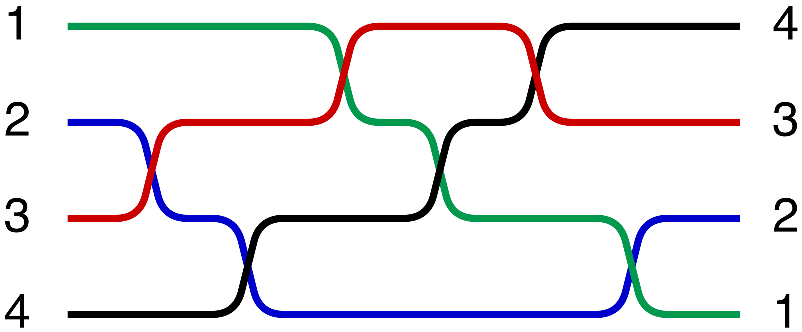
# The Archimedean Limit of Random Sorting Networks

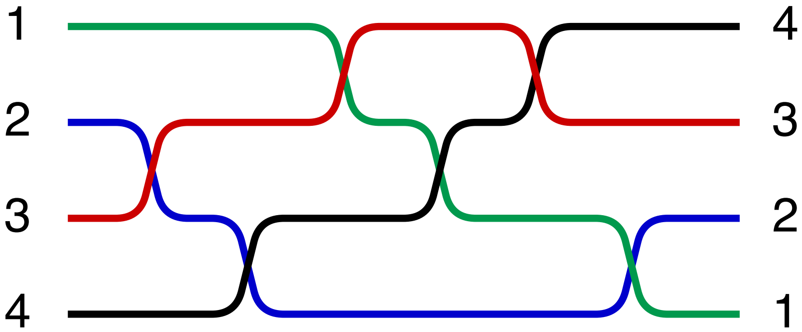
Duncan Dauvergne

March 11, 2019

# How to (naively) sort a list?

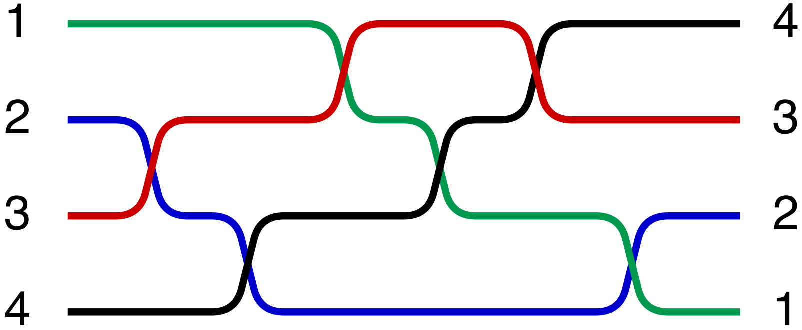


# How to (naively) sort a list?




An *n*-element **sorting network** is a way of sorting a list of *n* numbers from increasing to decreasing order using a minimal number of adjacent swaps.

# How to (naively) sort a list?




An $n$-element **sorting network** is a way of sorting a list of $n$ numbers from increasing to decreasing order using $N = \binom{n}{2}$ adjacent swaps.

# In terms of $S_n$:

$\Gamma(S_n)$: Cayley graph of $S_n$ with generators

$$\{\pi_i = (i, i+1) : i \in \{1, \ldots, n-1\}\}.$$

A sorting network $\sigma$ is shortest path in $\Gamma(S_n)$ from the identity $1 \cdots n$ to the reverse permutation $n \cdots 1$.

# In terms of $S_n$:

$\Gamma(S_n)$: Cayley graph of $S_n$ with generators

$$\{\pi_i = (i, i+1) : i \in \{1, \ldots, n-1\}\}.$$

A sorting network $\sigma$ is shortest path in $\Gamma(S_n)$ from the identity $1 \cdots n$ to the reverse permutation $n \cdots 1$. We can write

$$n \cdots 1 = \pi_{k_N} \cdots \pi_{k_1} :$$

a **reduced decomposition** of $n \cdots 1$.

# History:

# History:

- Stanley, 1984: The number of $n$-element sorting networks is equal to the number of Young tableaux of staircase shape $(n-1, n-2, \cdots, 1)$.

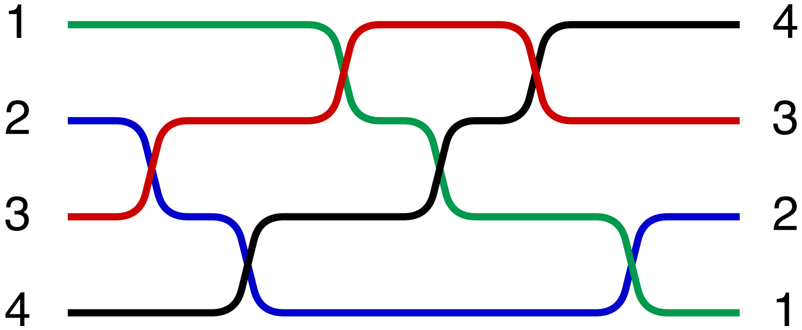
# History:

- Stanley, 1984: The number of $n$-element sorting networks is equal to the number of Young tableaux of staircase shape $(n-1, n-2, \cdots, 1)$.
- Edelman-Greene, 1987: Bijective proof

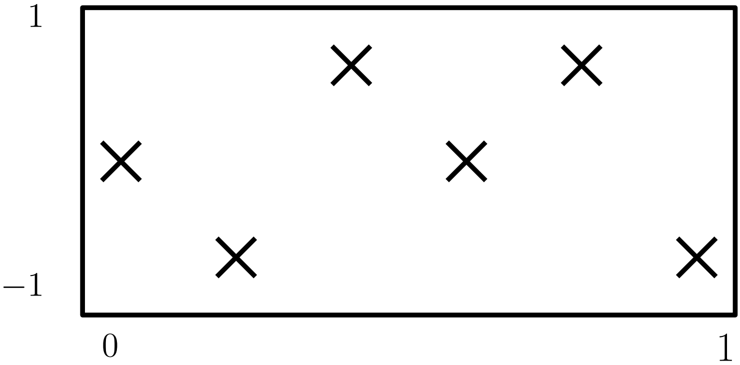
# History:

- Stanley, 1984: The number of $n$-element sorting networks is equal to the number of Young tableaux of staircase shape $(n-1, n-2, \cdots, 1)$.
- Edelman-Greene, 1987: Bijective proof
- Angel-Holroyd-Romik-Virág, 2007: What does a uniform **random sorting network** look like?

# Swap distribution:

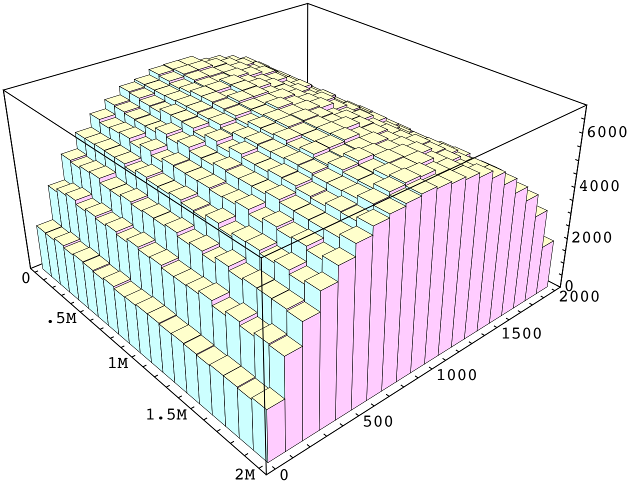


# Swap distribution:

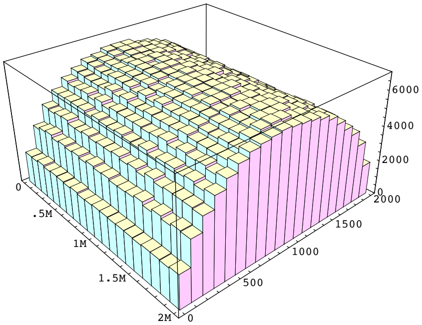



Just look at positions of the swaps and rescale space and time.

# Swap distribution:



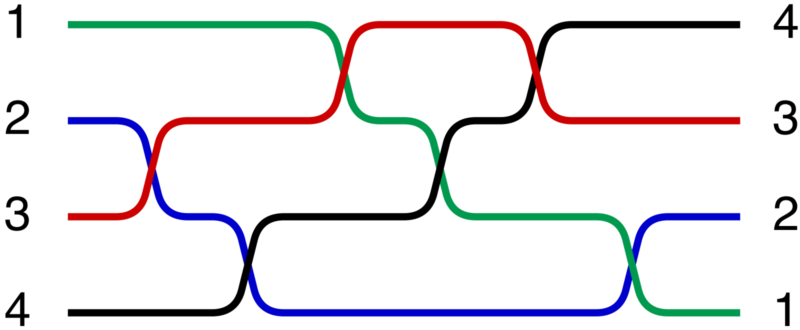
# Swap distribution:




### Theorem (AHRV, 07)

*The swap distribution of a random sorting network converges to* $\mathfrak{Leb} \times \mathfrak{semi}$.

# Trajectories:



# Trajectories:

AHRV conjectured that with high probability, all trajectories in a random sorting network are close to sine curves (with a random amplitude and phase shift): $t \mapsto A \sin(\pi t + \Theta)$.

# Permutation Matrices:

For a sorting network $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_N)$, how can we geometrically describe the half-way permutation $\sigma_{N/2}$ ?

# Permutation Matrices:

Example: $\tau = 54132$

# Permutation Matrices:

Example: $\tau = 54132$

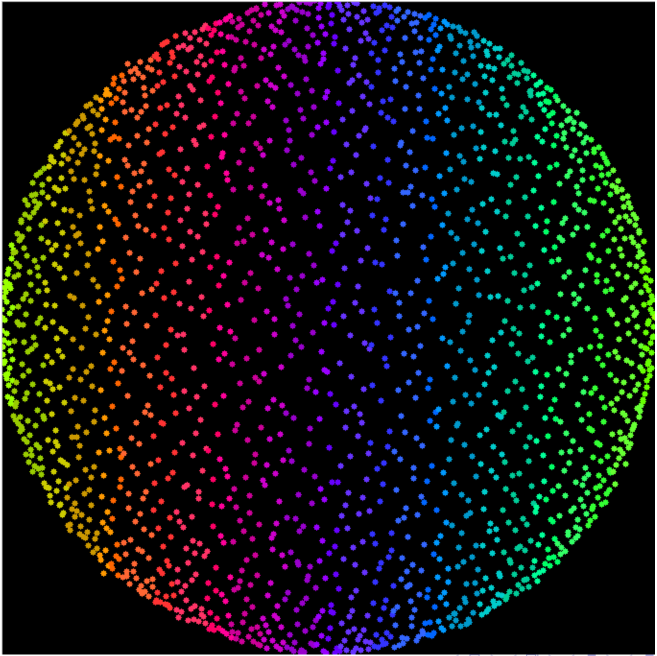
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

# Permutation Matrices:

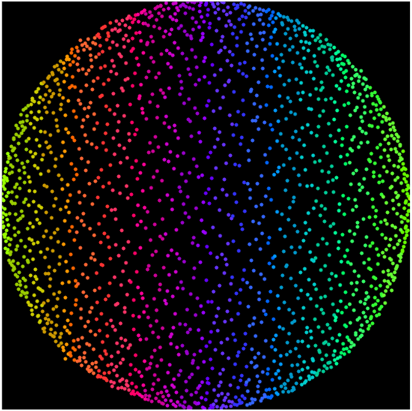
Example: $\tau = 54132$

$$\begin{bmatrix} & & & & \bullet \\ & & & \bullet & \\ \bullet & & & & \\ & & \bullet & & \\ & \bullet & & & \end{bmatrix}$$

Can think of this as a random measure on the square $[-1, 1]^2$ with $\delta$-masses of size $1/n$ at the locations of the 1s.

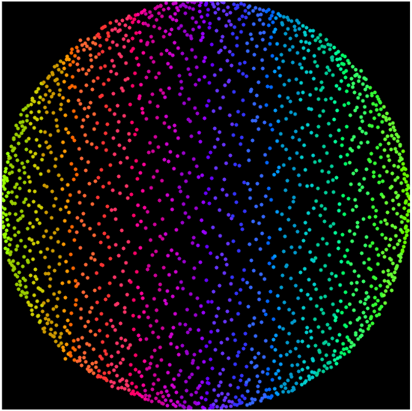


# Permutation Matrices:



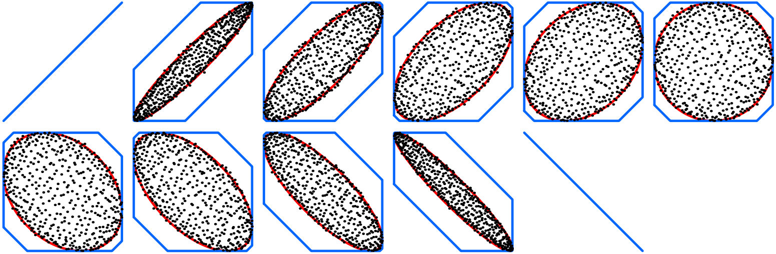
AHRV conjectured that the halfway permutation matrix measure converges to the projected surface area measure of the 2-sphere

# Permutation Matrices:



AHRV conjectured that the halfway permutation matrix measure converges to the projected surface area measure of the 2-sphere: the **Archimedean measure** $\mathfrak{Arch}$.

# Permutation Matrices at other times:



# Permutation Matrix Evolution:

For $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_N)$, look at the increment evolution for the halfway matrix:

$$t \mapsto \sigma_{N/2+Nt}\sigma_{Nt}^{-1}$$

# Permutation Matrix Evolution:

For $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_N)$, look at the increment evolution for the halfway matrix:

$$t \mapsto \sigma_{N/2+Nt}\sigma_{Nt}^{-1}$$

Sine curve conjecture suggests that the movement of a fixed particle under this evolution looks like

$$(a\sin(\pi t + \theta), a\sin(\pi t + \pi/2 + \theta)) = a(\sin(\pi t + \theta), \cos(\pi t + \theta))$$

# Permutation Matrix Evolution:

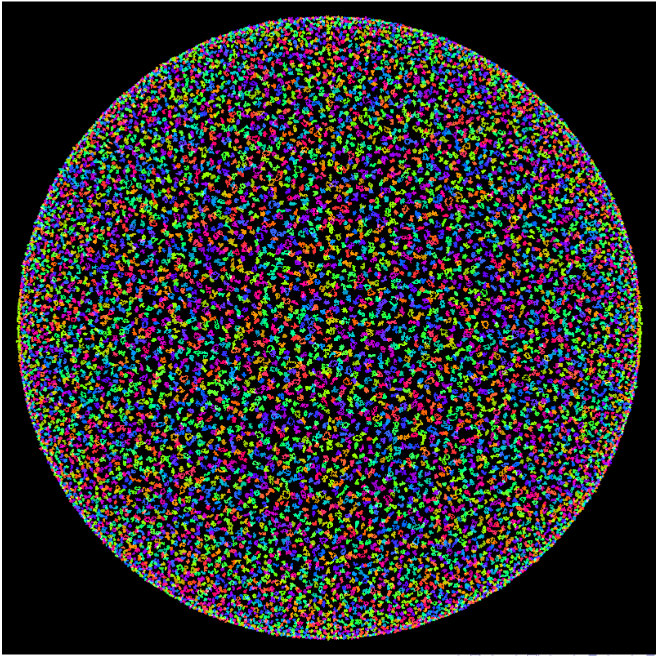
For $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_N)$, look at the increment evolution for the halfway matrix:

$$t \mapsto \sigma_{N/2+Nt}\sigma_{Nt}^{-1}$$

Sine curve conjecture suggests that the movement of a fixed particle under this evolution looks like

$$(a\sin(\pi t + \theta), a\sin(\pi t + \pi/2 + \theta)) = a(\sin(\pi t + \theta), \cos(\pi t + \theta))$$

Can plot these curves after subtracting the rotation!



# Back to $\Gamma(S_n)$:

We can embed $\Gamma(S_n)$ into $\mathbb{R}^n$ by the map:

$$\tau \mapsto (\tau(1), \ldots, \tau(n))$$

# Back to $\Gamma(S_n)$:

We can embed $\Gamma(S_n)$ into $\mathbb{R}^n$ by the map:

$$\tau \mapsto (\tau(1), \ldots, \tau(n)) \in \text{Hyperplane}$$

# Back to $\Gamma(S_n)$:

We can embed $\Gamma(S_n)$ into $\mathbb{R}^n$ by the map:

$$\tau \mapsto (\tau(1), \ldots, \tau(n)) \in \text{Hyperplane} \cap \text{Sphere}$$

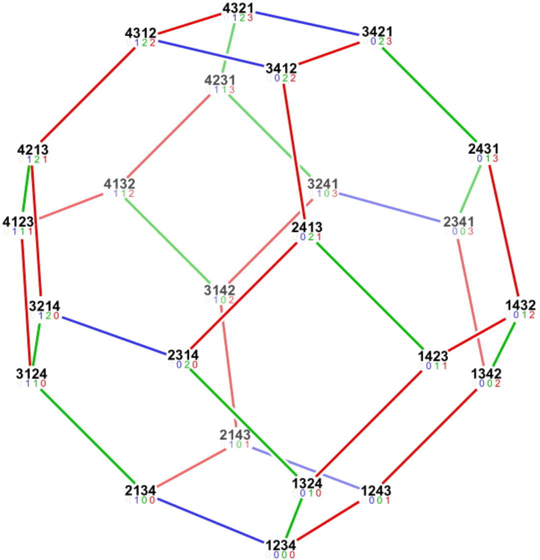
# Back to $\Gamma(S_n)$:

We can embed $\Gamma(S_n)$ into $\mathbb{R}^n$ by the map:

$$\tau \mapsto (\tau(1), \ldots, \tau(n)) \in \text{Hyperplane} \cap \text{Sphere}$$

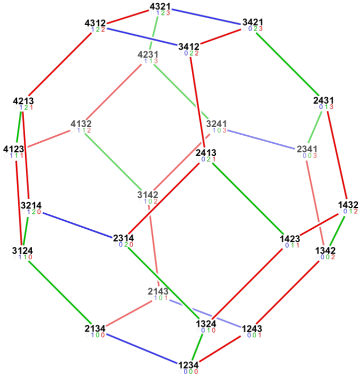
$\Gamma(S_n)$ lives in an $(n-2)$-dimensional sphere $\mathcal{S}^{n-2}$.

# $\Gamma(S_4)$:



4321
0123
4312
0122
3421
0023
3412
0022
4231
0113
4213
0121
2431
0013
4132
0112
3241
0103
4123
0111
2413
0021
2341
0003
3142
0102
3214
0120
1432
0012
2314
0020
1423
0011
3124
0110
1342
0002
2143
0101
2134
0100
1324
0010
1243
0001
1234
0000

## $\Gamma(S_4)$:




AHRV conjectured that a random sorting network is close to a (random) great circle on $\mathcal{S}^{n-2}$.

# The weak limit:

All conjectures follow from the following weak limit theorem:

## Theorem (D. 2018)

*Let* $Y_n : [0,1] \to [-1,1]$ *be a uniform scaled n-element sorting network trajectory. Then*

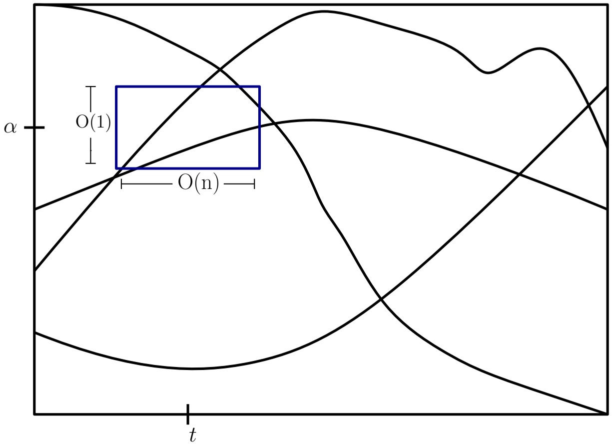
$$Y_n \xrightarrow{d} Y$$

*where*

$$Y(t) = X \cos(\pi t) + Z \sin(\pi t), \qquad (X, Z) \sim \mathfrak{Arch}.$$

# Local Limit:

Angel-D.-Holroyd-Virag and Gorin-Rahman constructed the **local limit** of random sorting networks:



# Local Limit at the centre ($\alpha = 0, t = 0$):

Define

$$U^n(x,t) = \sigma_{\lfloor Nt \rfloor}(x + \lfloor n/2 \rfloor) - \lfloor n/2 \rfloor.$$

# Local Limit at the centre ($\alpha = 0, t = 0$):

Define

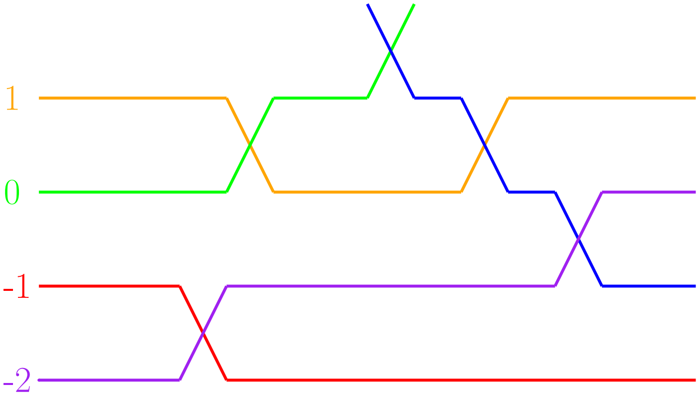
$$U^n(x,t) = \sigma_{\lfloor Nt \rfloor}(x + \lfloor n/2 \rfloor) - \lfloor n/2 \rfloor.$$

Then

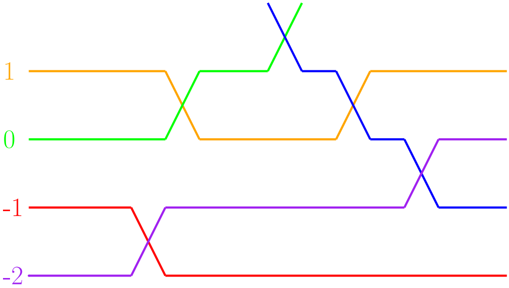
$$U^n \xrightarrow{d} U,$$

where $U : \mathbb{Z} \times [0,\infty) \to \mathbb{Z}$ is a random function: a **swap process** on the integers.

# Local Limit:

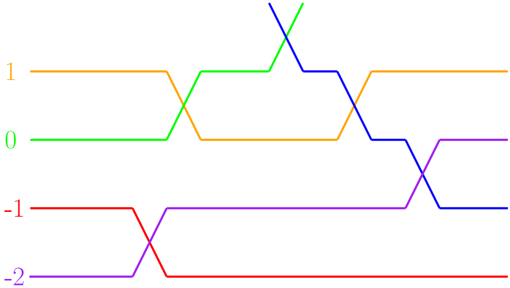


# Local Limit:




- $U$ is stationary in time and ergodic in space

# Local Limit:

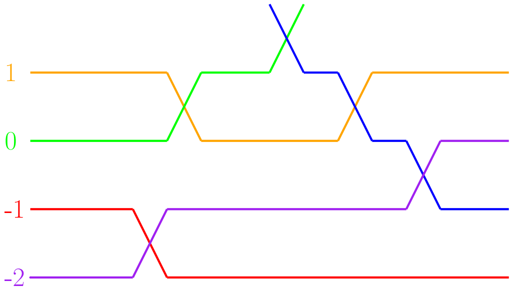



- $U$ is stationary in time and ergodic in space
- Away from the centre: for $(\alpha, t) \in (-1, 1) \times [0, 1)$ we get the limit

$$U_{t,\alpha}(x, s) = U(x, \sqrt{1-\alpha^2}s).$$

The only time/space dependence is by a semicircle rescaling

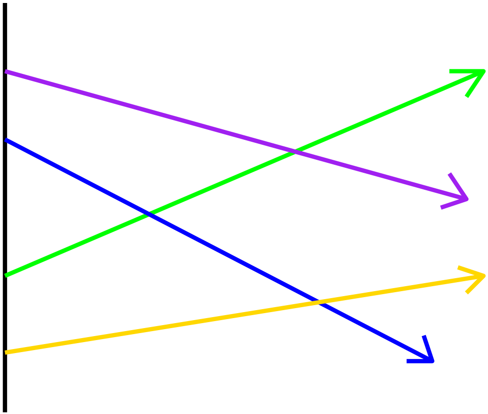
# Local Limit:




- Stationarity in space/time implies that particles have asymptotic speeds:

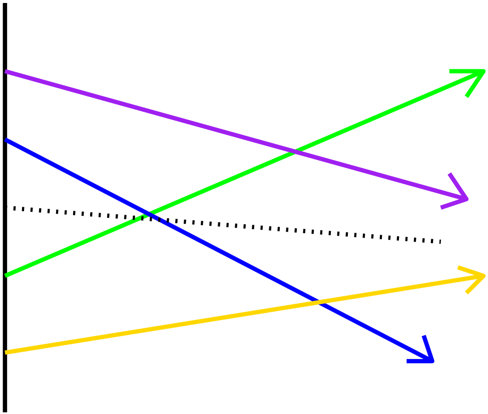
$$\lim_{t \to \infty} \frac{U(x,t) - U(x,0)}{t} = S(x) \quad \text{a.s.}$$

# Swap rates in the local limit:



Heuristically, particles in $U$ can be thought of as moving along lines with independent slopes drawn from the local speed distribution $\mu$.

# Swap rates in the local limit:



Heuristically, particles in $U$ can be thought of as moving along lines with independent slopes drawn from the local speed distribution $\mu$.

# Swap rates in the local limit:

Theorem (D.,Virag 2018)

1. *Let* $L(t) = ct + d$, *and let* $N(L, t)$ *be the number of particles that have crossed* $L$ *by time* $t$. *Then*

$$\lim_{t \to \infty} \frac{N(L, t)}{t} = \int |y - c| d\mu(y) \quad a.s.$$

# Swap rates in the local limit:

**Theorem (D.,Virag 2018)**

1. *Let $L(t) = ct + d$, and let $N(L, t)$ be the number of particles that have crossed $L$ by time $t$. Then*

$$\lim_{t\to\infty} \frac{N(L,t)}{t} = \int |y - c| d\mu(y) \quad a.s.$$

2. *Let $M(x, t)$ be the number of particles that particle $x$ has swapped with by time $t$. Then*

$$\lim_{t\to\infty} \frac{M(x,t)}{t} = \int |y - S(x)| d\mu(y) \quad a.s.$$

# Local $\to$ Global

Let $h : [0,1] \to [-1,1]$ be a (Lipschitz) path.
The number of particles that cross $h$ (counting **global** multiplicities) should be roughly $nJ(h)$, where

$$J(h) := \frac{1}{2}\int_0^1 D_\mu\left(\frac{h'(t)}{\sqrt{1-h^2(t)}}\right)\sqrt{1-h^2(t)}dt,$$

and

$$D_\mu(x) = \int |y-x| d\mu(y)$$

# Local $\to$ Global

Let $h : [0, 1] \to [-1, 1]$ be a (Lipschitz) path.
The number of particles that cross $h$ (counting **global** multiplicities) should be roughly $nJ(h)$, where

$$J(h) := \frac{1}{2}\int_0^1 D_\mu\left(\frac{h'(t)}{\sqrt{1-h^2(t)}}\right)\sqrt{1-h^2(t)}dt,$$

and

$$D_\mu(x) = \int |y-x| d\mu(y)$$

The quantity $J(h)$ is the **particle flux** across $h$.

# Local $\rightarrow$ Global

# Local $\rightarrow$ Global

**Theorem (D. 2018)**

1. *If* $h : [0,1] \rightarrow [-1,1]$ *is Lipschitz, and* $h(0) = -h(1)$, *then* $J(h) \geq 1$.

# Local $\to$ Global

Theorem (D. 2018)

1. *If* $h : [0, 1] \to [-1, 1]$ *is Lipschitz, and* $h(0) = -h(1)$*, then* $J(h) \geq 1$*.*
2. *If* $Y$ *is any subsequential limit of the uniform sorting network trajectory* $Y_n$*, then* $J(Y) = 1$*.*

# Local $\to$ Global

**Theorem (D. 2018)**

1. *If* $h : [0,1] \to [-1,1]$ *is Lipschitz, and* $h(0) = -h(1)$*, then* $J(h) \geq 1$.
2. *If* $Y$ *is any subsequential limit of the uniform sorting network trajectory* $Y_n$*, then* $J(Y) = 1$.

Upshot: Limits of particle trajectories minimize flux!

# Using the minimization property:

Problem: we don't know $\mu$.

# Using the minimization property:

Problem: we don't know $\mu$. We do know:

1. Since $\mu$ is symmetric, $D_\mu$ is even

# Using the minimization property:

Problem: we don't know $\mu$. We do know:

1. Since $\mu$ is symmetric, $D_\mu$ is even
2. $D_\mu$ is minimized at any median of $\mu$, hence $D_\mu$ is minimized at 0

# Using the minimization property:

Problem: we don't know $\mu$. We do know:

1. Since $\mu$ is symmetric, $D_\mu$ is even
2. $D_\mu$ is minimized at any median of $\mu$, hence $D_\mu$ is minimized at 0

- We can use these properties to narrow down the set of possible minimizers of flux

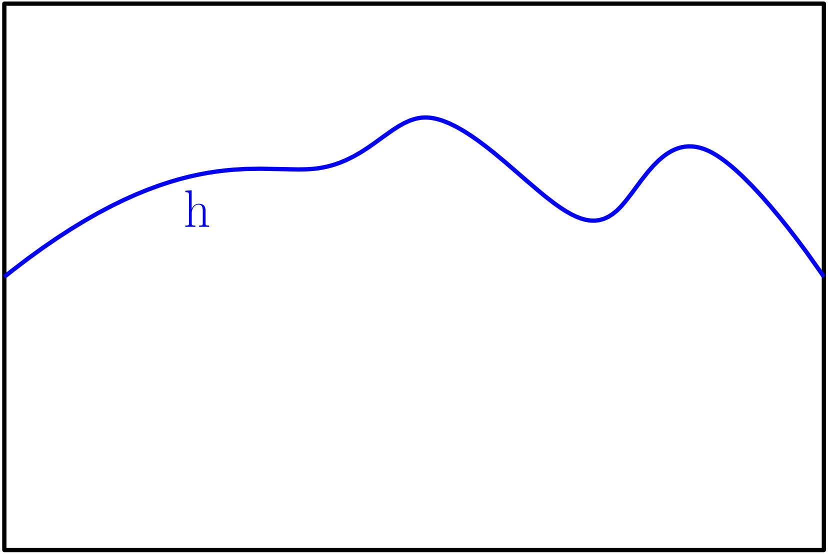
# Using the minimization property:

Problem: we don't know $\mu$. We do know:

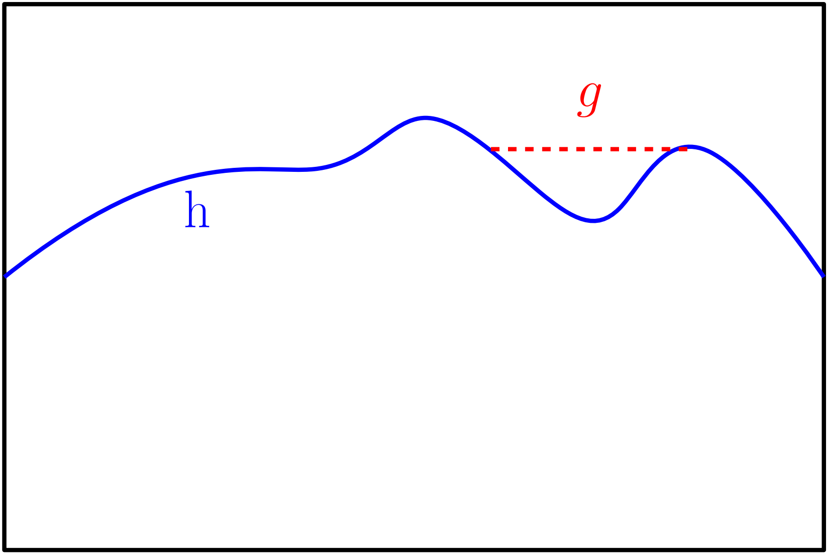
1. Since $\mu$ is symmetric, $D_\mu$ is even
2. $D_\mu$ is minimized at any median of $\mu$, hence $D_\mu$ is minimized at 0

- We can use these properties to narrow down the set of possible minimizers of flux
- Note: By shifting, it is enough to consider paths $h$ with $h(0) = -h(1) = 0$

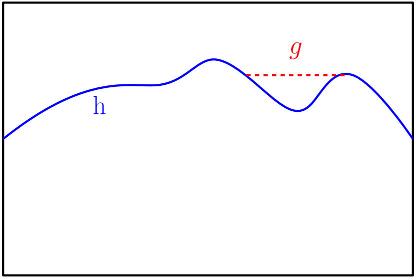
# Example:



# Example:

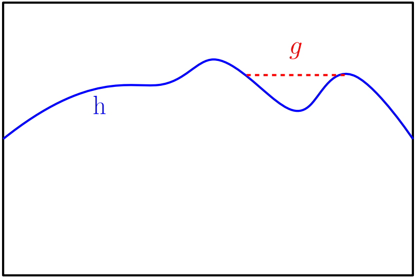


# Example:



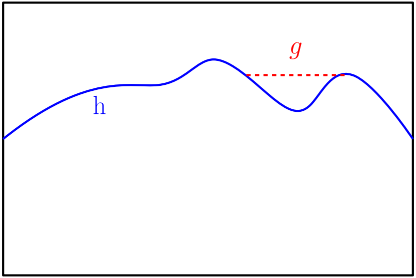

- $D_\mu(g'/\sqrt{1-g^2}) \leq D_\mu(h'/\sqrt{1-h^2})$ on the region where they differ

# Example:



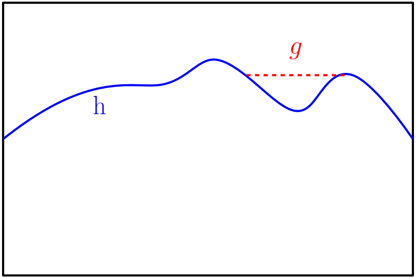

- $D_\mu(g'/\sqrt{1-g^2}) \leq D_\mu(h'/\sqrt{1-h^2})$ on the region where they differ
- $\sqrt{1-g^2} < \sqrt{1-h^2}$ on the region where they differ

# Example:



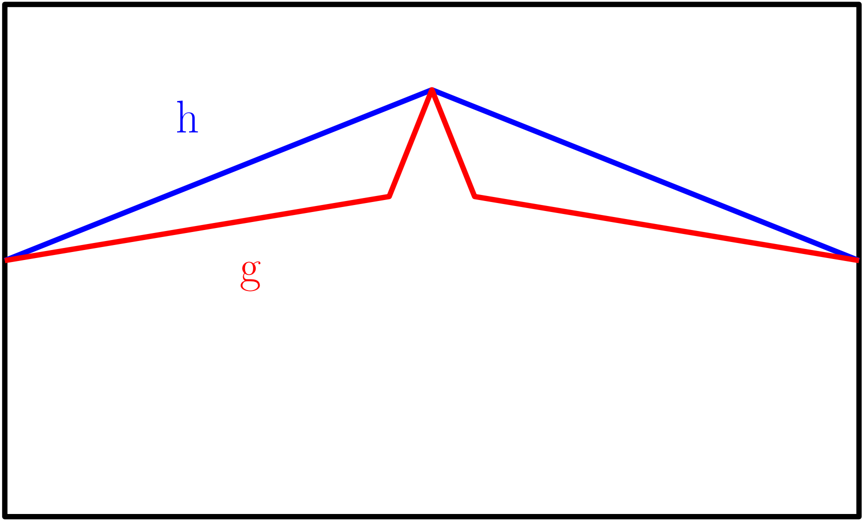

- $D_\mu(g'/\sqrt{1-g^2}) \leq D_\mu(h'/\sqrt{1-h^2})$ on the region where they differ
- $\sqrt{1-g^2} < \sqrt{1-h^2}$ on the region where they differ
- Hence $J(g) < J(h)$

# Example:

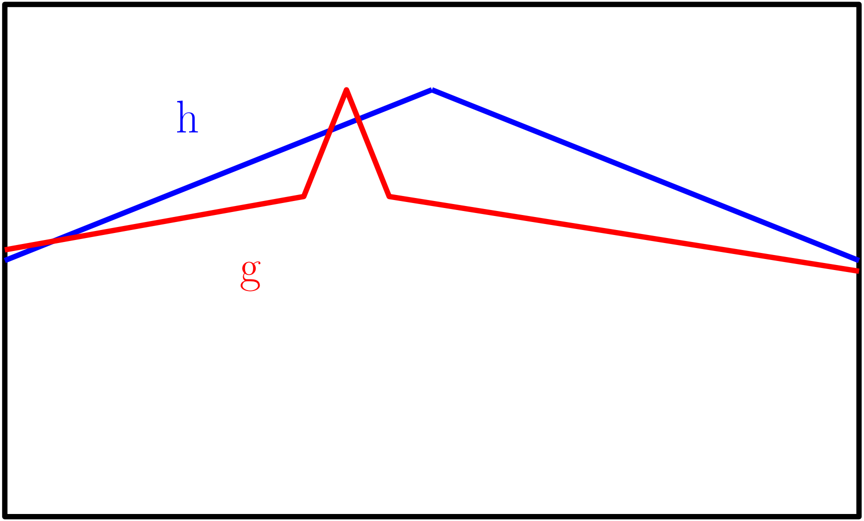



Hence if $h(0) = -h(1) = 0$ is a minimal flux path with $h \geq 0$, it must be unimodal! By using symmetry arguments, we can get that any minimal flux path with $h(0) = -h(1) = 0$ must be unimodal.

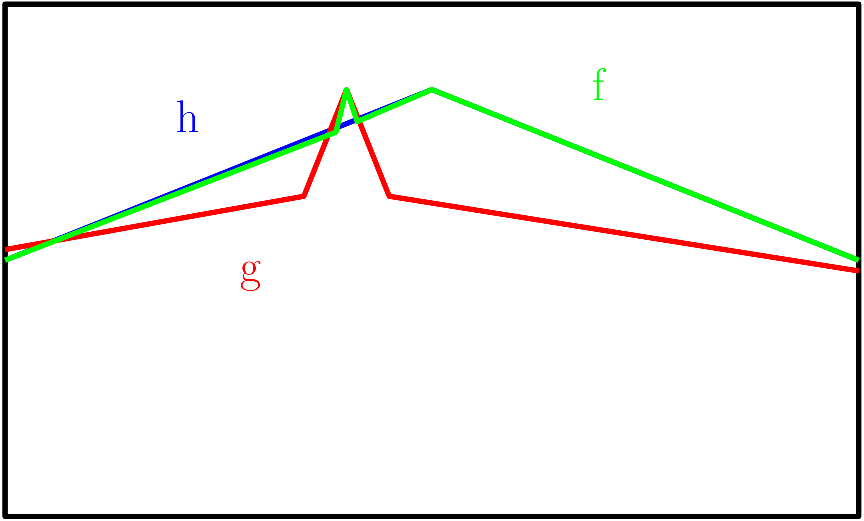
# Uniqueness at a fixed maximum value? Suppose not.



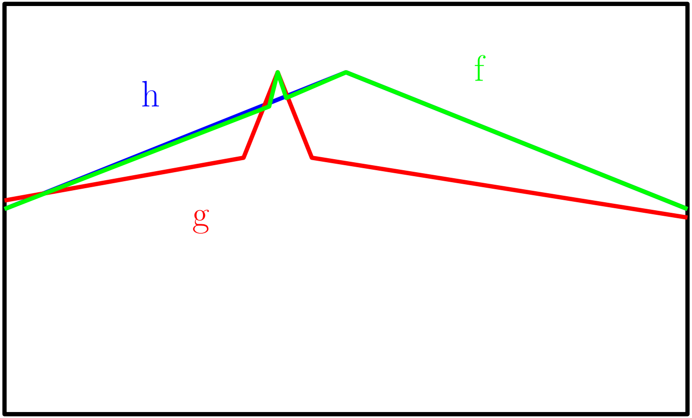
# Uniqueness at a fixed maximum value? Suppose not.



# Uniqueness at a fixed maximum value? Suppose not.



# Uniqueness at a fixed maximum value? Suppose not.




$f$ is a minimal flux path that contradicts unimodality!