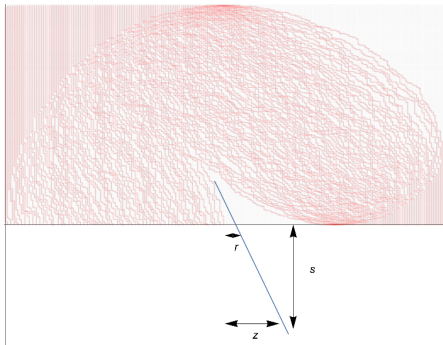
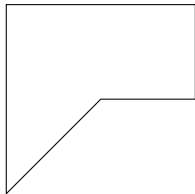
$$\lambda = (n, n-1, \dots, 1)$$



Cusps

Occurs when λ has a macroscopic jump. For example

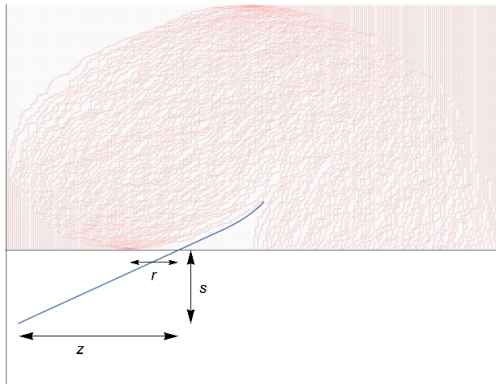
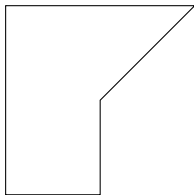
$$\lambda = (2n, \dots, 2n, n, n-1, \dots, 1), t = n$$



Cusps

Occurs when λ has a macroscopic flat section. For example

$$\lambda = (2n, 2n - 1, \dots, n + 1, n, \dots, n), t = n$$



In general

For $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $n + \lambda_i - i = n\alpha(\frac{i}{n})$ for some piecewise differentiable α , we have

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For $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $n + \lambda_i - i = n\alpha(\frac{i}{n})$ for some piecewise differentiable α , we have

“Theorem”

The arctic curve can be parametrized by

$$X(x) = \frac{x^2 I'(x)}{I(x) + x I'(x)}$$
$$Y(x) = \tau \frac{1}{I(x) + x I'(x)}$$

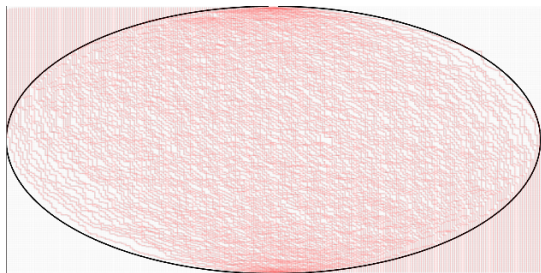
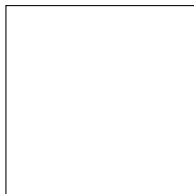
for an appropriate range of x , where $I(x) = e^{-\int_0^1 \frac{1}{x-\alpha(u)} du}$ and $\alpha(u)$ is the limiting profile.

Some Examples

$$\lambda = (n, \dots, n)$$

Parametrization:

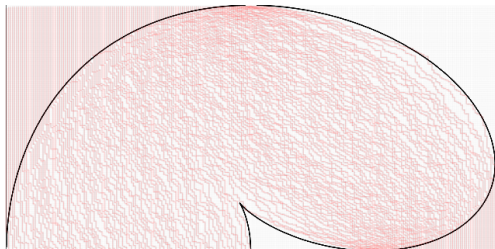
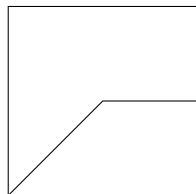
$$X(x) = \frac{x^2}{x^2 - 2x + 2}$$
$$Y(x) = \frac{\tau(x-1)^2}{x^2 - 2x + 2}$$



$$\lambda = (2n, \dots, 2n, n, n-1, \dots, 1)$$

Parametrization:

$$X(x) = \frac{x(2x^2 - 9x + 12)}{x^3 - 7x^2 + 17x - 12}$$
$$Y(x) = \frac{\tau(x-3)^2 \sqrt{x(x-2)}}{x^3 - 7x^2 + 17x - 12}$$

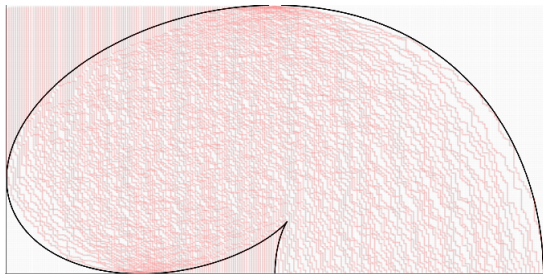
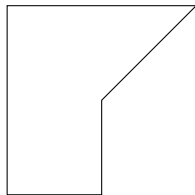


$$\lambda = (2n, \dots, n + 1, n, \dots, n)$$

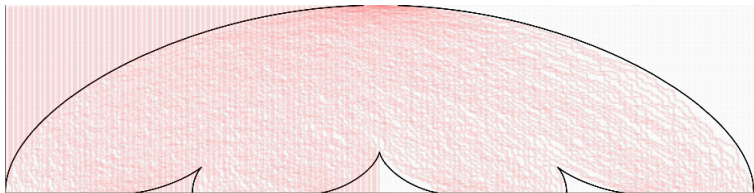
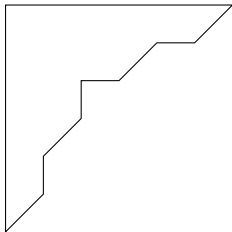
Parametrization:

$$X(x) = \frac{x^2(2x - 5)}{x^3 - 5x^2 + 9x - 8}$$

$$Y(x) = \frac{\tau(x - 1)^2 \sqrt{(x - 4)(x - 2)}}{x^3 - 5x^2 + 9x - 8}$$



$$\lambda = (6n, \dots, 5n + 1, 4n, \dots, 3n + 1, 2n, \dots, 2n, 2n, \dots, n + 1, n, \dots, n, n, \dots, 1)$$



Further Questions

- Rigorous proof (A. Aggarwal)
- Skew-tableaux
- q -weighted LHT
- Full limit shape

End!

Thank You!