Boundaries of branching graphs from 80s till present

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Large Unitary groups

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \qquad u \in U(3), \quad u^* u = I$$

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What is the representation theory of $U(\infty)$?

- The group is not locally compact.
- Too many possible actions in vector spaces.

Representations through characters

For finite or compact group G:

$$\text{Representations } \pi : G \mapsto GL(V)$$

$$\updownarrow$$

$$\text{Normalized characters } \chi(g) = \frac{\mathrm{Trace}(\pi(g))}{\dim(\pi)}.$$

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Example:

$$\text{Representations of } U(N) \text{ with highest weight}$$
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \in \mathbb{Z}$$
$$\updownarrow$$
$$\text{Normalized Schur polynomials } \frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(1,\dots,1)},$$
$$\{x_i\} \text{ — eigenvalues of } u \in U(N).$$

Reminder:

$$s_\lambda(x_1,\dots,x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}.$$

Characters of $U(\infty)$

Definition. Character of $U(\infty)$ is a continuous $\chi : U(\infty) \to \mathbb{C}$

- $\chi(gug^{-1}) = \chi(u)$ (central)
- $\sum_{i,j=1}^{n} z_i \bar{z}_j \chi(u_i u_j^{-1}) \geq 0$ (positive-definite)
- $\chi(I) = 1$ (normalized)

Like a **trace**, but without actual representation.

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Irreducible representation

$\updownarrow$

Extreme character = extreme point of the convex set.

Characters of $U(\infty)$

Theorem. (Voiculescu-1977; Edrei-1953) The extreme characters of $U(\infty)$ are parameterized by the points ω of domain

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spec}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \cdot \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)},$$

$$\gamma^\pm = \delta^\pm - \sum (\alpha_i^\pm + \beta_i^\pm).$$

Characters of $U(\infty)$

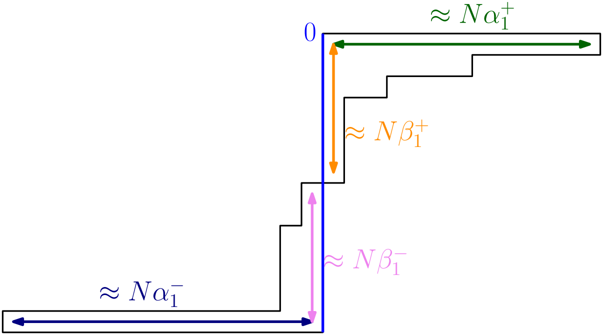
Proposition.(Vershik–Kerov, 70s) Extreme characters χ of $U(\infty)$ are $N \to \infty$ **limits of restrictions** on $U(k)$ of characters of $U(N)$

$$\chi(x_1, \dots, x_k, 1, \dots) = \lim_{N\to\infty} \frac{s_{\lambda(N)}(x_1, \dots, x_k, 1^{N-k})}{s_{\lambda(N)}(1^N)},$$

where $(x_1, \dots, x_k, 1, \dots)$ are eigenvalues of $u \in U(\infty)$.

Approximation of characters of $U(\infty)$

Theorem. (Vershik–Kerov-1982) The sequence of signatures $\lambda(N)$ with the following properties **approximates** the character of $U(\infty)$ with parameters $(\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$.



$$\frac{\lambda(N)_i^{\pm}}{N} \to \alpha_i^{\pm}, \quad \frac{\lambda(N)_i^{\pm'}}{N} \to \beta_i^{\pm},$$

$$\frac{\text{Sum of positive/negative } \lambda_i}{N} \to \pm\delta^{\pm}.$$

Infinite–dimensional unitary group $U(\infty)$

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How to study such limits?

Infinite–dimensional unitary group $U(\infty)$

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1. Binomial theorem (Lascoux, Macdonald, Okounkov–Olshanski)

$$\frac{s_\lambda(1+x_1,\dots,1+x_k,1^{N-k})}{s_\lambda(1^N)}=\sum_\mu\frac{s^*_\mu(\lambda_1,\dots,\lambda_N)\,s_\mu(x_1,\dots,x_k,0^{N-k})}{c(N;\mu)}$$

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2. *Normalized* skew-dimensions (Borodin–Olshanski, Petrov)

$$\frac{s_\lambda(x_1,\dots,x_k,1^{N-k})}{s_\lambda(1^N)} = \sum_\mu \frac{s_\mu(x_1,\dots,x_k)}{s_\mu(1^k)} \cdot \mathrm{D}_{\lambda/\mu},$$

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3. Contour integrals (G.–Panova)

$$\frac{s_\lambda(x,1^{N-1})}{s_\lambda(1^N)} = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\mathbf{i}} \oint_\infty \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))} \mathrm{d}z$$

Characters of $U(\infty)$ via combinatorial probability

Gelfand–Tsetlin graph

Level 0	$\emptyset$	
Level 1	λ_1	Integral coordinates
Level 2	$\mu_1 \geq \mu_2$	Edges = interlacing
Level 3	$\nu_1 \geq \nu_2 \geq \nu_3$	$\mu \prec \nu$ if
Level 4	$\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \kappa_4$	$\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \nu_3$.
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- Dimension of T_λ of $U(N) = s_\lambda(1^N) =$#SSYT of shape λ = number of directed paths from $\emptyset$ to λ at level N.
- Skew dimension $= s_{\lambda/\mu}(1^{N-k}) =$#skew SSYT of shape λ/μ = number of directed paths from μ at level k to λ at level N.

Example of a **branching graph.**

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$\mathcal{T}$ — directed paths from $\emptyset$ towards infinity.

Character χ on $U(\infty)$

$\updownarrow$

Gibbs probability measure $\mathbb{P}$ on $\mathcal{T}$

Gibbs property of $\emptyset \prec \lambda \prec \mu \prec \nu \prec \kappa \prec \dots$. Level 3 example: Conditionally on ν, the distribution of $\emptyset \prec \lambda \prec \mu$ is uniform.

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$$\chi\big|_{U(N)} = \sum_{\lambda} \frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1^N)} \mathbb{P}(\text{random path passes through } \lambda)$$

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Classification of characters of $U(\infty)$

$\updownarrow$

Classification of **Gibbs** probability measures on $\mathcal{T}$

$\updownarrow$

(Markov or minimal) **boundary** of Gelfand–Tsetlin graph

Quantum groups $U_q(\mathfrak{u}_N)$

Representation theory of $U(N)$ admits a **q–deformation**.

A quantization of the universal enveloping algebra of $U(N)$
$U_q(\mathfrak{u}_N)$ given by generators and relations.

Another approach: quantization of $\mathbb{C}[U(N)]$.

- A class of representations is parallel to $U(N)$–theory. Still parameterized by $\lambda_1 \geq \cdots \geq \lambda_N$, same dimensions.
- $\mathrm{Trace}(g) \longrightarrow \mathrm{Trace}(q^{2\rho} g)$ **quantum trace.**
- Dimension $\longrightarrow$ **quantum dimension**

$$\mathrm{Dim}(\lambda) = s_\lambda(1^N) = \prod_{i<j} \frac{\lambda_i - i - (\lambda_j - j)}{j - i}$$

$$\mathrm{Dim_q}(\lambda) = s_\lambda(q^{1-N}, q^{3-N}, \ldots, q^{N-3}, q^{N-1}) = \ldots$$

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Question: q–deformation of $U(\infty)$ characters/representations?

q-Characters of $U(\infty)$ via combinatorial probability

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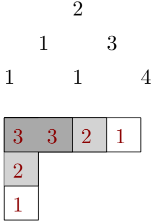

$$\mathcal{T} \text{ — paths } \emptyset \to \infty.$$

***q*–Gibbs property** of $\emptyset \prec \lambda \prec \mu \prec \nu \prec \kappa \prec \ldots$.

Level 4 example: Conditionally on κ, the distribution of $\emptyset \prec \lambda \prec \mu \prec \nu$ is

$$\sim q^{\text{volume}} = q^{\lambda_1 + (\mu_1+\mu_2) + (\nu_1+\nu_2+\nu_3)}$$

Definition of (G.-10) for q–characters of $U(\infty)$

$\updownarrow$

They are ***q*–Gibbs** probability measures on $\mathcal{T}$

q-Characters of $U(\infty)$ via combinatorial probability

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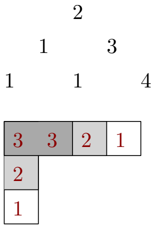

Extreme q–**Gibbs** probability measures on $\mathcal{T}$

$\updownarrow$

All possible limits

$$\lim_{N\to\infty} \frac{s_{\lambda(N)}(x_1, \dots, x_k, q^{1-k}, \dots, q^{1-N})}{s_{\lambda(N)}(1, q^{-1}, \dots, q^{1-N})}$$

Limits of q–specialized Schur polynomials

$$\lim_{N\to\infty} \frac{s_\lambda(x_1,\dots,x_k,q^{1-k},\dots,q^{1-N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})}?$$

Theorem. (G.-10) For $0 < q < 1$ the limit exists if and only if $\lambda_{N-j+1} \to \nu_j$ for every j. The limit is parameterized by

$$\nu_1 \le \nu_2 \le \nu_3 \le \dots \in \mathbb{Z}^\infty.$$

- Continuous parameters at $q = 1$
- Discrete parameters for $0 < q < 1$

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- Continuous parameters at $q = 1$
- Discrete parameters for $0 < q < 1$
- No explicit construction for corresponding representations so far. Some progress in (Sato-18)

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Most methods were eventually generalized

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1. q–Binomial theorem (Okounkov, G.)

$$\frac{s_\lambda(x_1,\ldots,x_k,q^{1-k},\ldots,q^{1-N})}{s_\lambda(1,q^{-1},\ldots,q^{1-N})} = \sum_{\ell(\mu)\le k} \frac{s^*_\mu(q^{\lambda-\delta};q)}{s^*_\mu(q^{\mu-\delta};q)} \, \frac{s^*_\mu(x_1,\ldots,x_k,q^{1-k},\ldots,q^{1-N};q)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}$$

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2. *Normalized* q-skew-dimensions (Petrov)

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Latest q–deformation question

$$\lim_{N\to\infty} \frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{1-k},\dots,q^{1-N})}{s_{\lambda(N)}(1,q^{-1},\dots,q^{1-N})}$$

- (G.-Cuenca-18) There is $q \leftrightarrow q^{-1}$ symmetry in the definition of $U_q(\mathfrak{u}_N)$. Can we **restore the symmetry** in the limit?
- (G.-Cuenca-18) Is q–deformation an artifact of A_N–type, or does it generalize to **orthogonal and symplectic** groups?
- (G.-Olshanski-14) Is there an $N=\infty$ q–version of the **harmonic analysis**? Decomposition of "natural"' representations into irreducibles?

Symmetric q–deformations

$$\lim_{N\to\infty} \frac{s_{\lambda(N)}(q^{-N}, q^{1-N}, \dots, q^{-k-1}, x_{-k}, \dots, x_k, q^{k+1}, \dots, q^N)}{s_{\lambda(N)}(q^{-N}, q^{1-N}, \dots, q^{N-1}, q^N)}$$

$\lambda(N)$ has $2N+1$ rows: $\lambda(N)_{-N} \geq \lambda(N)_{1-N} \geq \cdots \geq \lambda(N)_N$

- Symmetric setup does not distinguish between q and q^{-1}.
- For orthogonal and symplectic groups **only** symmetric setup is possible, as eigenvalues of matrices come in pairs z, z^{-1}.

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Theorem.(G.-Cuenca-18) The limit exists if and only if for every j $\lim_{N\to\infty} \lambda(N)_j = \nu_j$. Parameterized by **two-sided sequence**:

$$\cdots \geq \nu_{-2} \geq \nu_{-1} \geq \nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots \in \mathbb{Z}^\infty$$

Symmetric q–deformations

How to analyze?

$$\frac{s_\lambda(q^{-N}, q^{1-N}, \dots, q^{-k-1}, x_{-k}, \dots, x_k, q^{k+1}, \dots, q^N)}{s_\lambda(q^{-N}, q^{1-N}, \dots, q^{N-1}, q^N)}$$

So far, none of the previous methods generalize directly.

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So far, none of the previous methods generalize directly. However,

Proposition. (G.-Sun-18, G.-Cuenca-18) **Double** contour integral

$$\frac{s_\lambda(q^{-N}, \ldots, q^{-1}, q^x, q^1, \ldots, q^N)}{s_\lambda(q^{-N}, q^{1-N}, \ldots, q^{N-1}, q^N)} = \prod_{i=1}^{N} \frac{(1-q^i)^2}{(q^x - q^i)(1-q^x q^i)}$$

$$\times \oint_{\{q^{\lambda_i+N-i}\}} \frac{\mathrm{d}z}{2\pi\mathbf{i}} \oint_{\infty} \frac{\mathrm{d}w}{2\pi\mathbf{i}} \frac{z^{x+b} w^{-b-1}}{z-w} \prod_{i=-N}^{N} \frac{w - q^{\lambda_i+N-i}}{z - q^{\lambda_i+N-i}}$$

Symmetric q–deformations

How to analyze?

$$\frac{s_\lambda(q^{-N}, q^{1-N}, \ldots, q^{-k-1}, x_{-k}, \ldots, x_k, q^{k+1}, \ldots, q^N)}{s_\lambda(q^{-N}, q^{1-N}, \ldots, q^{N-1}, q^N)}$$

So far, none of the previous methods generalize directly. However,

Proposition. (G.-Sun-18, G.-Cuenca-18) **Double** contour integral

$$\frac{s_\lambda(q^{-N}, \ldots, q^{-1}, q^x, q^1, \ldots, q^N)}{s_\lambda(q^{-N}, q^{1-N}, \ldots, q^{N-1}, q^N)} = \prod_{i=1}^{N} \frac{(1-q^i)^2}{(q^x - q^i)(1-q^x q^i)}$$

$$\times \oint_{\{q^{\lambda_i+N-i}\}} \frac{\mathrm{d}z}{2\pi\mathbf{i}} \oint_{\infty} \frac{\mathrm{d}w}{2\pi\mathbf{i}} \frac{z^{x+b} w^{-b-1}}{z-w} \prod_{i=-N}^{N} \frac{w - q^{\lambda_i+N-i}}{z - q^{\lambda_i+N-i}}$$

Similar formulas exist and lead to asymptotic analysis for q–characters of **orthogonal and symplectic** groups.

Harmonic analysis

Decomposition of a representation into irreducibles

$\updownarrow$

Decomposition of a character into a combination of extremes

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Decomposition of a Gibbs measure into an integral of extremes
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One can define representations in **projective limit** of $L_2(U(N))$.
(Borodin–Olshanski): branching graphs $\to$ decompositions

Harmonic analysis: (z, w)–measures

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Level 2	$\mu_1 \geq \mu_2$
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A measure is determined by

- Projection to each level *and*
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$q = 1$: (z, w)–measures of (Borodin–Olshanski) with $\ell_i = \lambda_i - i$.

$$\frac{1}{Z} \prod_{i<j} (\ell_i - \ell_j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(z - \ell_i)\Gamma(z' - \ell_i)\Gamma(w + N + \ell_i)\Gamma(w' + N + \ell_i)}$$

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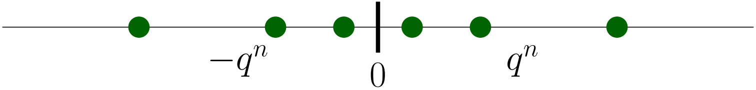
$0 < q < 1$: q-(z, w) measures of (G.–Olshanski) on **double lattice**



- Like orthogonality sets for q–Hypergeometric polynomials.
- Representation-theoretic meaning unclear.

Review

- **Character theory** for $U(\infty)$ is very rich.
- Language 1: limits of **symmetric polynomials** as $N \to \infty$.
- Language 2: **Gibbs measures** on Gelfand–Tsetlin graph.
- q–**deformations** lead to very different results.
- q–deformations for $B - C - D$ **series** lack some tools.
- **Representation–theoretic** meanings of q–deformations are not yet properly understood.

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