Lecture Hall Tableaux

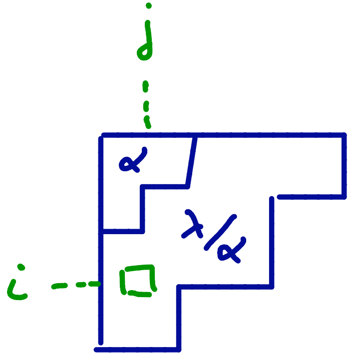
Sylvie Corteel - CNRS U. Paris Diderot

Banff March 15^{th}, 2019

Lecture Hall Tableaux

Two partitions λ, α

An integer n



Fill the cell (i,j) with $T_{i,j}$

$$\begin{cases} \dfrac{T_{ij}}{n-i+j} \geqslant \dfrac{T_{i,j+1}}{n-i+j+1} \\[2ex] \dfrac{T_{ij}}{n-i+j} > \dfrac{T_{i+1,j}}{n-1-i+j} \end{cases}$$

<u>Ex</u> $\lambda = (6,6,4,3)$

$\alpha = (3,1)$ $\qquad n = 5$

			9	4	3
	5	6	4	3	1
2	2	1	0		
1	0	0			

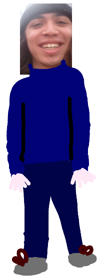
$\geqslant$

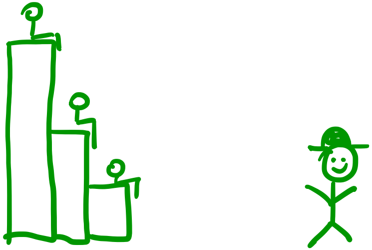
			$\frac{9}{8}$	$\frac{4}{9}$	$\frac{3}{10}$
	$\frac{5}{5}$	$\frac{6}{6}$	$\frac{4}{7}$	$\frac{3}{8}$	$\frac{1}{9}$
$\frac{2}{3}$	$\frac{2}{4}$	$\frac{1}{5}$	$\frac{0}{6}$		
$\frac{1}{2}$	$\frac{0}{3}$	$\frac{0}{4}$			

Plan

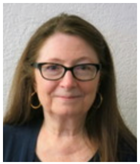
1. Lecture Hall partitions
2. Orthogonal polynomials: univariate and multivariate for "$q=t$".
3. Multivariate moments
4. Little q-Jacobi polynomials and LHT
5. Arctic curves for bounded LHT

by D. Keating 9:30





① Lecture Hall partitions



(See Savage "The mathematics of LHP")

<u>Eriksson</u>^{2} (98) $\quad w \in \tilde{C}_n$ affine hyperoctahedral group

$[w_1, \ldots, w_n]$

$\pm w_1, \ldots, \pm w_n$ are all distinct mod $(2n+2)$

$w \in \tilde{C}_n / C_n$ iff $0 < w_1 < \ldots < w_n$

$1 \leq j \leq i \leq n \qquad I_{ij} = \left\lfloor \dfrac{w_i - w_j}{2n+2} \right\rfloor + \left\lfloor \dfrac{w_i + w_j}{2n+2} \right\rfloor$

<u>Theorem</u> (Eriksson^2) $\quad \ell(w) = \sum_i I_i \; ; \quad I_i = \sum_j I_{ij}$

$$w \in \tilde{C}_n / C_n \iff \frac{I_n}{n} \geq \frac{I_{n-1}}{n-1} \geq \frac{I_{n-2}}{n-2} \geq \ldots \geq \frac{I_1}{1} \geq 0$$

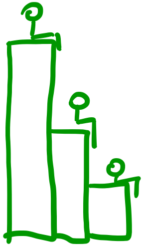
Bousquet-Mélou & Eriksson (97)

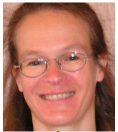
Lecture Hall partitions

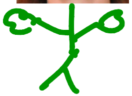
$$L_n = \left\{ \lambda \,\middle|\, \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \frac{\lambda_3}{n-2} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0 \right\}$$

Thm

$$\sum_{\lambda \in L_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$







$$|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

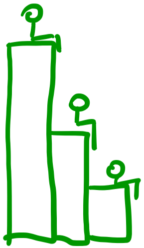
Bousquet-Mélou & Eriksson (97)

Lecture Hall partitions

$$L_n = \left\{ \lambda \;\middle|\; \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \frac{\lambda_3}{n-2} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0 \right\}$$

Thm

$$\sum_{\lambda \in L_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$



LHP in L_n $\longleftrightarrow$ Partitions into odd parts $< 2n$

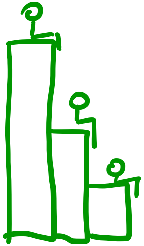
Bousquet-Mélou & Eriksson (97)

Lecture Hall partitions

$$L_n = \left\{ \lambda \,\middle|\, \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \frac{\lambda_3}{n-2} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0 \right\}$$

Thm

$$\sum_{\lambda \in L_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$



LHP in $L_n \longleftrightarrow$ Partitions into odd parts $< 2n$

$n \to \infty$ Euler

Distinct $\longleftrightarrow$ Odd

Back to $\tilde{C}_n / C_n$

Refined Bott's formula

(MacDonald)

$$\sum_{\pi \in \tilde{C}_n / C_n} q^{\ell(\pi)} \, a^{\# S_0(\pi)} \, b^{\# S_n(\pi)} = \prod_{i=1}^{n} \frac{1 + b q^i}{1 - a b q^{n+i}}$$

Back to $\tilde{C}_n/C_n$

Refined Bott's formula

(MacDonald)

$$\sum_{\pi \in \tilde{C}_n/C_n} q^{\ell(\pi)}\, b^{\#S_0(\pi)}\, a^{\#S_n(\pi)} = \prod_{i=1}^{n} \frac{1+aq^i}{1-abq^{n+i}}$$

Translate to statistics on Lecture Hall Partitions (BME 99, Hanusa & Savage 17)

$$\lceil \lambda \rceil = \left(\left\lceil \frac{\lambda_1}{n} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{1} \right\rceil \right) \qquad o(\lambda) = \#\text{ odd parts}$$

$$\sum_{\lambda \in L_n} q^{|\lambda|}\, u^{|\lceil \lambda \rceil|}\, v^{o(\lceil \lambda \rceil)} = \prod_{i=1}^{n} \frac{1+uvq^i}{1-u^2 q^{n+i}}$$

Back to $\tilde{C}_n/C_n$

Refined Bott's formula (MacDonald)

$$\sum_{\pi \in \tilde{C}_n/C_n} q^{\ell(\pi)} \, b^{\#S_0(\pi)} \, a^{\#S_n(\pi)} = \prod_{i=1}^{n} \frac{1+aq^i}{1-abq^{n+i}}$$

Translate to statistics on Lecture Hall Partitions (BME 99, Hanusa & Savage 17)

$$\lceil \lambda \rceil = \left(\left\lceil \frac{\lambda_1}{n} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{1} \right\rceil \right) \qquad o(\lambda) = \#\text{ odd parts}$$

$$\sum_{\lambda \in L_n} q^{|\lambda|} \, u^{|\lceil\lambda\rceil|} \, v^{o(\lceil\lambda\rceil)} = \prod_{i=1}^{n} \frac{1+uvq^i}{1-u^2q^{n+i}}$$

$$uv = a \qquad u^2 = b$$

2 Orthogonal polynomials

Univariate & multivariate
case

$b = (b_0, b_1, \dots)$ $\qquad$ $\lambda = (\lambda_1, \lambda_2, \dots)$

$$p_{n+1}(x) = (x - b_n)\,p_n(x) - \lambda_n\, p_{n-1}(x), \quad n \geq 0$$

$p_0(x) = 1, \quad p_{-1}(x) = 0$ $\qquad$ (Favard's theorem)

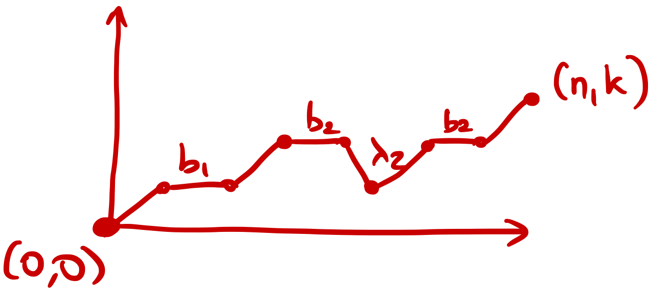
$b = (b_0, b_1, \ldots)$ $\qquad$ $\lambda = (\lambda_1, \lambda_2, \ldots)$

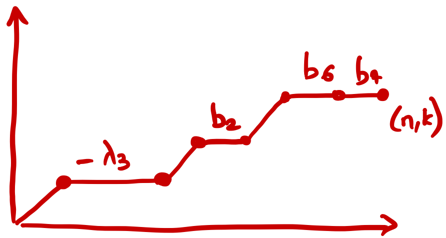
$$p_{n+1}(x) = (x - b_n)\,p_n(x) - \lambda_n\, p_{n-1}(x), \quad n \geq 0$$

$$p_0(x) = 1, \quad p_{-1}(x) = 0$$

Viennot (80s)

$\mu_{n,k}$ $\qquad\qquad$ $\nu_{n,k}$





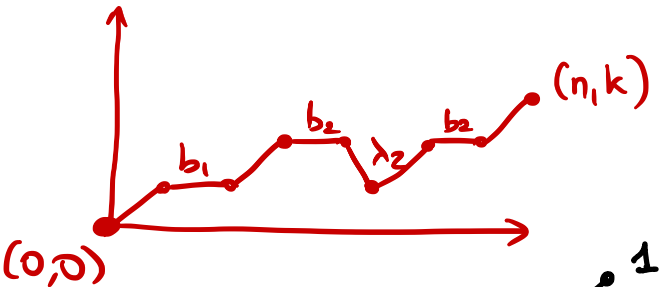
$b = (b_0, b_1, \dots)$ $\lambda = (\lambda_1, \lambda_2, \dots)$

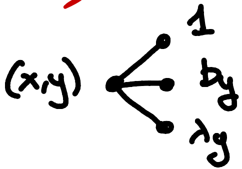
$$p_{n+1}(x) = (x - b_n)\, p_n(x) - \lambda_n\, p_{n-1}(x), \quad n \geq 0$$

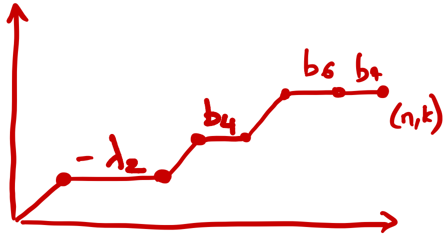
$$p_0(x) = 1, \quad p_{-1}(x) = 0$$

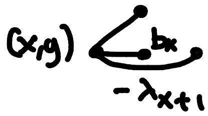
Viennot (80s)

$\mu_{n,k}$ $\nu_{n,k}$



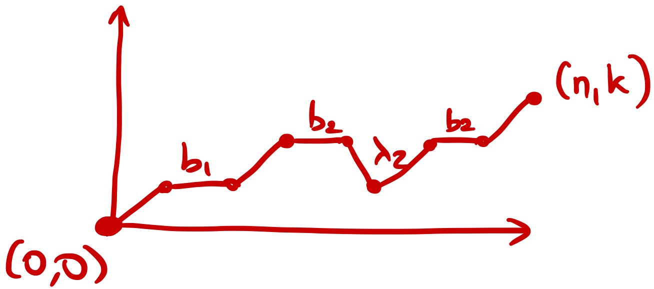


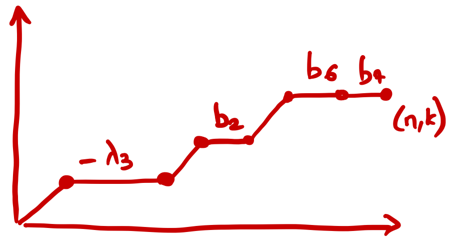




Viennot (80s)

$\mu_{n,k}$ $\qquad\qquad$ $\nu_{n,k}$





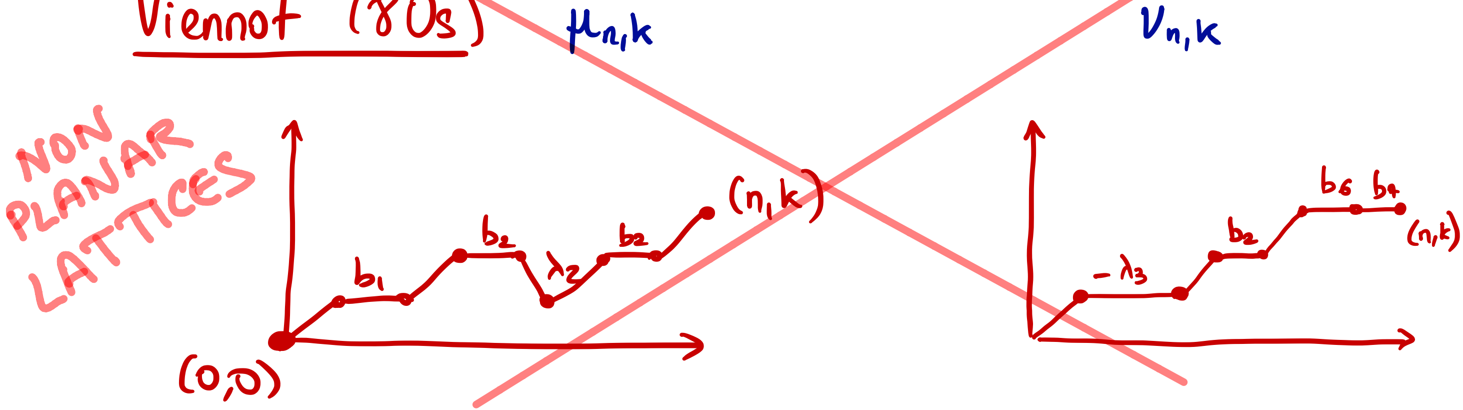
Theorem

① $$x^n = \sum_{k=0}^{n} \mu_{n,k}\, p_k(x)$$

② $$p_n(x) = \sum_{k=0}^{n} (-1)^{n-k}\, \nu_{n,k}\, x^k$$

$\mu_{n,k}$: moments , $\nu_{n,k}$: dual moments

Viennot (80s)



Theorem

① $$x^n = \sum_{k=0}^{n} \mu_{n,k} \, P_k(x)$$

② $$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \, \nu_{n,k} \, x^k$$

$\mu_{n,k}$: moments , $\nu_{n,k}$: dual moments

Multivariate polynomials at "$q=t$"

$\lambda = (\lambda_1, \ldots, \lambda_n)$ $\qquad$ $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$

Multivariate polynomials at "q=t"

$\lambda = (\lambda_1, \dots, \lambda_n)$ $\qquad$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$

$$P_\lambda(x_1, \dots, x_n) = \frac{\det\left(p_{n+\lambda_j - j}(x_i)\right)}{\det\left(x_j^{n-i}\right)}$$

Multivariate polynomials at "$q=t$"

$\lambda = (\lambda_1, \dots, \lambda_n)$ $\quad\quad$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$

$$P_\lambda(x_1, \dots, x_n) = \frac{\det\left(p_{n+\lambda_j - j}(x_i)\right)}{\det\left(x_i^{n-i}\right)}$$

Ex: $p_n(x) = x^n$ $\quad\quad$ $P_\lambda(x_1, \dots, x_n) = \underbrace{\text{Schur pols}}_{s_\lambda(x_1, \dots, x_n)}$

Multivariate polynomials at "$q=t$"

$\lambda = (\lambda_1, \dots, \lambda_n)$ $\qquad$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$

$$P_\lambda(x_1, \dots, x_n) = \frac{\det\left(p_{n+\lambda_j - j}(x_i)\right)}{\det\left(x_j^{n-i}\right)}$$

Ex: $p_n(x) = x^n$ $\qquad$ $P_\lambda(x_1, \dots, x_n) = \underbrace{\text{Schur pols}}_{s_\lambda(x_1, \dots, x_n)}$

$n=1$ $\qquad$ Back to univariate case

③ What are multivariate moments?

<u>Univariate</u> $\int p_n(x)\, p_m(x)\, w(x)\,dx = 0$ if $n \neq m$

moments $\mu_n = \int x^n\, w(x)\, dx$

③ What are multivariate moments?

Univariate

$$\int p_n(x)\, p_m(x)\, w(x)\, dx = 0 \quad \text{if } n \neq m$$

moments

$$\mu_n = \int x^n\, w(x)\, dx$$

Multivariate

$(\lambda_1, \dots, \lambda_n) \quad (\mu_1, \dots, \mu_n)$

$$\int p_\lambda(x_1, \dots, x_n)\, p_\mu(x_1, \dots, x_n)\, w(x_1, \dots, x_n)\, dx_1 \dots dx_n = 0 \quad \text{if } \lambda \neq \mu$$

③ What are multivariate moments?

<u>Univariate</u> $\int p_n(x)\, p_m(x)\, w(x)\, d_q x = 0$ if $n \neq m$

moments $\mu_n = \int x^n\, w(x)\, d_q x$

<u>Multivariate</u> $\int p_\lambda(x_1, \dots, x_n)\, p_\mu(x_1, \dots, x_n)\, w(x_1, \dots, x_n)\, d_q x_1 \dots d_q x_n = 0$ if $\lambda \neq \mu$

<u>C., Williams 2015</u> multivariate moments

$$M_\lambda = \int S_\lambda(x_1, \dots, x_n)\, w(x_1, \dots, x_n)\, d_q x_1 \dots d_q x_n$$

$$M_{\lambda/\alpha} = \int S_\lambda(x_1, \dots, x_n)\, p_\alpha(x_1, \dots, x_n)\, w(x_1, \dots, x_n)\, d_q x_1 \dots d_q x_n$$

In general

$$S_\lambda(x_1,\dots,x_n) = \sum_\alpha M_{\lambda/\alpha}\, P_\alpha(x_1,\dots,x_n)$$

$$P_\lambda(x_1,\dots,x_n) = \sum_\alpha (-1)^{|\lambda/\alpha|}\, N_{\lambda/\alpha}\, S_\alpha(x_1,\dots,x_n)$$

In general

$$S_\lambda(x_1,\dots,x_n) = \sum_\alpha M_{\lambda/\alpha}\, P_\alpha(x_1,\dots,x_n)$$

$$P_\lambda(x_1,\dots,x_n) = \sum_\alpha (-1)^{|\lambda/\alpha|} N_{\lambda/\alpha}\, S_\alpha(x_1,\dots,x_n)$$

Theorem (C., Williams 2015, C.-Kim 2018)

$$\begin{cases} M_{\lambda/\alpha} = \det\left(\mu_{\lambda_i+n-i,\,\alpha_j+n-j}\right)_{i,j} \\ N_{\lambda/\alpha} = \det\left(\nu_{\lambda_i+n-i,\,\alpha_j+n-j}\right)_{i,j} \end{cases}$$

In general

$$S_\lambda(x_1,\dots,x_n) = \sum_\alpha M_{\lambda/\alpha}\, P_\alpha(x_1,\dots,x_n)$$

$$P_\lambda(x_1,\dots,x_n) = \sum_\alpha (-1)^{|\lambda/\alpha|} N_{\lambda/\alpha}\, S_\alpha(x_1,\dots,x_n)$$

Theorem (C., Williams 2015, C.- Kim 2018)

$$\begin{cases} M_{\lambda/\alpha} = \det\left(\mu_{\lambda_i+n-i,\, \alpha_j+n-j}\right)_{i,j} \\ N_{\lambda/\alpha} = \det\left(\nu_{\lambda_i+n-i,\, \alpha_j+n-j}\right)_{i,j} \end{cases}$$

Cauchy Binet

④ Combinatorics of $N_{\lambda/\alpha}$ and $M_{\lambda/\alpha}$ for $\boxed{\text{Little } q\text{-Jacobi}}$ polynomials

$$p_n(x) = \sum_{k=0}^{n} (-1)^{n-k} x^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{(-aq^{k+1};q)_{n-k}}{(abq^{n+k+1};q)_{n-k}}$$

$$(a;q)_n = \prod_{i=0}^{n-1} (1-aq^i)$$

④ Combinatorics of $N_{\lambda/\alpha}$ and $M_{\lambda/\alpha}$ for Little q-Jacobi

$$p_n(x) = \sum_{k=0}^{n} (-1)^{n-k} x^k \underbrace{\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{(aq^{k+1};q)_{n-k}}{(abq^{n+k+1};q)_{n-k}}}_{\nu_{n,k}}$$

$$\mu_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1};q)_{n-k}}{(abq^{2k+2};q)_{n-k}}$$

Proposition (C., Kim 18)

① $\nu_{n,k} =$ generating function of

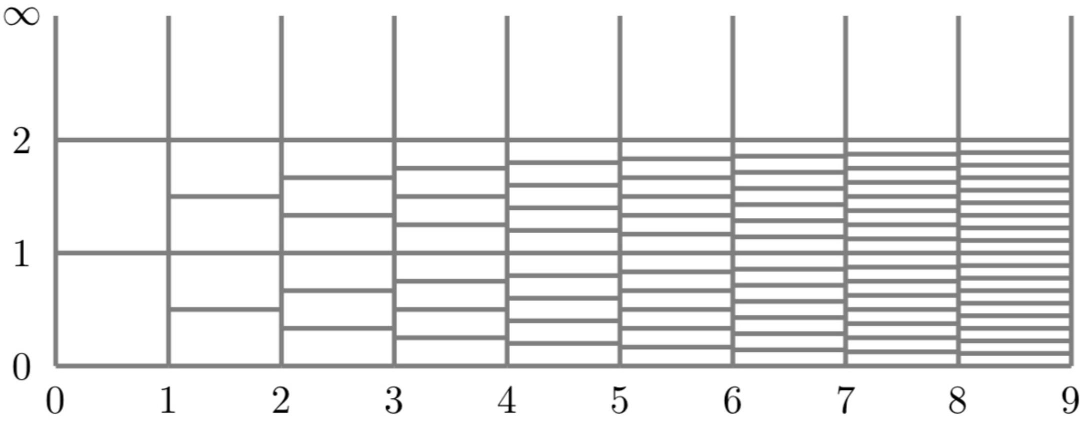
$$\frac{\lambda_1}{n} > \frac{\lambda_2}{n-1} > \cdots > \frac{\lambda_{n-k}}{k+1} \geq 0$$

② $\mu_{n,k} =$ generating function of

$$\frac{\lambda_1}{k+1} \geq \frac{\lambda_2}{k+2} \geq \cdots \geq \frac{\lambda_{n-k}}{n} \geq 0$$

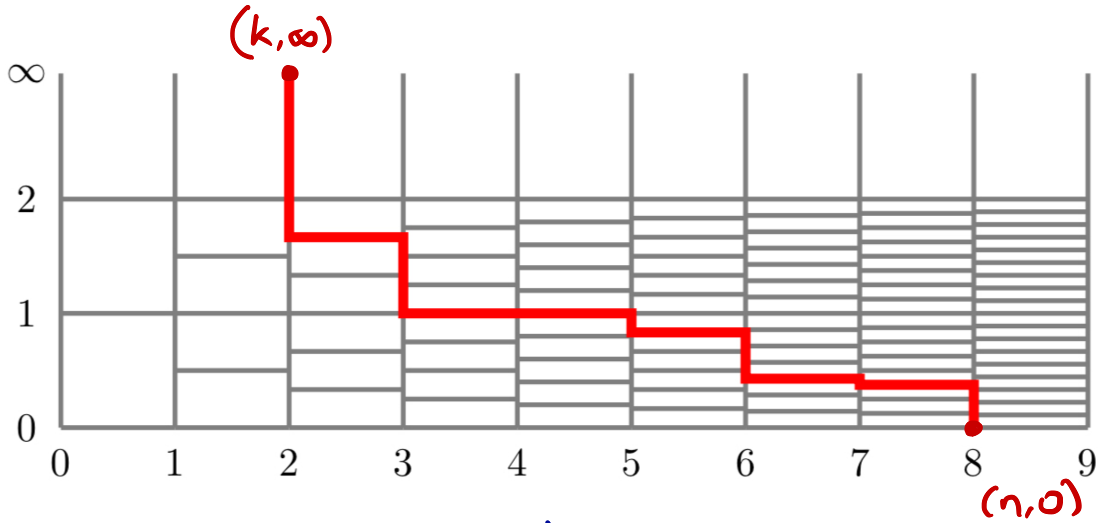
based on (C., Savage 04)

Combinatorics of $\mu_{n,k}$

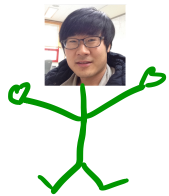


<u>Graph</u> • Vertices $\bigcup_{\substack{i \geq 1 \\ j \geq 0}} \left(i-1, \frac{j}{i}\right), \left(i, \frac{j}{i}\right)$

• Edges $(i,j) \to (i+1, j)$ $\qquad \left(i, \frac{j}{i}\right) \to \left(i, \frac{j+1}{i}\right)$



$\mu_{n,k}$

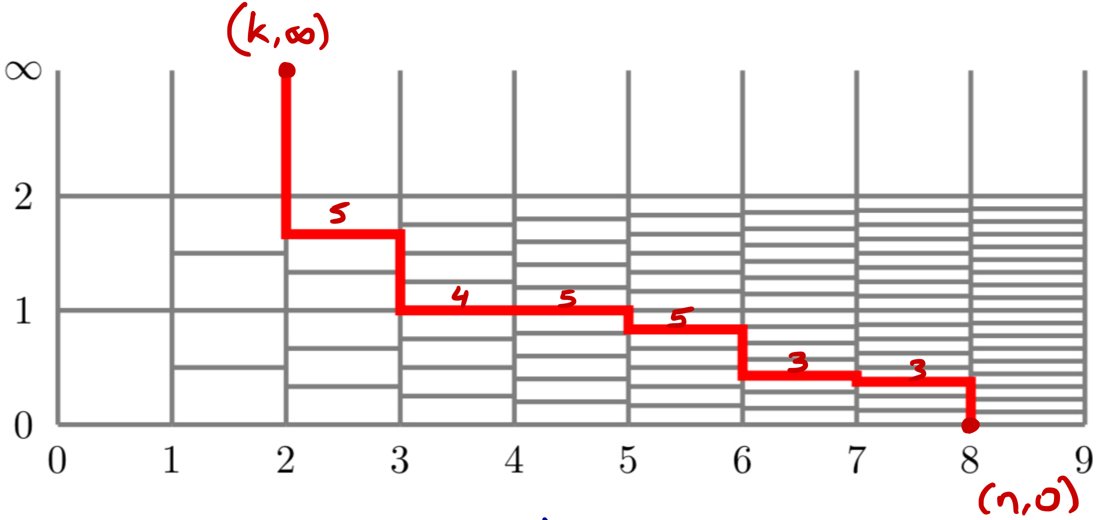


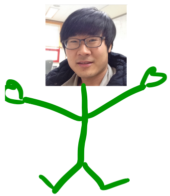
$$\frac{\lambda_1}{k+1} \geq \frac{\lambda_2}{k+2} \geq \cdots \geq \frac{\lambda_{n-k}}{n} \geq 0$$

$\Leftrightarrow$ Path from (k,∞) to $(n,0)$

$\lambda_i = \#$ rectangles under the i^{th} step

$\mu_{n,k}$





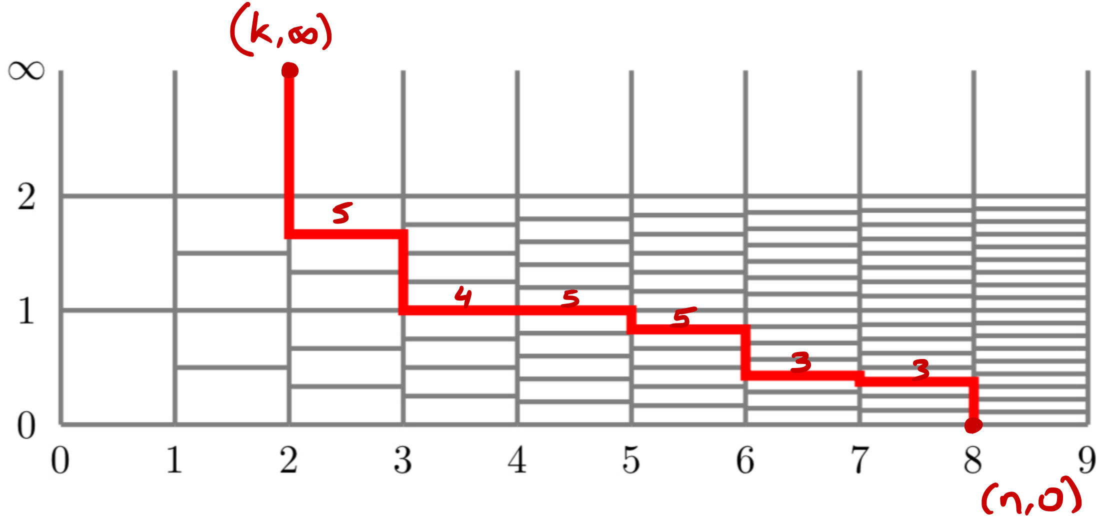
$$\frac{\lambda_1}{k+1} \geq \frac{\lambda_2}{k+2} \geq \cdots \geq \frac{\lambda_{n-k}}{n} \geq 0$$

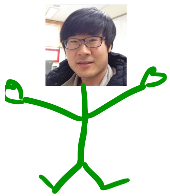
$\Leftrightarrow$ Path from (k, ∞) to $(n, 0)$

λ_i = # rectangles under the i^{th} step

$k=2, \; n=8$ $\qquad \lambda = (5, 4, 5, 5, 3, 3)$

$\mu_{n,k}$

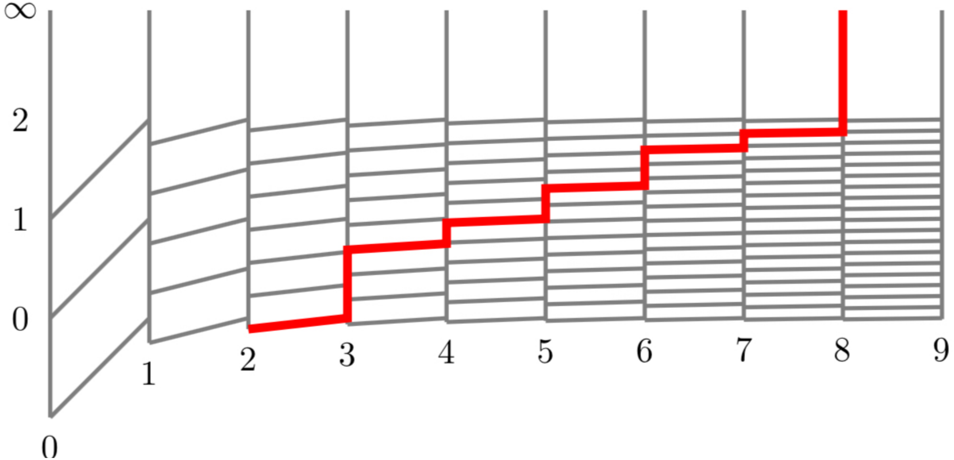




$$\mu_{n,k} = \sum_{\lambda} q^{|\lambda|} \, u^{|\lfloor \lambda \rfloor|} \, v^{o(\lfloor \lambda \rfloor)}$$

$$= \sum_{\text{paths}} q^{\#\text{cells}} \, u^{\#\text{unit cells}} \, v^{\#\text{odd}}$$

Combinatorics of $\mathcal{V}_{n,k}$

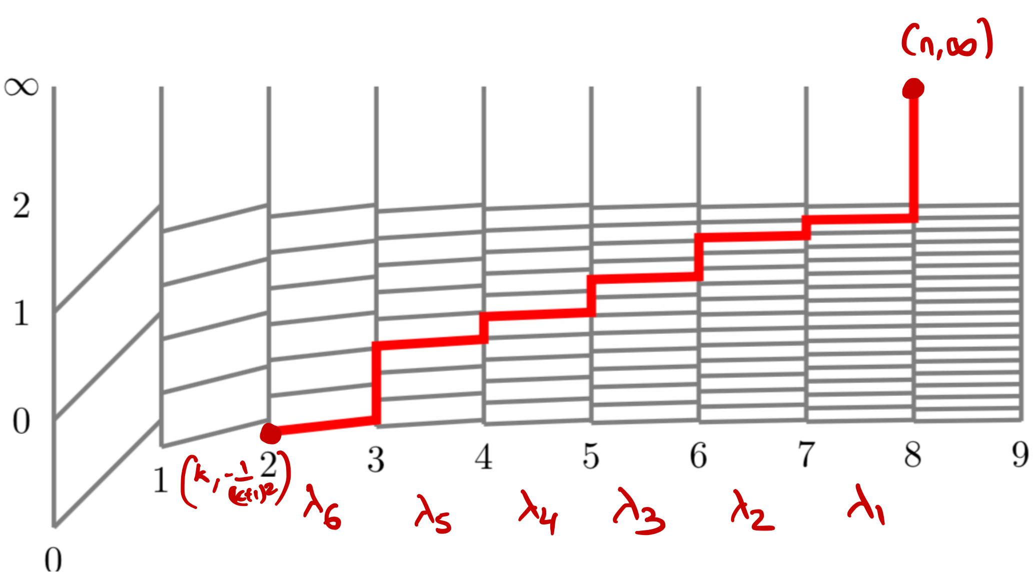


<u>Graph</u>

- Vertices $\displaystyle\bigcup_{\substack{i \geq 1 \\ j \geq 0}} \left(i-1, \frac{j}{i} - \frac{1}{i^2}\right), \left(i, \frac{j}{i}\right)$

- Edges $\left(i-1, \frac{j}{i} - \frac{1}{i^2}\right) \to \left(i, \frac{j}{i}\right) \quad \left(i, \frac{j}{i}\right) \to \left(i, \frac{j+1}{i}\right)$

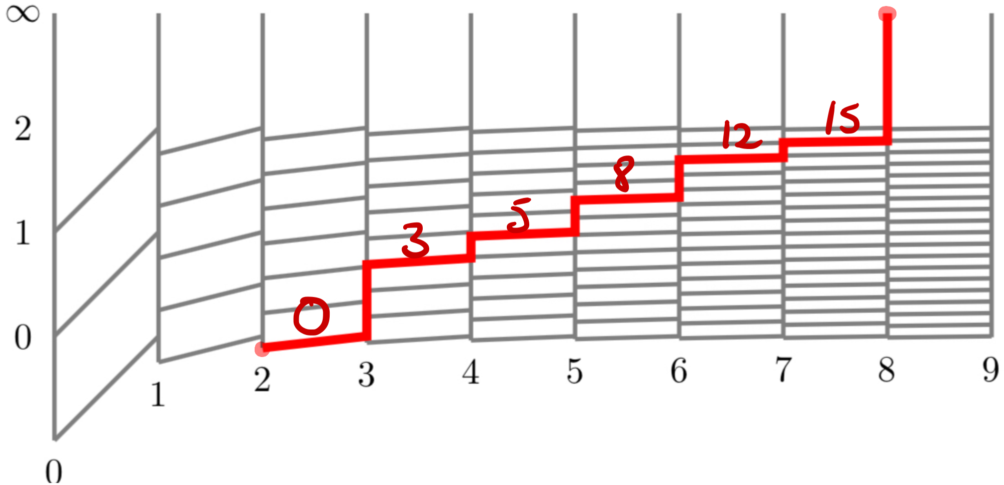
$V_{n,k}$



Paths $\left(k, -\frac{1}{(k+1)^2}\right) \longrightarrow (n, \infty)$

$$0 \leq \frac{\lambda_{n-k}}{k+1} < \frac{\lambda_{n-k-1}}{k+2} < \dots < \frac{\lambda_1}{n}$$

$V_{n,k}$



Paths $\left(k, -\frac{1}{(k+1)^2}\right) \longrightarrow (n, \infty)$

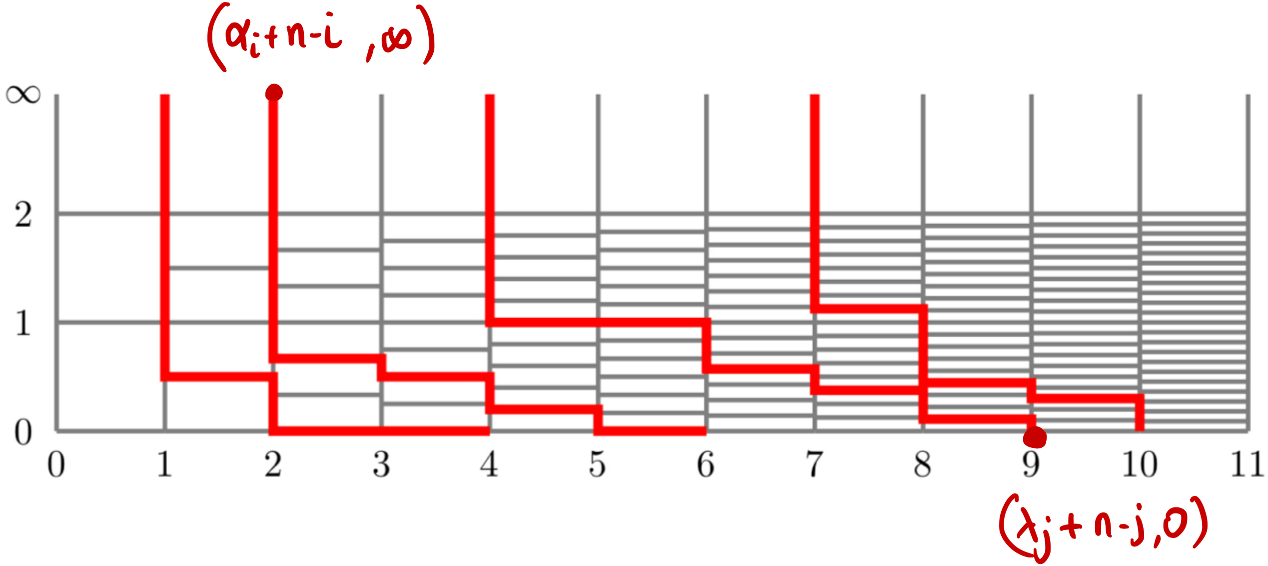
$$0 \le \frac{\lambda_{n-k}}{k+1} < \frac{\lambda_{n-k-1}}{k+2} < \dots < \frac{\lambda_1}{n}$$

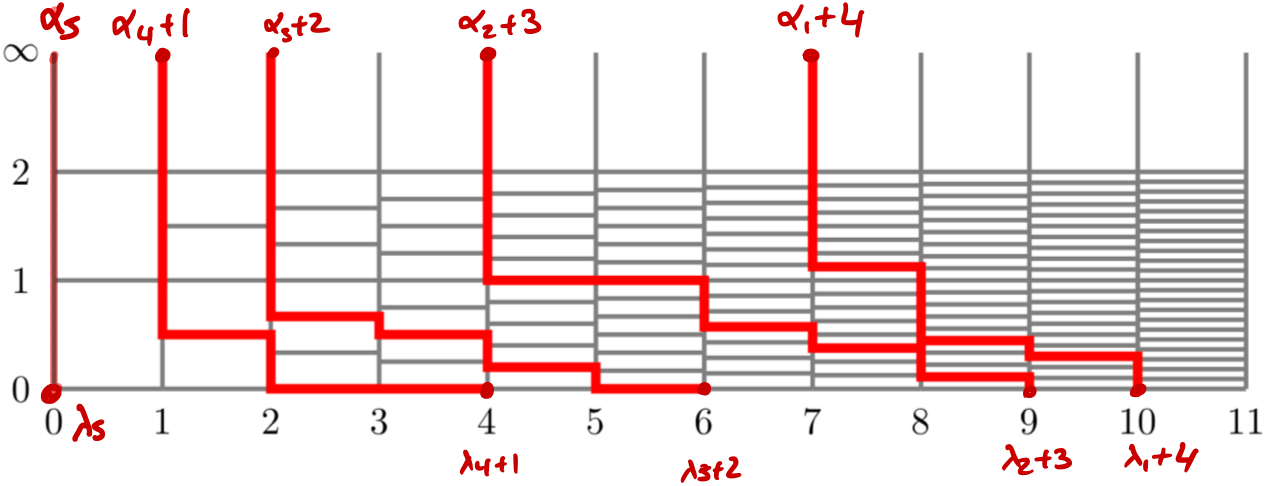
$k=2, \quad n=8 \qquad \lambda = (15, 12, 8, 5, 3, 0)$

Lecture Hall Tableaux

<u>Theorem</u> (C., Kim 18)

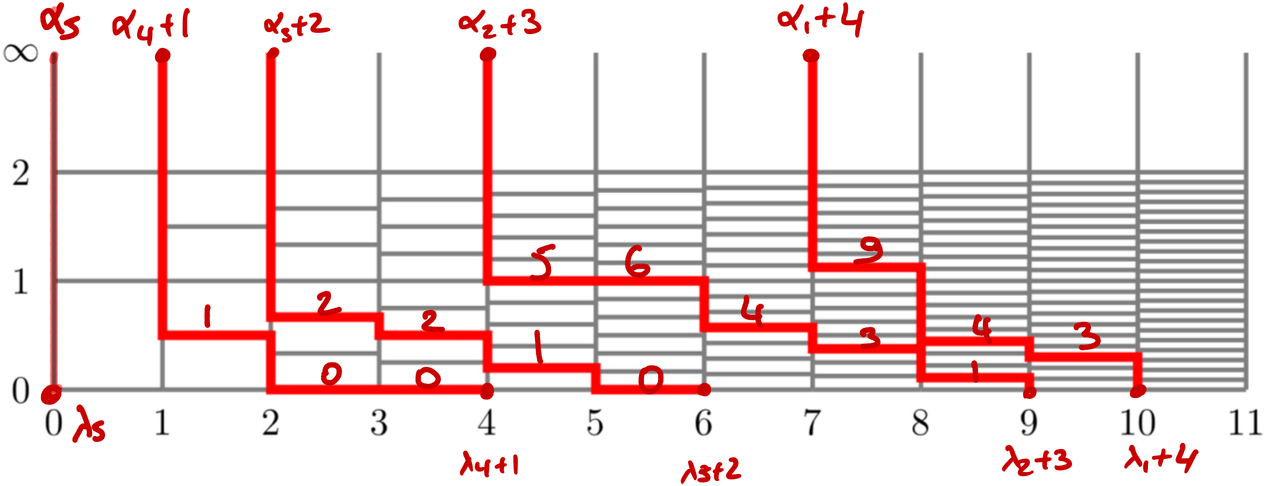
$$M_{\lambda/\alpha} = \det\left(\mu_{\lambda_i+n-i,\,\alpha_j+n-j}\right)_{i,j}$$





$\alpha = (3,1,0,0,0)$

$\lambda = (6,6,4,3,0)$



$$\alpha = (3,1,0,0,0)$$

$$\lambda = (6,6,4,3,0)$$

			9	4	3
	5	6	4	3	1
2	2	1	0		
1	0	0			

Theorem (C., Kim 18)

$$M_{\lambda/\emptyset} = s_\lambda(1,q,\dots,q^{n-1}) \prod_{i=1}^{n} \frac{(-aq^{n-i+1};q)_{\lambda_i}}{(abq^{2n-i+1};q)_{\lambda_i}}$$

Proof: q-Selberg integral

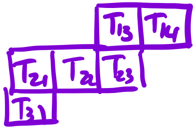
$$\frac{[n]_q!}{n!}\int_{[0,1]^n} s_\lambda(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^2 \prod_{i=1}^{n} x_i^{\alpha-1}\frac{(qx_i;q)_\infty}{(q^\beta x_i;q)_\infty} d_qx_1\dots d_qx_n$$

$$= q^{\alpha\binom{n}{2}+2\binom{n}{3}} \prod_{1\le i<j\le n} \frac{q^{\lambda_j+n-j}-q^{\lambda_i+n-i}}{q^{i-1}-q^{j-1}} \prod_{i=1}^{n} \frac{\Gamma_q(\alpha+n-i+\lambda_i)\Gamma_q(\beta+i-1)\Gamma_q(i+1)}{\Gamma_q(\alpha+\beta+2n-i-1+\lambda_i)}$$

or determinant (Krattenthaler)

"Reverse" Lecture Hall tableaux

$M_{\lambda/\alpha}$



$N_{\lambda/\alpha}$

$$\frac{T_{ij}}{n-i+j} \geq \frac{T_{i\,j+1}}{n-i+j+1} \qquad\qquad \frac{T_{ij}}{n-i+j} < \frac{T_{i\,j+1}}{n-i+j+1}$$

$$\frac{T_{ij}}{n-i+j} > \frac{T_{i+1,j}}{n-1-i+j} \qquad\qquad \frac{T_{ij}}{n-i+j} \leq \frac{T_{i+1,j}}{n-1-i+j}$$

Little q- Jacobi

$$P_\lambda = \sum_\alpha (-1)^{|\lambda/\alpha|} N_{\lambda/\alpha} S_\alpha$$

$$S_\lambda = \sum_\alpha M_{\lambda/\alpha} P_\alpha$$

$M_{\lambda/\emptyset}$, $N_{\lambda/\emptyset}$ NICE PRODUCTS

Bounded Lecture Hall Tableaux

(C., Keating, Nicoletti 2018)

Shape λ, α

Integers n, t

$$\boxed{T_{ij} < t\,(n-i+j)}$$

<u>Theorem</u> (C., Kim, Savage 18)

The number of BLHT of shape λ/α is

$$Z_{\lambda/\alpha}(t) = \prod_{(i,j)\in\lambda/\alpha} (n+j-i)\ \frac{f^{\lambda/\alpha}}{|\lambda/\alpha|!}\ t^{|\lambda/\alpha|}$$

q-analogue   Open question

Can we count those bounded tableaux T following $|T|$? $Z_{x/\alpha}(t,q) = \sum_T q^{|T|}$

Chen et al (2011) $t > \frac{T_1}{1} \geq \frac{T_2}{2} \geq \frac{T_3}{3} \geq \cdots$

$$Z_{\infty,\phi}(t) = \sum_T q^{|T|} = \frac{(-q;q)_\infty}{(q;q)_\infty} (q, q^{t+2}, q^{t+3}; q^{t+3})_\infty$$

(Infinity crystal $B(\infty)$ of $A_1^{(1)}$).

C., Lovejoy, Savage (2013)

$$Z_{n,\phi}(t) = \frac{(-q;q)_n}{(q^2;q)_n} \sum_{m=0}^{n} \frac{(1-q^{2m+1})}{(1-q)} \cdot \frac{(q^2;q)_m}{(q^{n+2};q)_m} \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^m q^{(t+1)\binom{m+1}{2}}$$

Open questions

1) LHT with other denominators

Bousquet-Melou & Eriksson (97)

$$\frac{\lambda_1}{a_n} \geqslant \frac{\lambda_2}{a_{n-1}} \geqslant \cdots \geqslant \frac{\lambda_n}{a_1} \geqslant 0$$

with $a_{i+1} = \ell a_i - a_{i-1}$ $\quad \ell \geqslant 2$

2) generalized LHT related to big q-Jacobi or Askey Wilson Polynomials.

3) Formulas for $M_{\lambda/\alpha}$ and $N_{\lambda/\alpha}$ for $\alpha \neq \emptyset$?

4) Asymptotics