Gaussian fluctuations of Jack-deformed random Young diagrams

(joint work with Piotr Śniady)

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Problem

β-ensembles: the probability distributions on $\mathbb{R}^n$ with the density

$$p(x_1, \dots, x_n) = \frac{1}{Z} e^{V(x_1)+\cdots+V(x_n)} \prod_{i<j} |x_i - x_j|^\beta,$$

- $V : \mathbb{R} \to \mathbb{R}$,
- Z - normalization constant.

Problem

What is the discrete counterpart of β-ensambles?

Solution

- No obvious unique way of defining the discrete counterpart of β-ensambles. ☹

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- Several alternative approaches are available (Borodin, Gorin and Guionnet, Moll). ☺
- Different approach viaJack characters with nice asymptotic properties, for instance double-scaling limit. 😎

Examples and the representation theory I

Idea (Kerov, Biane, Śniady, ...)

Random Young diagrams come from representation theory of the symmetric group $\mathfrak{S}_n$!

- ρ_n - a representation of $\mathfrak{S}_n$
- $\mathbb{P}_n$ - probability measure on $\mathbb{Y}_n$ associated with ρ_n:

$$\chi_n(\pi) := \frac{\mathrm{Tr}\ \rho_n(\pi)}{\mathrm{Tr}\ \rho_n(\mathrm{id})} = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_n(\lambda)\chi_\lambda(\pi),$$

where χ_λ - an irreducible character.

Definition

$\chi : \mathcal{P}_n \to \mathbb{R}$ is a convex character, if it is a convex combination of irreducible characters.

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Examples and the representation theory II

Example

- Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad \mathbb{P}_\chi(\lambda) := \frac{(\dim \rho_\lambda)^2}{n!}$$

- Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} \quad \leftrightarrow \quad \mathbb{P}_\chi(\lambda) := \frac{\dim E_\lambda}{N^n},$$

where $(\mathbb{C}^N)^{\otimes n} = \bigoplus_{\lambda \vdash n} E_\lambda$.

Jack deformation

Let $\alpha \in \mathbb{R}_{>0}$.

- $J_\lambda^{(\alpha)}$ - Jack polynomial ($J_\lambda^{(1)} = \frac{n!}{\dim \lambda} s_\lambda$ - normalized Schur polynomial)
- irreducible Jack character $\chi_\lambda^{(\alpha)}$:

$$\chi_\lambda^{(\alpha)}(\pi) := \alpha^{-\frac{\|\pi\|}{2}} \frac{z_\pi}{n!} \theta_\pi^{(\alpha)}(\lambda),$$

  where $\|\pi\| := |\pi| - \ell(\pi)$ and $J_\lambda^{(\alpha)} = \sum_\pi \theta_\pi^{(\alpha)}(\lambda) \; p_\pi$.
- deformation of irreducible characters: $\chi_\lambda^{(1)}(\pi) = \chi_\lambda(\pi)$.

We call $\chi : \mathcal{P}_n \to \mathbb{R}$ a convex Jack character, if it is a convex combination of irreducible Jack characters.

Jack deformation - examples

Example

- Jack-Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \mathbb{P}_\chi(\lambda) := \frac{n!}{\prod_{(x,y)\in\lambda} h_\alpha(x,y) h'_\alpha(x,y)}$$

- Jack-Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} = N^{-\|\pi\|} \quad \leftrightarrow$$

$$\mathbb{P}_\chi(\lambda) := n! \prod_{(x,y)\in\lambda} \frac{N + \sqrt{\alpha}(x-1) - \sqrt{\alpha}^{-1}(y-1)}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}$$

$$= n! \prod_{(x,y)\in\lambda} \frac{N + \underbrace{\left(\sqrt{\alpha}\, x - \sqrt{\alpha}^{-1}\, y\right)}_{c_\alpha(x,y)} + \underbrace{\left(\sqrt{\alpha}^{-1} - \sqrt{\alpha}\right)}_{\gamma}}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}.$$

Main result

Setup:

- $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ - convex Jack character, $n \in \mathbb{N}$
- $\alpha = \alpha(n) \in \mathbb{R}$ s.t. $\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$ for some $g, g' \in \mathbb{R}$.
- (χ_n) fulfills some technical assumptions about its asymptotic behavior; we will specify their details later.

Theorem (D., Śniady 2019)

Let λ_n be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with $\chi := \chi_n$. Then

- *λ_n converges to some limit shape when $n \to \infty$*
- *the fluctuations of λ_n around the limit shape are asymptotically Gaussian.*

α-anisotropic Young diagrams

Definition

Anisotropic Young diagram $T_{w,h}(\lambda)$ - polygon obtained from the Young diagram λ by a horizontal stretching of ratio w and a vertical stretching of ratio h (each box 1×1 is replaced by a box of dimension $w \times h$).

$$\lambda \mapsto T_{2,\frac{1}{2}}(\lambda) \Longrightarrow$$

In order to study the shape of random Young diagrams $\lambda_n \in \mathbb{Y}_n$ sampled by some Jack-deformed measure, the right scaling is the following:

$$\Lambda_n := T_{\sqrt{\frac{\alpha}{n}},\sqrt{\frac{1}{\alpha n}}}\lambda_n.$$

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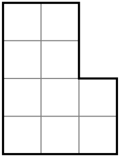

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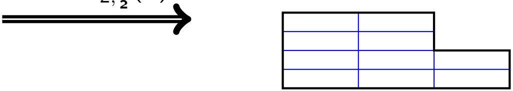

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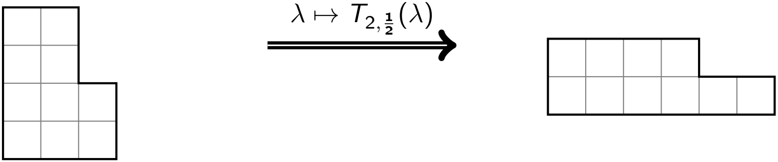

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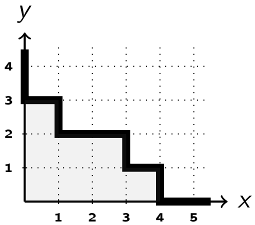
Young diagrams as continuous objects

French convention:



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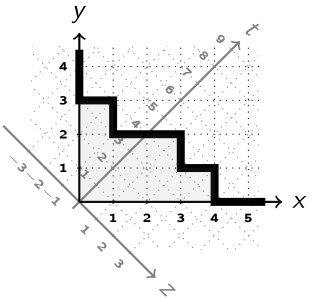

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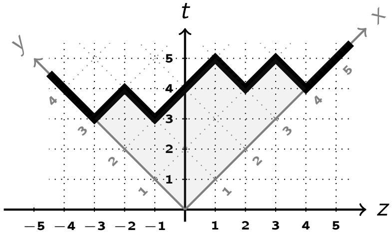

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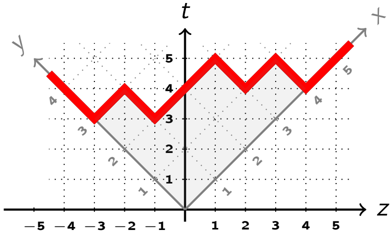

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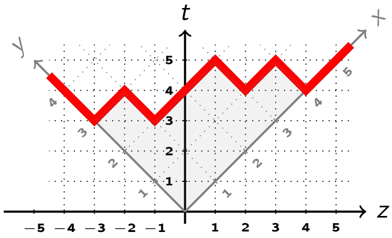

Definition

A profile of a Young diagram λ is a function $\omega_\lambda : \mathbb{R} \to \mathbb{R}_+$ such that its graph is a profile of λ drawn in Russian convention.

Young diagrams as continuous objects

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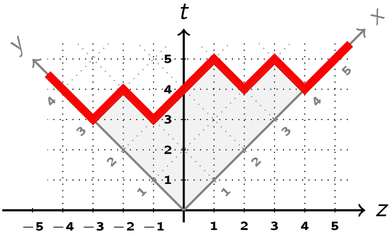

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When we claim that a sequence $(\lambda_n)_n$ of Young diagrams $\lambda_n \in \mathbb{Y}_n$ converges to some limit shape, we actually mean that the sequence of profiles ω_{Λ_n} converges.

Asymptotic shape of large Jack-deformed Young diagrams

Setup:

- $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ - convex Jack character, $n \in \mathbb{N}$
- $\alpha = \alpha(n) \in \mathbb{R}$ s.t. $\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$ for some $g, g' \in \mathbb{R}$.
- (χ_n) fulfills some technical assumptions about its asymptotic behavior; we will specify their details later.

Theorem (D., Śniady 2019; $\alpha = 1$ Biane 2002)

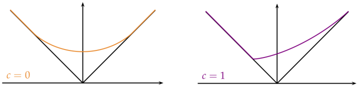
Let λ_n be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with $\chi := \chi_n$.
Then there exists some deterministic function $\omega_{\Lambda_\infty} : \mathbb{R} \to \mathbb{R}$ with the property that

$$\lim_{n\to\infty} \omega_{\Lambda_n} = \omega_{\Lambda_\infty},$$

where the convergence holds true with respect to the supremum norm, in probability.

Examples

We recall that $\gamma = g\sqrt{n} + g' + o(1)$.



Example

When $\alpha > 0$ is fixed, that is $g = 0$ then the limit shape ω_{Λ_∞} does not depend on α!.

- Jack–Plancherel measure (D., Féray 2016)

$$\omega_{\Lambda_\infty}(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin \frac{x}{2} + \sqrt{4 - x^2}\right) & \text{otherwise.} \end{cases}$$

- Jack–Schur–Weyl measure with $\sqrt{n} \sim cN$ (D., Śniady 2019)

$$\omega_{\Lambda_\infty}(x) - \text{explicit function depending on } c.$$

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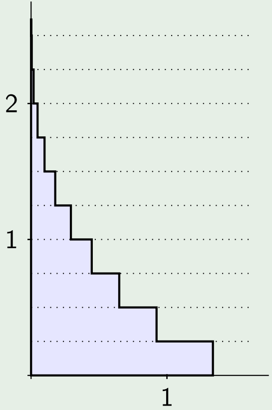
We recall that $\gamma = g\sqrt{n} + g' + o(1)$.

Example



- $\alpha(n) = \frac{1}{c^2 n}$ for some $c \in \mathbb{R}_+$,
- then $g = c, g' = 0$.

Then Λ_n = collection of rectangles $(\frac{1}{gn} \cdot \lambda_i, g)_i$ thus ω_{Λ_∞} clearly depends on g!

The limit shape of random Young diagrams distributed according to the Jack–Plancherel measure in the double scaling limit for $c = \frac{1}{4}$.

Fluctuations

Problem

How to "measure" fluctuations around the limit shape ω_{Λ_∞}?

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We know that $\omega_{\Lambda_n} \to \omega_{\Lambda_\infty}$, so we define a random vector

$$\Delta_n := \sqrt{n}\left(\omega_{\Lambda_n} - \omega_{\Lambda_\infty}\right).$$

We would like to show that Δ_n converges in distribution to some (non-centered) Gaussian random vector Δ_∞ in the space $(\mathbb{R}[x])'$ of distributions, so informally speaking,

$$\omega_{\Lambda_n} \approx \omega_{\Lambda_\infty} + \frac{1}{\sqrt{n}}\Delta_\infty.$$

We need to study suitable test functions:

$$Y_k := \frac{k-1}{2}\int u^{k-2}\,\Delta_n(u)du, \quad k \geq 2.$$

Central limit theorem

Setup:

- $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ - convex Jack character, $n \in \mathbb{N}$
- $\alpha = \alpha(n) \in \mathbb{R}$ s.t. $\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$ for some $g, g' \in \mathbb{R}$.
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Theorem (D., Śniady 2019; $\alpha = 1$ Śniady 2006)

Let λ_n be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with $\chi := \chi_n$.
Then the random vector Δ_n converges in distribution to some (non-centered) Gaussian random vector Δ_∞ in the space $(\mathbb{R}[x])'$ of distributions, the dual space to polynomials, as $n \to \infty$.

Equivalently, the family of random variables $(Y_k)_{k \geq 2}$ converges as $n \to \infty$ to a (non-centered) Gaussian distribution.

Question

Problem

What are the proper assumptions about asymptotic behavior of convex Jack characters which provide the law of large numbers and the central limit theorem?

Approximate factorization property

We extend the domain of $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ to the set $\bigsqcup_{0 \leq k \leq n} \mathcal{P}_k$ of partitions of sufficiently small numbers by setting

$$\chi_n(\pi) := \chi_n(\pi, 1^{n-|\pi|}) \qquad \text{for } |\pi| \leq n.$$

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The general idea of our assumptions is the following:

- the characters do not grow too fast:

$$\chi_n(\pi) = O(n^{-\frac{\|\pi\|}{2}}),$$

- characters on cycles have subleading terms of a proper order:

$$\chi_n((l))\ n^{\frac{l-1}{2}} = a_{l+1} + \frac{b_{l+1} + o(1)}{\sqrt{n}} \qquad \text{for } n \to \infty,$$

- the characters should approximately factorize, i.e.

$$\chi_n(\pi_1 \cdots \pi_\ell) \approx \chi_n(\pi_1) \cdots \chi_n(\pi_\ell).$$

Cumulants I

Note that $\chi_n(\pi) = \mathbb{E}\big(\chi_{(\circ)}(\pi)\big)$ is, by definition, the expectation of the irreducible Jack characters $\chi_\lambda(\pi)$ taken with the probability $\mathbb{P}_{\chi_n}(\lambda)$.

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Cumulants $\kappa_\ell^{\mathbb{E}}(x_1, \dots, x_\ell)$ of random variables $x_1, \dots, x_\ell$ - natural generalization of a variance:

$$\begin{cases} \mathbb{E}(x_1) = \kappa_1^{\mathbb{E}}(x_1), \\ \mathbb{E}(x_1 x_2) = \kappa_2^{\mathbb{E}}(x_1, x_2) + \kappa_1^{\mathbb{E}}(x_1)\kappa_1^{\mathbb{E}}(x_2), \\ \mathbb{E}(x_1 x_2 x_3) = \kappa_3^{\mathbb{E}}(x_1, x_2, x_3) + \kappa_1^{\mathbb{E}}(x_1)\kappa_2^{\mathbb{E}}(x_2, x_3) \\ \qquad + \kappa_1^{\mathbb{E}}(x_2)\kappa_2^{\mathbb{E}}(x_1, x_3) + \kappa_1^{\mathbb{E}}(x_3)\kappa_2^{\mathbb{E}}(x_1, x_2) \\ \qquad + \kappa_1^{\mathbb{E}}(x_1)\kappa_1^{\mathbb{E}}(x_2)\kappa_1^{\mathbb{E}}(x_3), \\ \qquad \vdots \end{cases}$$

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Cumulants $\kappa_\ell^\chi(\pi_1 \dots \pi_\ell)$ of random variables $\chi_{(\circ)}(\pi_1), \dots, \chi_{(\circ)}(\pi_\ell)$ - natural generalization of a variance:

$$\begin{cases} \chi(\pi_1) = \kappa_1^\chi(\pi_1), \\ \chi(\pi_1\pi_2) = \kappa_2^\chi(\pi_1,\pi_2) + \kappa_1^\chi(\pi_1)\ \kappa_1^\chi(\pi_2), \\ \chi(\pi_1\pi_2\pi_3) = \kappa_3^\chi(\pi_1,\pi_2,\pi_3) + \kappa_1^\chi(\pi_1)\ \kappa_2^\chi(\pi_2,\pi_3) \\ \qquad + \kappa_1^\chi(\pi_2)\ \kappa_2^\chi(\pi_1,\pi_3) + \kappa_1^\chi(\pi_3)\ \kappa_2^\chi(\pi_1,\pi_2) \\ \qquad + \kappa_1^\chi(\pi_1)\ \kappa_1^\chi(\pi_2)\ \kappa_1^\chi(\pi_3), \\ \vdots \end{cases}$$

Approximate factorization property revisited

$$\begin{cases} \chi_n(\pi) = O(n^{-\frac{\|\pi\|}{2}}), \\ \chi_n(\pi_1 \cdots \pi_\ell) \approx \chi_n(\pi_1) \cdots \chi_n(\pi_\ell) \end{cases}$$

Examples (Of measures with AFP, thus CLT)

- Jack–Plancherel measure ($\alpha > 0$ fixed, D., Féray 2016)

$$\chi_n(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \quad \kappa_\ell^\chi(\pi_1, \dots, \pi_\ell) = \begin{cases} 1 & \text{if } \ell = 1, \pi_1 = 1^k, \\ 0 & \text{otherwise} \end{cases}$$

- Jack–Schur–Weyl measure ($\sqrt{n} \sim cN$, D., Śniady 2019)

$$\chi_n(\pi) := N^{-\|\pi\|} \quad \kappa_\ell^\chi(\pi_1, \dots, \pi_\ell) = \begin{cases} N^{-\|\pi_\ell\|} & \text{if } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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$$\kappa_\ell^\chi(\pi_1,\dots,\pi_\ell) = O\left(n^{-\frac{\|\pi_1\|+\cdots+\|\pi_\ell\|-2(\ell-1)}{2}}\right).$$

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More examples

Theorem

$\left(\chi^1_n\right), \left(\chi^2_n\right)$ - *families of convex Jack characters with AFP. Then all the families have AFP.*

- *the* **restriction** $\left(\chi^i_{q,n}\right) := \left(\left(\chi^i_{q_n}\right)^{\downarrow^{q_n}_n}\right)$, *where* $q_n \geq n$ *and* $\lim_{n\to\infty} \frac{q_n}{n} = q$;
- *the* **induction** $\left(\chi^i_{q,n}\right) := \left(\left(\chi^i_{q_n}\right)^{\uparrow^{q_n}_n}\right)$, *where* $q_n \leq n$ *and* $\lim_{n\to\infty} \frac{q_n}{n} = q$;
- *the* **outer product**
$$(\chi_n) := \left(\chi^1_{q_n^{(1)}} \circ \chi^2_{q_n^{(2)}}\right),$$
where $q_n^{(1)} + q_n^{(2)} = n$ *and the limits* $q^{(i)} := \lim_{n\to\infty} \frac{q_n^{(i)}}{n}$ *exist;*
- *the* **tensor product**
$$(\chi_n) := \left(\chi^1_n \cdot \chi^2_n\right).$$

$J^{(\alpha)}_\lambda \cdot J^{(\alpha)}_\mu = \sum_\rho c^\rho_{\lambda\mu}(\alpha) J^{(\alpha)}_\rho$

Conjecture: (Stanley '89)

$c^\rho_{\lambda\mu}(\alpha) \cdot h_\alpha(\rho) h'_\alpha(\rho) \cdot \alpha^{|\rho|} \in \mathbb{N}_+[\alpha]$

The main tool - algebraic combinatorics

Our main tool for proving above theorems are certain results on the structure of the algebra of polynomial functions $\mathscr{P}$.

We define the normalized Jack character $\mathrm{Ch}_\pi^{(\alpha)} : \mathbb{Y} \to \mathbb{Q}[\sqrt{\alpha}, \sqrt{\alpha}^{-1}]$:

$$\mathrm{Ch}_\pi^{(\alpha)}(\lambda) := \begin{cases} |\lambda|^{\underline{|\pi|}} \, \chi_\lambda^{(\alpha)}(\pi) & \text{if } |\lambda| \geq |\pi|; \\ 0 & \text{if } |\lambda| < |\pi|. \end{cases}$$

- Basis: $\mathscr{P} = \mathrm{Span}\{\gamma^k \ \mathrm{Ch}_\pi : k \in \mathbb{N}, \pi \in \mathcal{P}\}$.
- Gradation: $\deg(\gamma^k \, \mathrm{Ch}_\pi) = k + \|\pi\|$.

Equivalent characterization of characters with AFP

Theorem (D., Śniady 2019; $\alpha = 1$ Śniady 2006)

- *for each integer* $\ell \geq 1$ *and all integers* $l_1, \dots, l_\ell \geq 2$ *the limit*

$$\lim_{n\to\infty} \kappa_\ell^{\chi_n}\big((l_1), \dots, (l_\ell)\big)\ n^{\frac{l_1+\cdots+l_\ell+\ell-2}{2}}$$

exists and is finite;

- *for each integer* $\ell \geq 1$ *and all* $x_1, \dots, x_\ell \in \mathscr{P}$ *the limit*

$$\lim_{n\to\infty} \kappa_\ell^{\chi_n}(x_1, \dots, x_\ell)\ n^{-\frac{\deg x_1+\cdots+\deg x_\ell-2(\ell-1)}{2}}$$

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Two different cumulants:

Question

Why this theorem is helpful? $\kappa_{\ell}^{\chi_n}$ *vs.* $\kappa_{\bullet\ell}^{\chi_n}$*?*

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Why this theorem is helpful? $\kappa_{\ell}^{\chi_n}$ *vs.* $\kappa_{\bullet\ell}^{\chi_n}$*?*

$\mathscr{P}$ and $\mathscr{P}_{\bullet}$ are the same rings, but the multiplication is different:

- $(\gamma^p \operatorname{Ch}_\pi) \cdot (\gamma^q \operatorname{Ch}_\sigma) = \gamma^{p+q} \sum_{???^{\tau}_{\pi,\sigma}} ???^{\tau}_{\pi,\sigma} \operatorname{Ch}_\tau,$
- $(\gamma^p \operatorname{Ch}_\pi) \bullet (\gamma^q \operatorname{Ch}_\sigma) := \gamma^{p+q} \operatorname{Ch}_{\pi\sigma}.$

Two different cumulants:

Question

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This multiplication gives a rule for computing cumulants!

- $\kappa_2^{\chi_n}(\pi, \sigma) = \sum_{???^{\tau}_{\pi,\sigma}} ???^{\tau}_{\pi,\sigma} \ n^{\underline{|\tau|}} \chi_n(\tau) - n^{\underline{|\pi|}} \cdot n^{\underline{|\sigma|}} \cdot \chi_n(\pi) \cdot \chi_n(\sigma),$
- $\kappa_{\bullet 2}^{\chi_n}(\pi, \sigma) = n^{\underline{|\pi\sigma|}} \chi_n(\pi\sigma) - n^{\underline{|\pi|}} \cdot n^{\underline{|\sigma|}} \cdot \chi_n(\pi) \cdot \chi_n(\sigma).$

Perspectives

- Limit shape of the Jack-Plancherel measure (or other measures given by convex characters) in the double scaling limit?
- Covariance of normal distribution in the double scaling limit = the top-degree of normalized Jack characters indexed by two rows = the combinatorics of unhandled maps with two faces.
- Joint distribution of properly normalized $(\lambda_{(n)})_1 \geq (\lambda_{(n)})_2 \geq \dots$ with respect to Jack-Plancherel measure = Tracy-Widom β (Guionnet, Huang 2019). What about convex characters with AFP?

Thank you

THANK YOU FOR YOUR ATTENTION!