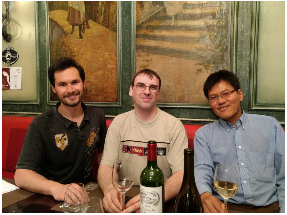
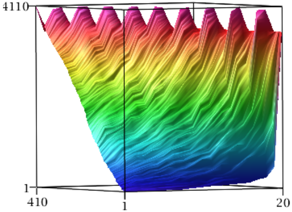
Periodic Pólya urns and asymptotics of triangular Young tableaux

Cyril Banderier, Philippe Marchal, Michael Wallner
(CNRS/Univ. Paris Nord/Univ. Bordeaux)





2019 March 12, BIRS Workshop on Asymptotic Algebraic Combinatorics

Introduction: what are Pólya urns?

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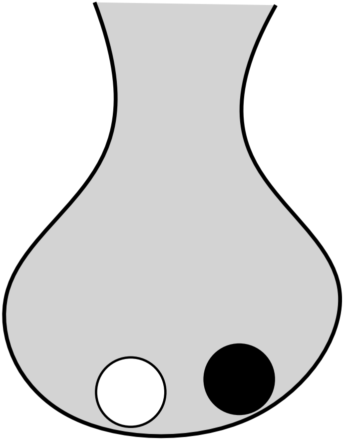
$=$ an urn $\qquad + \qquad$ a replacement rule



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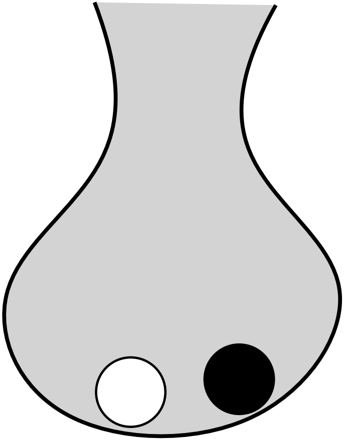
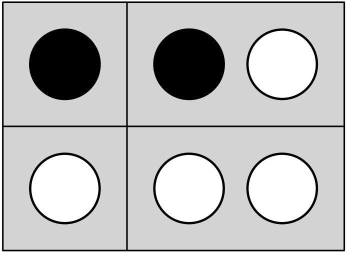


If ↑ drawn then ↓ added



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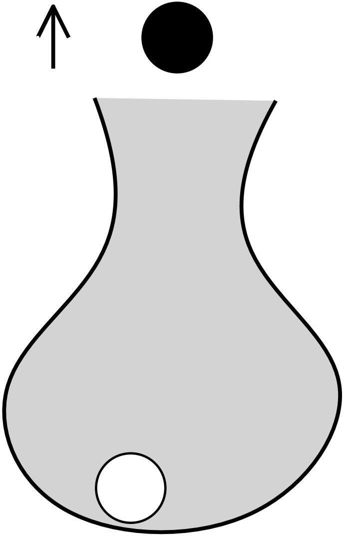

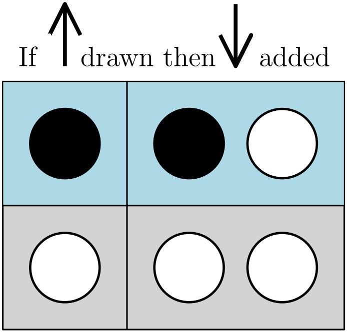

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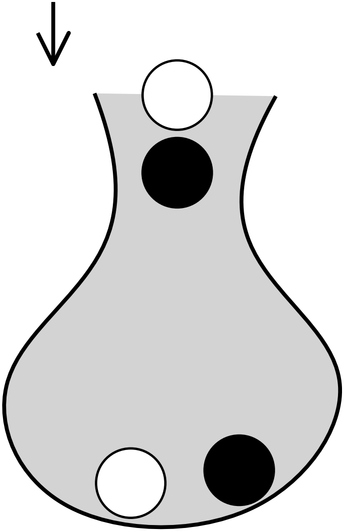

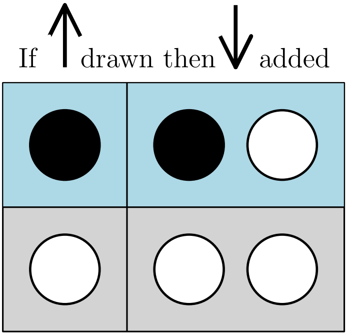

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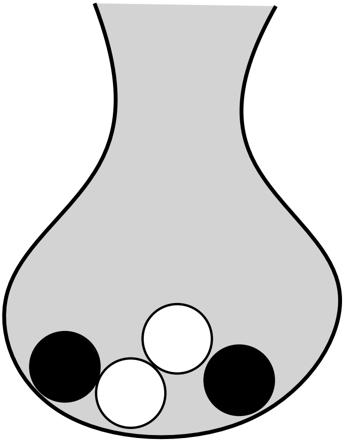

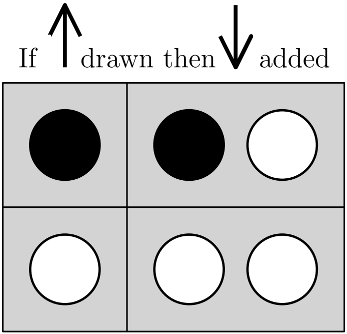

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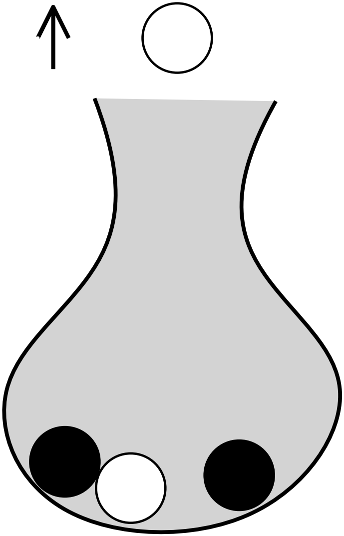

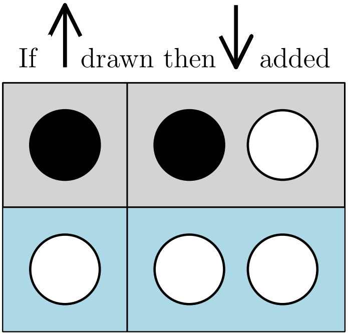

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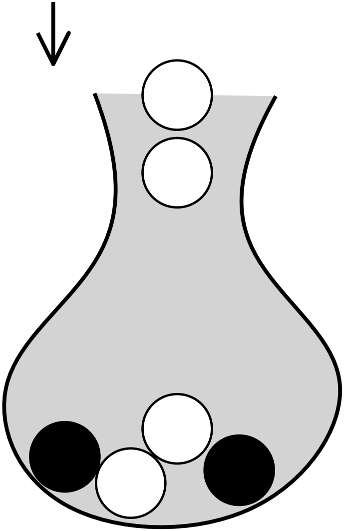

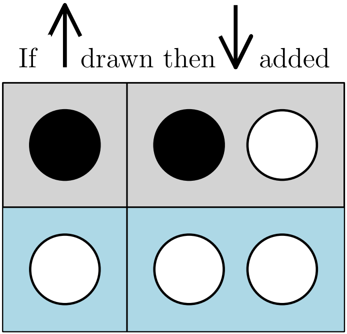

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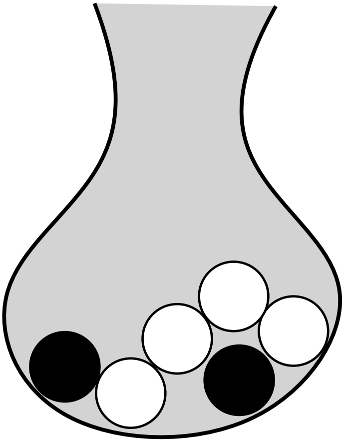

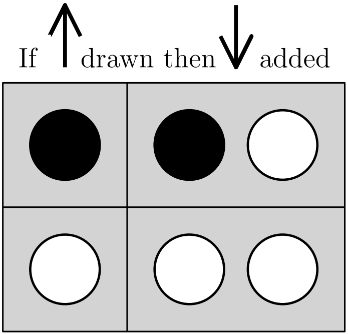

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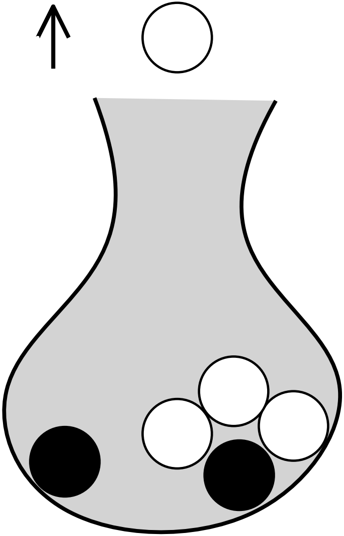

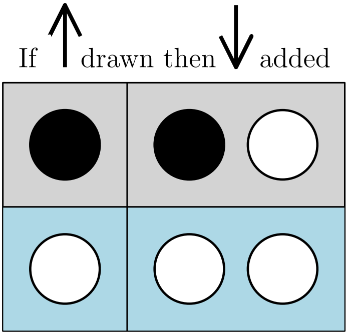

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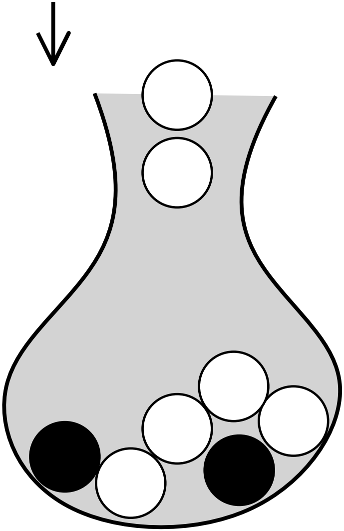

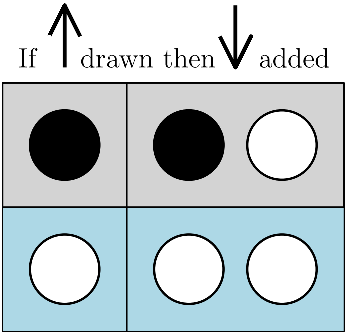

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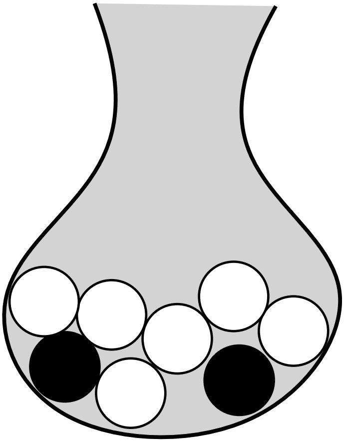

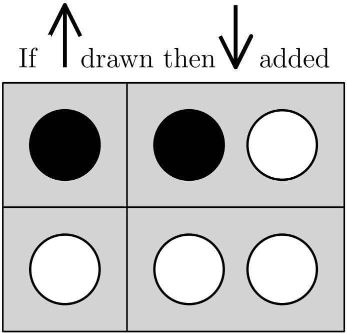

A formal definition

Replacement matrix

Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$.

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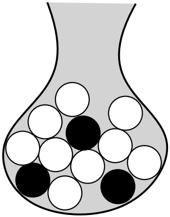

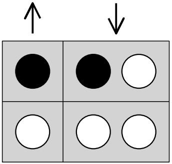

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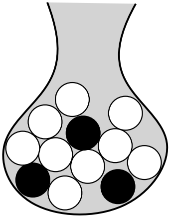

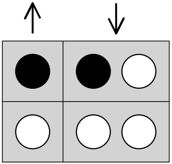

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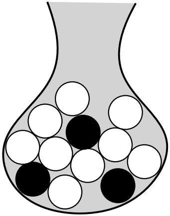

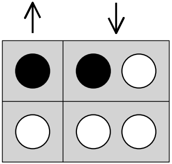

- Balanced urn: $K := a + b = c + d$ (above $K = 2$)
- Initial b_0 black ($\bullet$) and w_0 white ($\circ$)
- After n steps $b_0 + w_0 + Kn$ balls in the urn (deterministic!)

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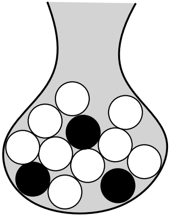

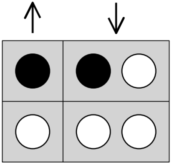

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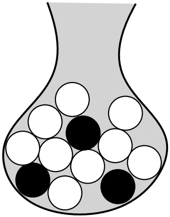

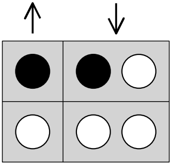

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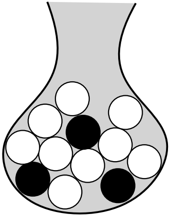

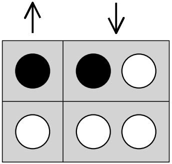

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Famous urn models [Flajolet, Dumas, Puyhaubert 2006]

Contagion urn (Pólya)	Adverse-campaign urn (Friedman)	Coupon collector's urn
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

Vast number of applications

- [Pólya, Eggenberger 1923-1930]: Disease infections $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$
- [Rivest 2012]: How to check election results? How to be sure your vote was counted?
- [Fanti, Viswanath 2017]: Deanonymization in Bitcoin's peer-to-peer network
- [Smythe, Mahmoud 1994, Holmgren, Janson 2016]: m-ary search trees
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Vivid field:

- many tools (martingales, analytic combinatorics, contraction, stochastic approximation, ...),
- many experts (see above, and more),
- many challenges: more colors, subset samplings, non tenable (non trivial positivity constraint), non time homogeneous, ...

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The *Young–Pólya urn* is a Pólya urn of period 2 with

replacement matrix $M_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for every odd step,

and replacement matrix $M_2 := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ for every even step.

Caveat: not modelizable as a classical Pólya urn (even with multidrawing)

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Histories of length n: A sequence of n drawings/evolutions.
$h_{n,k,\ell}$: Number of histories of length n, from (b_0, w_0) to (k, ℓ).

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$$H(x, y, z) = \sum_{n \geq 0} H_n(x, y) \frac{z^n}{n!} = \sum_{n,k,\ell \geq 0} h_{n,k,\ell} x^k y^\ell \frac{z^n}{n!}.$$

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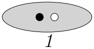

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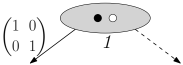

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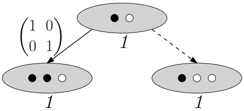

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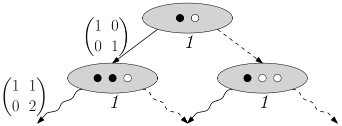

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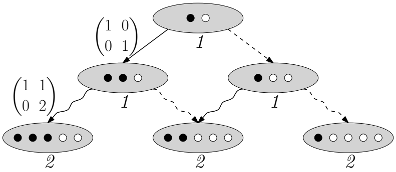

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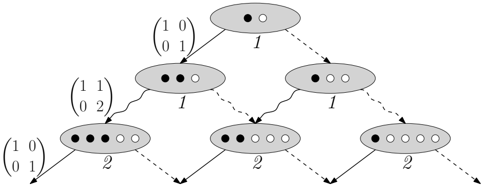

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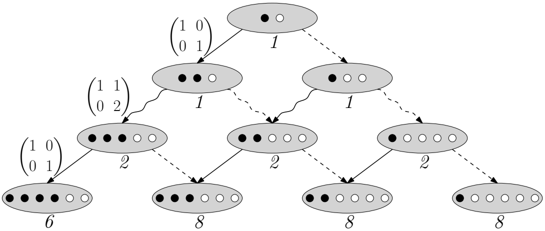

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$$H_2 = 2x^3y^2 + 2x^2y^3 + 2xy^4$$

$$H_3 = 6x^4y^2 + 8x^3y^3 + 8x^2y^4 + 8xy^5$$

Asymptotic distribution

Distribution of the urn

Main question:

How does the composition of the urn look like after n steps?

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- $\Rightarrow$ deterministic number of balls
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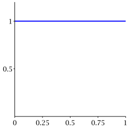
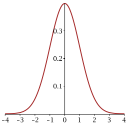
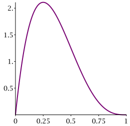
Many known distributions for (non-periodic) urn models:



Uniform distribution



Normal distribution



Beta distribution $\mathcal{B}(2,4)$

Periodic Pólya urns gives a new universal distribution

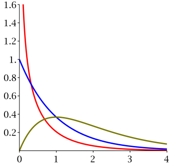
Generalized Gamma distribution GenGamma(α, β)

Let $\alpha, \beta > 0$ be real, then the density function with support $(0, +\infty)$ is

$$f(x;\alpha,\beta) := \frac{\beta\, x^{\alpha-1} \exp(-x^{\beta})}{\Gamma(\alpha/\beta)},$$

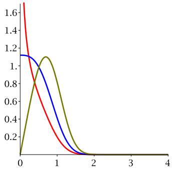
where Γ is the classical Gamma function $\Gamma(z) := \int_0^\infty t^{z-1} \exp(-t)\, dt$.

Some examples with red $\alpha = 0.5$, blue $\alpha = 1$, and green $\alpha = 2$:



$\beta = 1$

Gamma distributions



$\beta = 3$

This talk!

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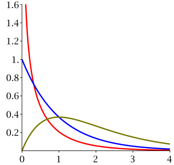

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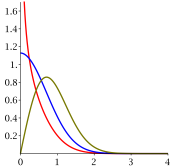
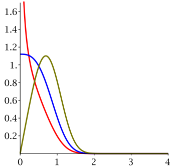


$\beta = 2$

Half-normal for $\alpha = 1$



$\beta = 3$

This talk!

- [Janson 2010]: area of the supremum process of the Brownian motion
- [Peköz, Röllin, Ross 2016]: preferential attachments in graphs
- [Khodabin, Ahmadabadi 2010]: generalization of special functions

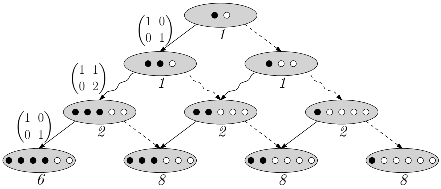
The number of black balls

Black balls in Young–Pólya urns

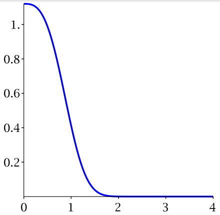
Theorem

The normalized random variable $\frac{2^{2/3}}{3}\frac{B_n}{n^{2/3}}$ *of the number of black balls in a Young–Pólya urn converges in law to a generalized Gamma distribution:*

$$\frac{2^{2/3}}{3}\frac{B_n}{n^{2/3}} \quad \xrightarrow{\mathcal{L}} \quad \mathrm{GenGamma}\,(1,3)\,.$$



The evolution of the Young–Pólya urn



Density: $f(x;1,3) = \frac{3\,e^{-x^3}}{\Gamma(1/3)}$

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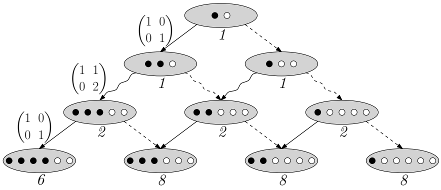

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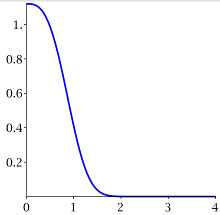

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1. Classical Pólya urn $\begin{pmatrix}1&0\\0&1\end{pmatrix}$: $c_1\frac{B_{1,n}}{n} \xrightarrow{\mathcal{L}}$ uniform distribution
2. Classical Pólya urn $\begin{pmatrix}1&1\\0&2\end{pmatrix}$: $c_2\frac{B_{2,n}}{\sqrt{n}} \xrightarrow{\mathcal{L}}$ half-normal distribution

Capture the evolution of the urn I

1. Split into even and odd steps

$$H_e(x,y,z) := \sum_{n\geq 0} H_{2n}(x,y)\frac{z^{2n}}{(2n)!}$$

$$H_o(x,y,z) := \sum_{n\geq 0} H_{2n+1}(x,y)\frac{z^{2n+1}}{(2n+1)!}$$

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2. Model the evolution using differential operators

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\mathcal{D}_1 := x^2\partial_x + y^2\partial_y \qquad \mathcal{D}_2 := x^2 y\partial_x + y^3\partial_y$$

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3. Link odd and even steps

$$\partial_z H_o(x,y,z) = \mathcal{D}_1 H_e(x,y,z) \qquad \partial_z H_e(x,y,z) = \mathcal{D}_2 H_o(x,y,z)$$

Capture the evolution of the urn II

4. As the urn is balanced, we have:

$$\text{number of balls after } n \text{ steps} = 2 + n + \left\lfloor \frac{n}{2} \right\rfloor.$$

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5. Algebra! Eliminate $\partial_y H_e$ and $\partial_y H_o$, then substitute $y = 1$
(We define $H(x,z) := H(x,1,z)$, $H_e(x,z) := H_e(x,1,z)$, and $H_o(x,z) := H_o(x,1,z)$)

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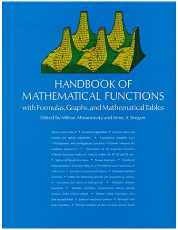
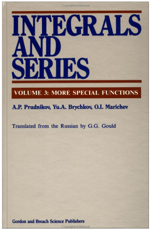
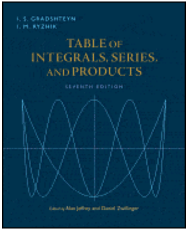
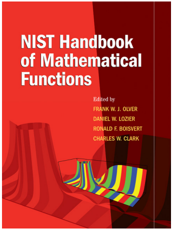
Moreover, they satisfy linear *differential equations, i.e., they are* D-finite.

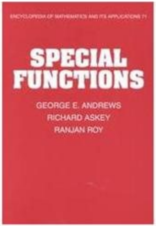
The nice world of D-finite functions

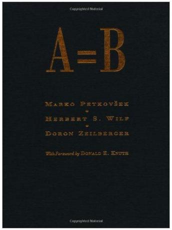


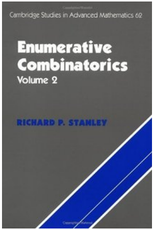


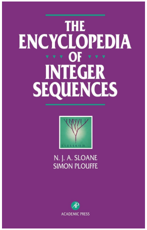


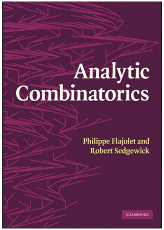












Holonomy is the key to handle sums, integrals, special functions, orthogonal polynomials, q-series. Allows proof of identities, asymptotic expansions, numerical values, structural properties. Applications: combinatorics, computer science, probability theory, engineering, physics...

D-finite world, computer algebra, and formulas

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i}=(2n+1)\binom{2n}{n}^2$$

$$\int_0^{+\infty} x\,J_1(ax)\,I_1(ax)\,Y_0(x)\,K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\int_{-1}^{+1}\frac{e^{-px}\,T_n(x)}{\sqrt{1-x^2}}\,dx=(-1)^n\pi\,I_n(p)$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}}$$

$$\zeta(3)=1.2020569031595942853997381615114499907649862923404 98\ldots$$

$$[z^n]\left\{y+6\,y'+(-54+54\,z)y''+(27-54\,z+27\,z^2)y'''=0,y(0)=\mathrm{e},y'(0)=\frac{-\mathrm{e}}{3},y''(0)=\frac{-\mathrm{e}}{9}\right\}$$

$$\sim -1/3\,\frac{n^{-4/3}}{\Gamma(2/3)}-1/6\,\frac{\sqrt{3}\Gamma(2/3)n^{-5/3}}{\pi}-1/18\,\frac{n^{-7/3}}{\Gamma(2/3)}-\frac{19}{216}\,\frac{\sqrt{3}\Gamma(2/3)n^{-8/3}}{\pi}+O(n^{-3})$$

$${}_2F_1\left([\frac{1}{12},\frac{5}{12}],[1];1728\,\frac{z}{(z+16)^3}\right)=\left(\frac{z+256}{16\,z+256}\right)^{-1/4}\cdot\ {}_2F_1\left([\frac{1}{12},\frac{5}{12}],[1];1728\,\frac{z^2}{(z+256)^3}\right)$$

Periodic Pólya urns are D-finite, with a rich structure, and offer a lot of identities with similar flavors.

Urns are D-finite!

Let $\widetilde{H}(x,z) := \sum_{n\geq 0} \frac{H_n(x)}{H_n(1)} z^n$ be the probability generating function. Then we have

$$L\widetilde{H}(x,z) = (\cdots \partial_z^3 + \cdots \partial_z^2 + \cdots \partial_z)\widetilde{H}(x,z) = 0.$$

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$$\begin{aligned}
L = {} & 9\,z(z-1)(z+1)\left(x^3z^2+2\,x^3z+3\,x^2-3\,x+1\right)\left(x^3z^2-2\,x^3z+3\,x^2-3\,x+1\right)\left(15\,x^7z^6+36\,x^7z^5+45\,x^7z^4-3\,x^6z^5+15\,x^6z^4+180\,x^6z^3-24\,x^5z^4-210\,x^5z^3+8\,x^4z^4\right. \\
& \left.-108\,x^6z+63\,x^5z^2+90\,x^4z^3+180\,x^5z-108\,x^4z^2-10\,x^3z^3-81\,x^5-135\,x^4z+81\,x^3z^2+189\,x^4+54\,x^3z-30\,x^2z^2-189\,x^3-9\,x^2z+5\,xz^2+99\,x^2-2\,zx-27\,x+z+3\right)\partial_z^3 \\
& +3\,\left(375\,x^{13}z^{12}+1188\,x^{13}z^{11}+912\,x^{13}z^{10}-84\,x^{12}z^{11}-2646\,x^{13}z^9+1419\,x^{12}z^{10}-4995\,x^{13}z^8+9822\,x^{12}z^9-1734\,x^{11}z^{10}-486\,x^{13}z^7+5853\,x^{12}z^8-10974\,x^{11}z^9\right. \\
& +578\,x^{10}z^{10}+1620\,x^{13}z^6-24138\,x^{12}z^7-1902\,x^{11}z^8+4506\,x^{10}z^9-7938\,x^{12}z^6+53940\,x^{11}z^7-5102\,x^{10}z^8-424\,x^9z^9+24138\,x^{12}z^5+5292\,x^{11}z^6-65778\,x^{10}z^7+5958\,x^9z^8 \\
& +3240\,x^{12}z^4-62748\,x^{11}z^5+11358\,x^{10}z^6+49444\,x^9z^7-2472\,x^8z^8-7776\,x^{12}z^3+13932\,x^{11}z^4+102168\,x^{10}z^5-29508\,x^9z^6-21948\,x^8z^7+412\,x^7z^8+27054\,x^{11}z^3 \\
& -46224\,x^{10}z^4-120504\,x^9z^5+33384\,x^8z^6+4918\,x^7z^7-8748\,x^{11}z^2-55242\,x^{10}z^3+69579\,x^9z^4+102294\,x^8z^5-22658\,x^7z^6-360\,x^6z^7+25434\,x^{10}z^2+85158\,x^9z^3 \\
& -77346\,x^8z^4-61030\,x^7z^5+9456\,x^6z^6+2430\,x^{10}z-46332\,x^9z^2-104976\,x^8z^3+71694\,x^7z^4+24626\,x^6z^5-2286\,x^5z^6-13770\,x^9z+76869\,x^8z^2+100998\,x^7z^3-52878\,x^6z^4 \\
& -6102\,x^5z^5+254\,x^4z^6+5103\,x^9+35964\,x^8z-109242\,x^7z^2-72612\,x^6z^3+29151\,x^5z^4+718\,x^4z^5-22113\,x^8-56646\,x^7z+116856\,x^6z^2+37458\,x^5z^3-11534\,x^4z^4-8\,x^3z^5 \\
& +44226\,x^7+59346\,x^6z-89910\,x^5z^2-13098\,x^4z^3+3144\,x^3z^4-53298\,x^6-43092\,x^5z+49137\,x^4z^2+2698\,x^3z^3-540\,x^2z^4+42525\,x^5+21906\,x^4z-18726\,x^3z^2-144\,x^2z^3 \\
& \left.+45\,xz^4-23247\,x^4-7674\,x^3z+4758\,x^2z^2-66\,xz^3+8694\,x^3+1764\,x^2z-728\,xz^2+12\,z^3-2142\,x^2-238\,zx+51\,z^2+315\,x+14\,z-21\right)\partial_z^2+2\,\left(1020\,x^{13}z^{11}+4032\,x^{13}z^{10}\right. \\
& +7461\,x^{13}z^9-276\,x^{12}z^{10}+972\,x^{13}z^8+1317\,x^{12}z^9-9315\,x^{13}z^7+29340\,x^{12}z^8-2559\,x^{11}z^9-5346\,x^{13}z^6+27990\,x^{12}z^7-34260\,x^{11}z^8+853\,x^{10}z^9-52974\,x^{12}z^6 \\
& -19935\,x^{11}z^7+14352\,x^{10}z^8-40743\,x^{12}z^5+113634\,x^{11}z^6-5916\,x^{10}z^7-1466\,x^9z^8+40338\,x^{12}z^4+25839\,x^{11}z^5-127818\,x^{10}z^6+13083\,x^9z^7+19440\,x^{12}z^3-150660\,x^{11}z^4 \\
& +40608\,x^{10}z^5+89886\,x^9z^6-5442\,x^8z^7-11664\,x^{12}z^2-4617\,x^{11}z^3+240894\,x^{10}z^4-80883\,x^9z^5-38142\,x^8z^6+907\,x^7z^7+58806\,x^{11}z^2-92340\,x^{10}z^3-206514\,x^9z^4 \\
& +69570\,x^8z^5+8198\,x^7z^6-17496\,x^{11}z-115182\,x^{10}z^2+227124\,x^9z^3+93150\,x^8z^4-38517\,x^7z^5-526\,x^6z^6+58563\,x^{10}z+127008\,x^9z^2-289710\,x^8z^3-12354\,x^7z^4 \\
& +14154\,x^6z^5-10206\,x^{10}-105462\,x^9z-93312\,x^8z^2+232929\,x^7z^3-9330\,x^6z^4-3231\,x^5z^5+27216\,x^9+139482\,x^8z+50922\,x^7z^2-122508\,x^6z^3+5958\,x^5z^4+359\,x^4z^5 \\
& -20412\,x^8-150903\,x^7z-20358\,x^6z^2+41499\,x^5z^3-1632\,x^4z^4-19278\,x^7+134298\,x^6z+4050\,x^5z^2-8526\,x^4z^3+194\,x^3z^4+55566\,x^6-94770\,x^5z+1350\,x^4z^2+948\,x^3z^3 \\
& \left.-57834\,x^5+50715\,x^4z-1416\,x^3z^2-60\,x^2z^3+35910\,x^4-19620\,x^3z+540\,x^2z^2+5\,xz^3-14490\,x^3+5148\,x^2z-110\,xz^2+3780\,x^2-819\,zx+10\,z^2-588\,x+60\,z+42\right)\partial_z \\
& +17010\,x^{10}-61236\,x^9+102060\,x^8-103194\,x^7+70308\,x^6-34020\,x^5+11970\,x^4-3024\,x^3 \\
& +504\,x^2-42\,x+12\,x\left(270\,x^6-261\,x^5+126\,x^4+21\,x^3-69\,x^2+30\,x-5\right)\left(3\,x^2-3\,x+1\right)^2z \\
& +2\,x\left(3\,x^2-3\,x+1\right)\left(3240\,x^9-2673\,x^8-6129\,x^7+16254\,x^6-16101\,x^5+8010\,x^4-1923\,x^3+78\,x^2+45\,x-5\right)z^2 \\
& +4\,x^3\left(3\,x^2-3\,x+1\right)\left(1134\,x^7-4995\,x^6+5886\,x^5-2841\,x^4+129\,x^3+366\,x^2-159\,x+28\right)z^3 \\
& -6\,x^4\left(3\,x^2-3\,x+1\right)\left(1485\,x^6-468\,x^5-927\,x^4+1150\,x^3-623\,x^2+162\,x-27\right)z^4 \\
& -4\,x^6\left(3\,x^2-3\,x+1\right)\left(1107\,x^4-3093\,x^3+1863\,x^2-565\,x-28\right)z^5+6\,x^7\left(405\,x^6+5178\,x^5-4335\,x^4-128\,x^3+1619\,x^2-666\,x+111\right)z^6 \\
& +20\,x^9\left(405\,x^4+1053\,x^3-1335\,x^2+597\,x-76\right)z^7+10\,x^{10}\left(783\,x^3-57\,x^2-69\,x+23\right)z^8+240\,x^{12}\left(12\,x-1\right)z^9+600\,x^{13}z^{10}
\end{aligned}$$

Urns are D-finite!

Let $\widetilde{H}(x,z) := \sum_{n\geq 0} \frac{H_n(x)}{H_n(1)} z^n$ be the probability generating function. Then we have

$$L\widetilde{H}(x,z) = (\cdots \partial_z^3 + \cdots \partial_z^2 + \cdots \partial_z)\widetilde{H}(x,z) = 0.$$

$$
\begin{aligned}
L = {} & 9\,z(z-1)(z+1)\left(x^3z^2+2\,x^3z+3\,x^2-3\,x+1\right)\left(x^3z^2-2\,x^3z+3\,x^2-3\,x+1\right)\left(15\,x^7z^6+36\,x^7z^5+45\,x^7z^4-3\,x^6z^5+15\,x^6z^4+180\,x^6z^3-24\,x^5z^4-210\,x^5z^3+8\,x^4z^4\right.\\
& \left.-108\,x^6z+63\,x^5z^2+90\,x^4z^3+180\,x^5z-108\,x^4z^2-10\,x^3z^3-81\,x^5-135\,x^4z+81\,x^3z^2+189\,x^4+54\,x^3z-30\,x^2z^2-189\,x^3-9\,x^2z+5\,xz^2+99\,x^2-2\,zx-27\,x+z+3\right)\partial_z^3\\
& +3\,\left(375\,x^{13}z^{12}+1188\,x^{13}z^{11}+912\,x^{13}z^{10}-84\,x^{12}z^{11}-2646\,x^{13}z^9+1419\,x^{12}z^{10}-4995\,x^{13}z^8+9822\,x^{12}z^9-1734\,x^{11}z^{10}-486\,x^{13}z^7+5853\,x^{12}z^8-10974\,x^{11}z^9\right.\\
& +578\,x^{10}z^{10}+1620\,x^{13}z^6-24138\,x^{12}z^7-1902\,x^{11}z^8+4506\,x^{10}z^9-7938\,x^{12}z^6+53940\,x^{11}z^7-5102\,x^{10}z^8-424\,x^9z^9+24138\,x^{12}z^5+5292\,x^{11}z^6-65778\,x^{10}z^7+5958\,x^9z^8\\
& +3240\,x^{12}z^4-62748\,x^{11}z^5+11358\,x^{10}z^6+49444\,x^9z^7-2472\,x^8z^8-7776\,x^{12}z^3+13932\,x^{11}z^4+102168\,x^{10}z^5-29508\,x^9z^6-21948\,x^8z^7+412\,x^7z^8+27054\,x^{11}z^3\\
& -46224\,x^{10}z^4-120504\,x^9z^5+33384\,x^8z^6+4918\,x^7z^7-8748\,x^{11}z^2-55242\,x^{10}z^3+69579\,x^9z^4+102294\,x^8z^5-22658\,x^7z^6-360\,x^6z^7+25434\,x^{10}z^2+85158\,x^9z^3\\
& -77346\,x^8z^4-61030\,x^7z^5+9456\,x^6z^6+2430\,x^{10}z-46332\,x^9z^2-104976\,x^8z^3+71694\,x^7z^4+24626\,x^6z^5-2286\,x^5z^6-13770\,x^9z+76869\,x^8z^2+100998\,x^7z^3-52878\,x^6z^4\\
& -6102\,x^5z^5+254\,x^4z^6+5103\,x^9+35964\,x^8z-109242\,x^7z^2-72612\,x^6z^3+29151\,x^5z^4+718\,x^4z^5-22113\,x^8-56646\,x^7z+116856\,x^6z^2+37458\,x^5z^3-11534\,x^4z^4-8\,x^3z^5\\
& +44226\,x^7+59346\,x^6z-89910\,x^5z^2-13098\,x^4z^3+3144\,x^3z^4-53298\,x^6-43092\,x^5z+49137\,x^4z^2+2698\,x^3z^3-540\,x^2z^4+42525\,x^5+21906\,x^4z-18726\,x^3z^2-144\,x^2z^3\\
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& +40608\,x^{10}z^5+89886\,x^9z^6-5442\,x^8z^7-11664\,x^{12}z^2-4617\,x^{11}z^3+240894\,x^{10}z^4-80883\,x^9z^5-38142\,x^8z^6+907\,x^7z^7+58806\,x^{11}z^2-92340\,x^{10}z^3-206514\,x^9z^4\\
& +69570\,x^8z^5+8198\,x^7z^6-17496\,x^{11}z-115182\,x^{10}z^2+227124\,x^9z^3+93150\,x^8z^4-38517\,x^7z^5-526\,x^6z^6+58563\,x^{10}z+127008\,x^9z^2-289710\,x^8z^3-12354\,x^7z^4\\
& +14154\,x^6z^5-10206\,x^{10}-105462\,x^9z-93312\,x^8z^2+232929\,x^7z^3-9330\,x^6z^4-3231\,x^5z^5+27216\,x^9+139482\,x^8z+50922\,x^7z^2-122508\,x^6z^3+5958\,x^5z^4+359\,x^4z^5\\
& -20412\,x^8-150903\,x^7z-20358\,x^6z^2+41499\,x^5z^3-1632\,x^4z^4-19278\,x^7+134298\,x^6z+4050\,x^5z^2-8526\,x^4z^3+194\,x^3z^4+55566\,x^6-94770\,x^5z+1350\,x^4z^2+948\,x^3z^3\\
& \left.-57834\,x^5+50715\,x^4z-1416\,x^3z^2-60\,x^2z^3+35910\,x^4-19620\,x^3z+540\,x^2z^2+5\,xz^3-14490\,x^3+5148\,x^2z-110\,xz^2+3780\,x^2-819\,zx+10\,z^2-588\,x+60\,z+42\right)\partial_z\\
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& +20\,x^9\left(405\,x^4+1053\,x^3-1335\,x^2+597\,x-76\right)z^7+10\,x^{10}\left(783\,x^3-57\,x^2-69\,x+23\right)z^8+240\,x^{12}(12\,x-1)\,z^9+600\,x^{13}z^{10}
\end{aligned}
$$

NB: D-finiteness allows "time travelling": composition at time n in time $O(\sqrt{n})$!

An ugly D-finite equation, but nice moments

Let $m_r(n)$ be the r-th factorial moment of the distribution of black balls after n steps, i.e.

$$\begin{aligned} m_r(n) &:= \mathbb{E}\left(B_n(B_n-1)\cdots(B_n-r+1)\right) \\ &= [z^n]\partial_x^r \widetilde{H}(x,z)_{|x=1} \\ &= \frac{[z^n]\ \frac{\partial^r}{\partial x^r} H(x,z)\Big|_{x=1}}{[z^n]H(1,z)} \end{aligned}$$

Theorem

The r-th factorial moment satisfies

$$m_r(n) = \frac{3^r}{2^{2r/3}} \frac{\Gamma\left(\frac{r}{3}+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} n^{2r/3} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

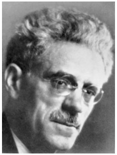
For n large, this gives:

$$\lim_{n\to\infty} \frac{2^{2r/3}}{3^r} \frac{m_r(n)}{n^{2r/3}} = \frac{\Gamma\left(\frac{r}{3}+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} =: m_r.$$

Is this the moment of some known law? Is there only one such possible law?

Moments and generalized gamma distributions

Torsten Carleman
(1892-1949)



Maurice Fréchet
(1878-1973)

Theorem

The distribution of Young–Pólya urn is characterized by its moments.

Proof.

[Carleman 1923] & [Fréchet, Shohat 1930] $\Rightarrow$ the moments determine the distribution uniquely.
The support of the distribution plays a role. There is a unique distribution with such moments if the Carleman's condition holds:
– for support $[0, \infty)$ (Stieljes moment problem): if $\sum 1/m_r^{1/2r} = \infty$
– for support $(-\infty, \infty)$ (Hamburger moment problem): if $\sum 1/m_{2r}^{1/2r} = \infty$ $\square$

Young–Pólya urn of period p and parameter ℓ

The same method of differential operators allows us to obtain the distribution of black balls for the urn with replacement matrices

$$M_1 = M_2 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}.$$

We call this model the *Young–Pólya urn of period p and parameter ℓ*.

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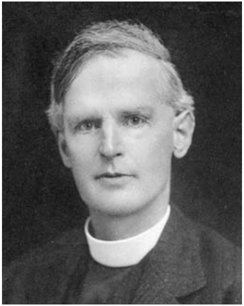
Theorem (Product of GenGamma)

Let b_0 be the initial black and w_0 the initial white balls. Then,

$$\frac{p^\delta}{p+\ell}\frac{B_n}{n^\delta} \quad \xrightarrow{\mathcal{L}} \quad \mathrm{Beta}(b_0, w_0) \prod_{i=0}^{\ell-1} \mathrm{GenGamma}(b_0 + w_0 + p + i, p + \ell),$$

with $\delta = p/(p+\ell)$, and where $\mathrm{Beta}(b_0, w_0)$ *is the law with support* $[0,1]$ *and density* $\frac{\Gamma(b_0+w_0)}{\Gamma(b_0)\Gamma(w_0)} x^{b_0-1}(1-x)^{w_0-1}$.

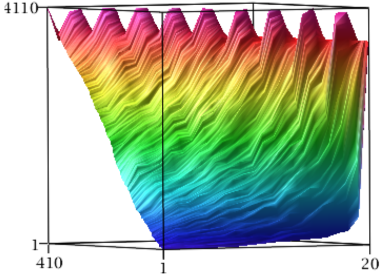
Quotes



"After this the reader who wishes to do so will have no difficulty in developing the theory of urns when they are regarded as differential operators."
[From the wording of Alfred Young (1873–1940) in Grace & Young, *The algebra of invariants*, 1903, p. 366.]

"A method is a trick used twice."
[From the wording of George Pólya (1887–1985) in *How to solve it.*, 1957, p. 208.]

What is the density method?



[ancestors (playing with posets, volume of polytopes, and probability or enumeration): Pak 2001, Elkies 2003, Baryshnikov & Romik 2010]

Young tableaux with local decreases

7	18	19	12	21	20	17
2	6	8	9	10	14	16
1	3	4	5	11	13	15

We consider Young tableaux in which some pairs of (horizontally or vertically) consecutive cells are allowed to have decreasing labels.
Places where a decrease is allowed (but not compulsory) are drawn by a bold red edge, which we call a "wall".

14	13
10	12
9	11
8	7
4	6
3	5
2	1

Nice formulae for some specific tableaux of shape $n \times 2$:

- Walls everywhere: $(2n)!$
- Horizontal walls everywhere:

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- Horizontal walls everywhere: $\frac{(2n)!}{2^n}$
- Horizontal walls everywhere in left column:

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- Vertical walls everywhere:

Young tableaux with local decreases

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- Horizontal walls everywhere: $\frac{(2n)!}{2^n}$
- Horizontal walls everywhere in left column: $\frac{(2n)!}{2^n n!}$
- Vertical walls everywhere: $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
- k vertical walls:

Young tableaux with local decreases

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- Horizontal walls everywhere in left column: $\frac{(2n)!}{2^n n!}$
- Vertical walls everywhere: $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
- k vertical walls: $\frac{1}{n+1-k}\binom{n}{k}\binom{2n}{n}$.

Young tableaux with local decreases

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1	3	4	5	11	13	15

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- Horizontal walls everywhere: $\frac{(2n)!}{2^n}$
- Horizontal walls everywhere in left column: $\frac{(2n)!}{2^n n!}$
- Vertical walls everywhere: $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
- k vertical walls: $\frac{1}{n+1-k}\binom{n}{k}\binom{2n}{n}$.

For shape $n \times m$ with k long walls at distance λ_i:

$$\frac{(m-1)!}{(mn+m-1)_{m-1}} \left(\prod_{i=1}^{k+1}\prod_{j=1}^{m-2}\binom{\lambda_i+j}{j}^{-1}\right)\left(\prod_{i=1}^{k+1}\binom{m(\lambda_1+\dots\lambda_i)+m-1}{\lambda_i,\dots,\lambda_i}\right).$$

Uniform random generation and enumeration

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

We now show how to generate the above tableaux via the density method. This example is "without loss of generality" (i.e., our method works also for non-periodic shapes). 🙂

The density method will give thousands of coefficients in a few seconds.

$f_n = (6n+1)! \int_0^1 p_n(z)$, with

$$p_{n+1}(z) = \int_0^z \frac{1}{24}(z-1)(x-z)(3x^3 - 7x^2z - xz^2 - z^3 - 2x^2 + 4xz + 4z^2)p_n(x)\,dx.$$

$\{f_n\}_{n\geq 0}=\{$1, 12, 8550, 39235950, 629738299350, 26095645151941500, 23234979501013722232250, 392833430654718548673344250, 115375222087417545717234273063750, 55038140590519890608190921051205837500, $\ldots\}$.

From tableaux to tuples of real numbers, and polytopes

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

7	16	17
2	14	15
9	11	19
4	10	13
5	8	18
3	6	12
	1	

.74	.96	.97
.25	.94	.95
.85	.91	.99
.42	.90	.93
.54	.82	.98
.35	.57	.92
	.06	

S	Z	W
R	Y	V
	X	

S < Z < W
R < Y < V
X

Left: 2$n \times 3$ Young tableau with walls.
Centre: A related tableau Polyo$_n$ with one more cell (removing this cell + relabel: bijection with left tableau).
Our algorithm generates real numbers between 0 and 1, with same relative order.
All possible values = a polytope $\mathcal{P} \in [0,1]^{6n+1}$.
Right: The "building block" of 7 cells. Each polyomino Polyo$_n$ is made of the overlapping of n such building blocks.

Density method: key ideas

• geometric point of view:
Associate with a poset of size N its "order polytope" $\mathcal{P}$ which is a subset of $[0,1]^N$. Generate a random element of the polytope slice by slice using conditional densities.
In the present example, $N = 6n+1$ and the slices are the building blocks of size 6 (except for the first one).

• sequence of densities: sequence of polynomials $p_n(x)$, defined by the following recurrence (which in fact encodes the full structure of the problem, building block after building block): $p_0 = 1$ and by induction,

$$p_{n+1}(z) = \int_0^z \int_x^z \int_0^y \int_r^z \int_z^1 \int_y^w p_n(v)\, dv\, dw\, ds\, dr\, dy\, dx$$

$$p_{n+1}(z) = \int_{0<x<z} \int_{x<y<z} \int_{0<r<y} \int_{r<s<z} \int_{z<w<1} \int_{y<v<w} p_n(v)\, dv\, dw\, ds\, dr\, dy\, dx.$$

The density method algorithm

1 Initialization: Order the building blocks from $k = n - 1$ to $k = 0$ (top to bottom). Start at the top, i.e. $k := n - 1$. Put into the top cell Z a random number z with density $p_n(z) / \int_0^1 p_n(t)\,dt$.

2 Filling: Now that Z is known, put into the cells X, Y, R, S, V, W random numbers x, y, r, s, v, w with conditional density

$$g_{k,z}(x, y, r, s, v, w) := \frac{1}{p_{k+1}(z)} p_k(x) \mathbf{1}_{\mathcal{P}z},$$

where $\mathbf{1}_{\mathcal{P}z}$ is the indicator function of the k-th building block (with value z in cell Z):

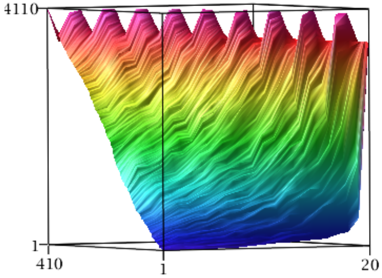
$$\mathbf{1}_{\mathcal{P}z} := \mathbf{1}_{\{0 \le x \le y \le z, 0 \le r \le y, r \le s \le z, z \le w \le 1, y \le v \le w\}}.$$

3 Iteration: Consider X as a the Z of the next building block. Set $k := k - 1$ and go to step 2 (until $k = 0$).

Theorem

The density method algorithm is a uniform random generation algorithm with quadratic time complexity and linear space complexity.

Limit surface of Young Tableaux

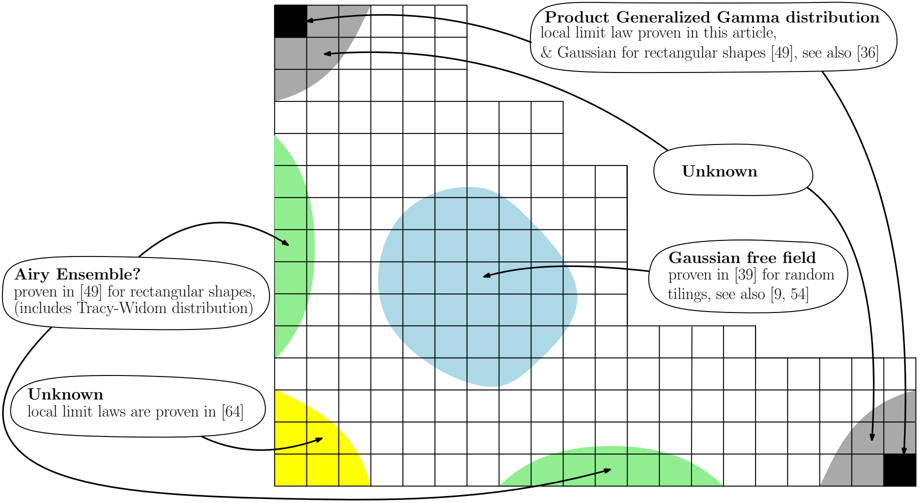


Limiting objects

Classical dream: universality via limiting objects for combinatorial stuctures:

- Dyck paths $\rightsquigarrow$ Brownian motion (Bachelier, Einstein)
- Trees $\rightsquigarrow$ Continuous random trees (Aldous)
- Domino tilings $\rightsquigarrow$ Gaussian Free Field (Kenyon)
- Planar maps $\rightsquigarrow$ Brownian map (Marckert, Mokkadem: existence,
  Le Gall: triangulations, Miermont: quadrangulations)
- Self-avoiding processes $\rightsquigarrow$ SLE (Schramm, Lawler, Werner, Smirnov...)
- Young tableaux $\rightsquigarrow$?
  - Fluctuations in the corners of rectangular shapes: Gaussian
  - Fluctuations along the edge of square shapes: Tracy–Widom limit law

Fluctuation on the limiting surface



Triangular Young tableaux

Triangular Young tableaux

A triangular Young tableau of slope $\alpha := -\frac{\ell}{p}$ and of size N is a classical Young tableau with N cells such that

- the first p rows (from the bottom) have length $n\ell$,
- the next p rows have length $(n-1)\ell$ and so on

6		
2	4	
1	3	5

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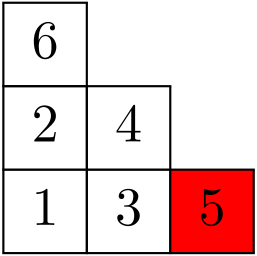

We are interested in $n \to \infty$.
In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 3$

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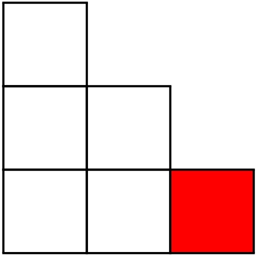

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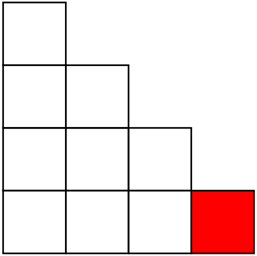

We are interested in $n \to \infty$. In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 4$

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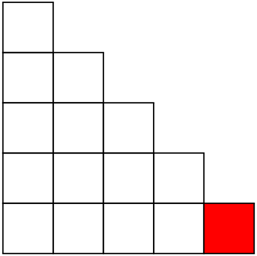

We are interested in $n \to \infty$.
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Here

- $p = 1, \ell = 1$
- $n = 5$

Triangular Young tableaux

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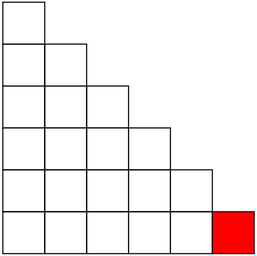

We are interested in $n \to \infty$.
In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 6$

Triangular Young tableaux

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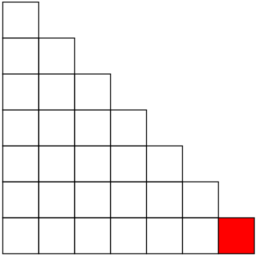

We are interested in $n \to \infty$.
In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 7$

Triangular Young tableaux

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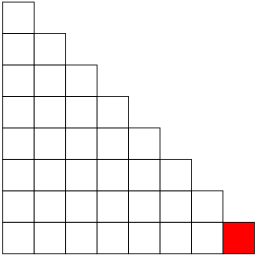

We are interested in $n \to \infty$. In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 8$

Triangular Young tableaux

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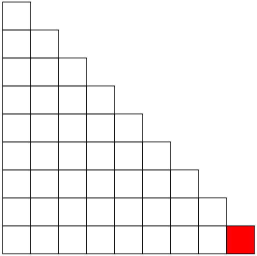

We are interested in $n \to \infty$. In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 9$

Triangular Young tableaux

Triangular Young tableaux

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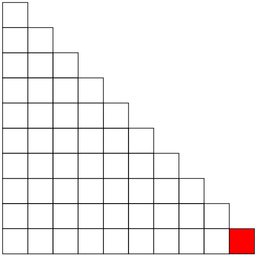

We are interested in $n \to \infty$. In particular: Distribution of the lower right corner.

Here

- $p = 1, \ell = 1$
- $n = 10$

Triangular Young tableaux

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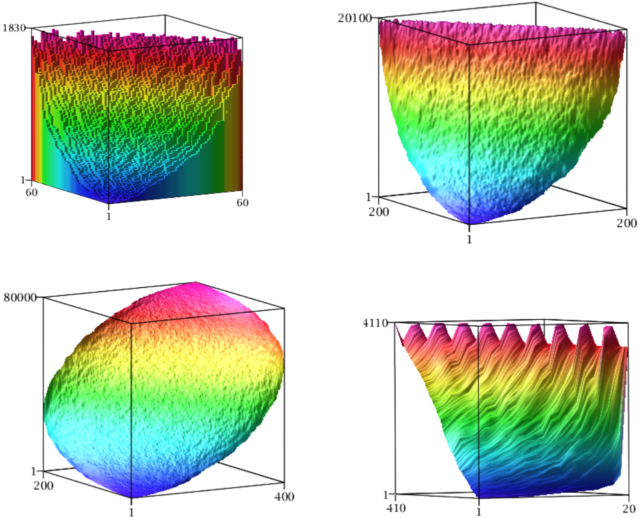
41	55	61	72									
31	44	60	71	p								
22	27	45	58									
18	25	32	43	46	57	59	68					
17	19	26	30	40	52	56	63	p				
12	14	20	29	38	39	51	62					
6	8	10	21	28	35	50	53	54	65	67	70	
3	5	7	13	15	24	47	48	49	64	66	69	p
1	2	4	9	11	16	23	33	34	36	37	42	
	ℓ				ℓ				ℓ			

We are interested in $n \to \infty$. In particular: Distribution of the lower right corner.

Here

- $p = 3, \ell = 4$
- $n = 3$

Random Young tableaux as random surfaces



Asymptotics of triangular Young tableaux

Asymptotic results for Young Tableaux:[Logan, Shepp 77], [Vershik, Kerov 77], [Cohn, Larsen, Propp 98], [Borodin, Okounkov, Olshanski 99], [Widom 01], [Okounkov, Reshetikhin 01], [Pittel, Romik 04], [Romik 15], [Marchal 15], [Morales, Pak, Panova 16], . . . Svante, Robin, Vadim, Alejandro, Jehanne, Valentin, Piotr, room!

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Theorem

Let X_n be the entry of the lower right corner of a uniform random triangular Young tableau of slope $\alpha = -\frac{\ell}{p}$ and of size N. Define $\delta = \frac{p}{p+\ell} = \frac{1}{1-\alpha}$.

,

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$$\frac{N - X_n}{n^{1+\delta}}$$

converges in law to the same limiting distribution as the number of black balls in a periodic Young–Pólya urn with initial conditions $w_0 = \ell$, $b_0 = p$ and with replacement matrices $M_1 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}$,

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$$\frac{2}{p\ell} \frac{N - X_n}{n^{1+\delta}} \xrightarrow{\mathcal{L}} \mathrm{Beta}(b_0, w_0) \prod_{i=0}^{\ell-1} \mathrm{GenGamma}(b_0 + w_0 + p + i, p + \ell).$$

Proof: **Young tableaux** $\to$ **trees** $\to$ periodic Pólya urns

Correspondence 1

41	55	61	72									
31	44	60	71	p								
22	27	45	58									
18	25	32	43	46	57	59	68					
17	19	26	30	40	52	56	63	p				
12	14	20	29	38	39	51	62					
6	8	10	21	28	35	50	53	54	65	67	70	
3	5	7	13	15	24	47	48	49	64	66	69	p
1	2	4	9	11	16	23	33	34	36	37	42	
	ℓ				ℓ				ℓ			

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31	44	60	71	p								
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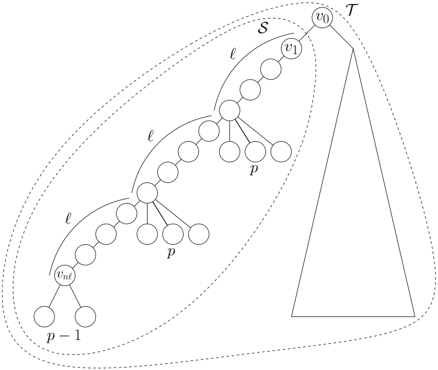
ℓ				ℓ				ℓ				



Proof: **Young tableaux** → **trees** → periodic Pólya urns

Correspondence 1

- Lower right cell of $\mathcal{Y}$ corresponds to node v in the tree $\mathcal{T}$

41	55	61	72									
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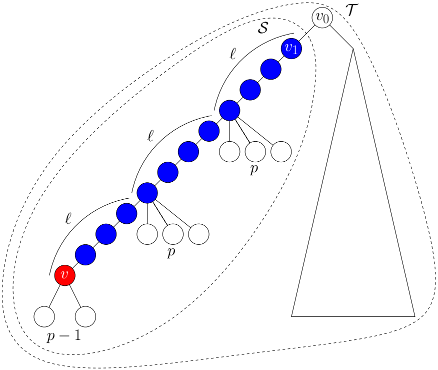

Proof: **Young tableaux** → **trees** → periodic Pólya urns

Correspondence 1

- Lower right cell of $\mathcal{Y}$ corresponds to node v in the tree $\mathcal{T}$
- First row corresponds to left-most branch of $\mathcal{S}$

41	55	61	72								
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22	27	45	58								
18	25	32	43	46	57	59	68				
17	19	26	30	40	52	56	63				
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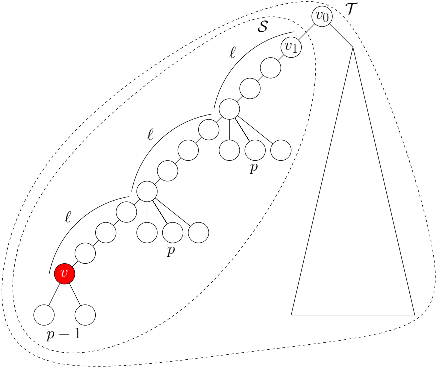
1	2	4	9	11	16	23	33	34	36	37	42

p p p

ℓ ℓ ℓ



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- Important: Hook lengths are the same!

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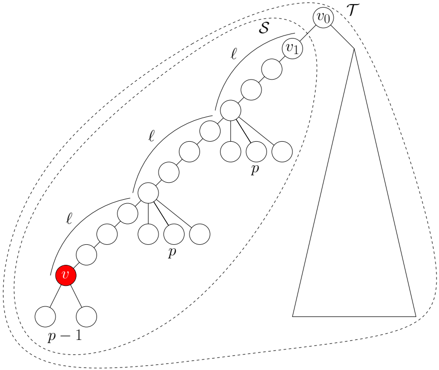

Proof: **Young tableaux** $\rightarrow$ **trees** $\rightarrow$ periodic Pólya urns

Correspondence 1

- Lower right cell of $\mathcal{Y}$ corresponds to node v in the tree $\mathcal{T}$
- First row corresponds to left-most branch of $\mathcal{S}$
- Important: Hook lengths are the same!

Key result (via density method)

Let $E_{\mathcal{T}}$ be a uniform random linear extension of $\mathcal{T}$, and X_n be the lower right entry of $\mathcal{Y}$.

$$1 + X_n \stackrel{\mathcal{L}}{=} E_{\mathcal{T}}(v).$$

41	55	61	72								
31	44	60	71								
22	27	45	58								
18	25	32	43	46	57	59	68				
17	19	26	30	40	52	56	63				
12	14	20	29	38	39	51	62				
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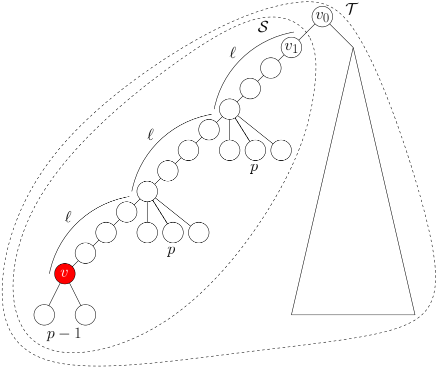
3	5	7	13	15	24	47	48	49	64	66	69
1	2	4	9	11	16	23	33	34	36	37	42

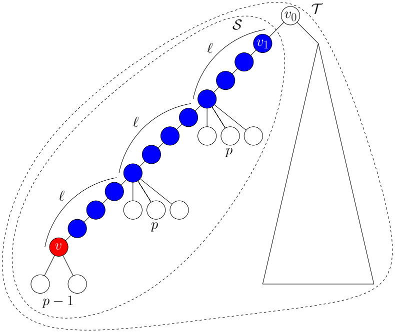
p p p

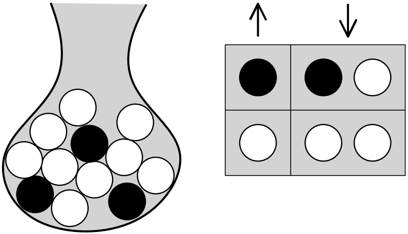
ℓ ℓ ℓ



Proof: Young tableaux → **trees** → **periodic Pólya urns**

Correspondence 2

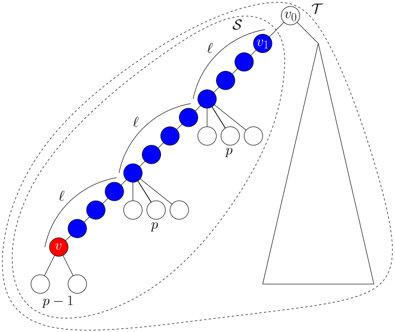


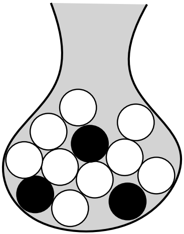


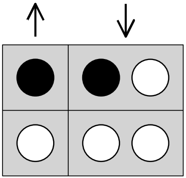
Correspondence 2

- Periodic Young–Pólya urn with period p and parameter ℓ with replacement matrices

$$M_1 = M_2 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}.$$







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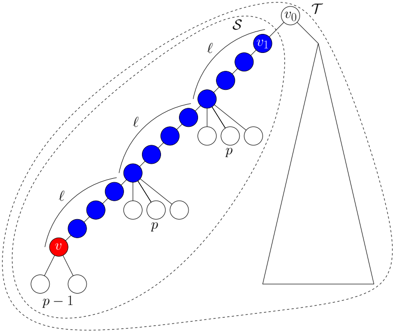
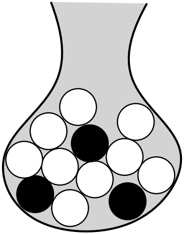
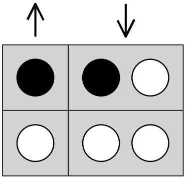
- Initial conditions $b_0 = p$ and $w_0 = \ell$







Proof: Young tableaux → **trees** → **periodic Pólya urns**

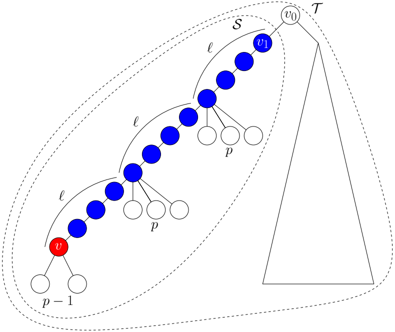
Correspondence 2

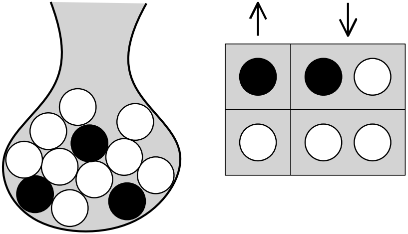
- Periodic Young–Pólya urn with period p and parameter ℓ with replacement matrices

$$M_1 = M_2 = \cdots = M_{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M_p = \begin{pmatrix} 1 & \ell \\ 0 & 1+\ell \end{pmatrix}.$$

- Initial conditions $b_0 = p$ and $w_0 = \ell$

$\Rightarrow$ $N - E_{\mathcal{S}}(v)$ is distributed like $B_{(n-1)p}$, the black balls after $(n-1)p$ steps





The general result for Pólya urns

Theorem (The product generalized gamma distribution for balanced periodic triangular urns)

Let $p \geq 1$ and $\ell_1, \ldots, \ell_p \geq 0$ be non-negative integers. Consider a periodic Pólya urn of period p with replacement matrices $M_1, \ldots, M_p$ given by

$$M_j := \begin{pmatrix} 1 & \ell_j \\ 0 & 1 + \ell_j \end{pmatrix}.$$

Then, the renormalized distribution of black balls is asymptotically for $n \to \infty$ given by the following product of independent distributions:

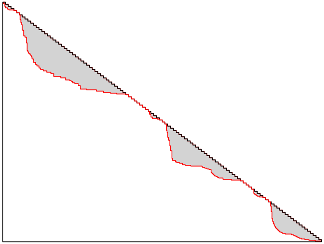
$$\frac{p^\delta}{p+\ell} \frac{B_n}{n^\delta} \quad \xrightarrow{\mathcal{L}} \quad \operatorname{Beta}(b_0, w_0) \quad \prod_{\substack{i=1 \\ i \neq \ell_1 + \cdots + \ell_j + j \text{ with } 1 \leq j \leq p-1}}^{p+\ell-1} \quad \operatorname{GenGamma}(b_0 + w_0 + i, p + \ell).$$

with $\ell = \ell_1 + \cdots + \ell_p$, $\delta = p/(p+\ell)$, and $\operatorname{Beta}(b_0, w_0) = 1$ when $w_0 = 0$.

Solves the law of the south-east corner for
a periodic triangular Young tableaux of any periodic shape.

Fluctuations of the area above the level line

44	55	61	72								
31	43	60	71								
22	25	32	39								
18	24	27	35	41	58	59	68				
17	19	26	30	40	52	56	63				
12	14	20	29	38	49	51	62				
6	8	10	21	28	46	50	53	57	65	67	70
3	5	7	13	15	45	47	48	54	64	66	69
1	2	4	9	11	16	23	33	34	36	37	42

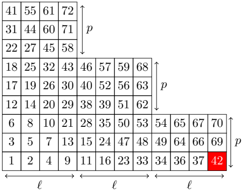


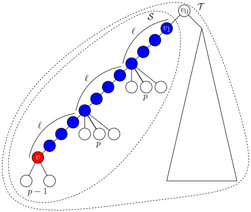
The level line (in red) of the south-east corner X_n: it separates all the entries smaller than X_n from the other ones.
On the left: one example with the level line of $X_n = 42$.
One the right: the level line of X_n, for a very large Young tableau of size N of triangular shape.

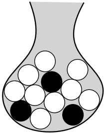
The area between this level line and the hypotenuse is the quantity $N - X_n$, its fluctuation are captured by our theorem.

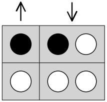
Funny: duality between corners gives a probabilistic proof of the Gauss multiplication formula!

Conclusion

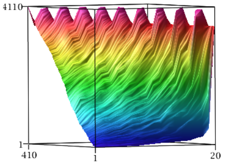


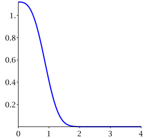




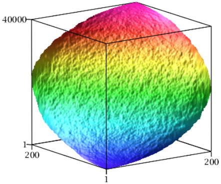


1. Powerful method to tackle Pólya urn problems (colors, multidrawing, time. . .)
2. Exact and asymptotic analysis of the urn composition
3. New universal limit law: product of generalized Gamma distributions
4. Computer algebra challenges (D-finiteness and pde's, sums of hypergeom)
5. En passant, density method to generate/enumerate combinatorial structures
6. Nice cherry: corners of triangular Young tableaux, duality





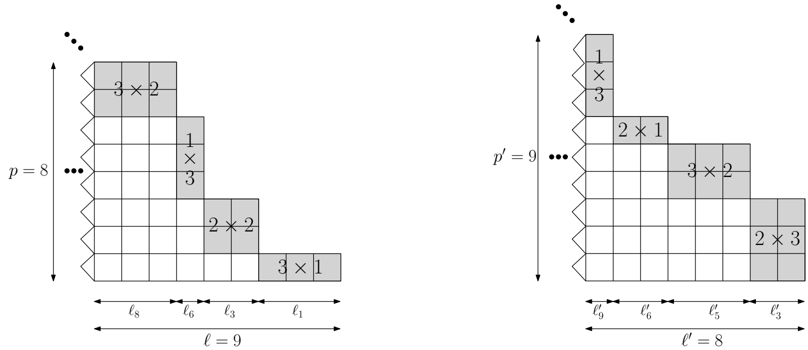
GenGamma$(1,3)$: $\frac{3\,e^{-x^3}}{\Gamma(1/3)}$



Bonus slides, just in case...

Factorisation of Gamma distributions via Young tableaux

$(i_1, j_1; \dots; i_4, j_4) = (\overset{\rightarrow}{3}, \overset{\downarrow}{2}; \overset{\rightarrow}{1}, \overset{\downarrow}{3}; \overset{\rightarrow}{2}, \overset{\downarrow}{2}; \overset{\rightarrow}{3}, \overset{\downarrow}{1}) \xrightarrow{\text{cyclic shift}} (j_4, i_1; \dots; j_3, i_4) = (\overset{\rightarrow}{1}, \overset{\downarrow}{3}; \overset{\rightarrow}{2}, \overset{\downarrow}{1}; \overset{\rightarrow}{3}, \overset{\downarrow}{2}; \overset{\rightarrow}{2}, \overset{\downarrow}{3})$

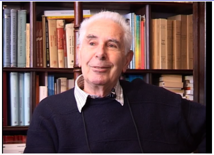


$$\mathrm{GenGamma}(s_0+1, 2) = \sqrt{m} \prod_{k=1}^{m} \mathrm{GenGamma}(s_0+2k-1, 2m).$$

$\mathrm{GenGamma}(a, 1/b) = \Gamma(ab)^b$ thus with $x := \frac{s_0+1}{2m}$:
$\Gamma(mx)^m = m^m \prod_{k=1}^m \Gamma\left(x + \frac{k-1}{m}\right) \longleftrightarrow$
probabilistic proof of Gauss multiplication formula!

Universality of the tails

Gösta Mittag-Leffler (1846-1927)



Jean-Pierre Kahane (1926-2017)

Theorem

ProdGenGamma *have tails similar to a Mittag-Leffler distribution:* $\frac{\log \frac{\mathbb{E}(X^r)}{\mathbb{E}(Y^r)}}{r} \to 0.$

Definition (Subgaussian tails Kahane, 1960)

A random variable X has subgaussian tails if there exist $c, C > 0$, such that

$$\mathbb{P}(|X| \geq t) \leq Ce^{-ct^2}, \qquad t > 0.$$

Theorem

The ProdGenGamma$([\ell_1, \ldots, \ell_p], b_0, w_0)$ *distributions have subgaussian tails if and only if* $p \geq \ell$, *where* $\ell = \ell_1 + \ldots + \ell_p$.

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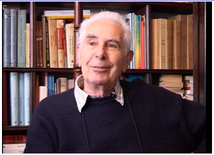

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The tails **depend** only on

- the slope δ
- the period length p.

The tails are **independent** of

- the initial conditions b_0 and w_0
- the periodic pattern $[\ell_1, \ldots, \ell_p]$.

Theorem (Limit law for the location of the maximum)

Choose a uniform random triangular Young tableau of parameters (ℓ, p, n).
$\mathrm{Posi}_n \in \{1, \ldots, \ell n\}$*:= x-coordinate of the cell containing the largest entry. Then,*

$$\frac{\mathrm{Posi}_n}{\ell n} \xrightarrow{\mathcal{L}} \mathrm{Arcsine}(\delta), \quad \textit{where } \delta := p/(p+\ell).$$

Proof.

$\mathcal{Y}^*$:=tableau where the cell containing the max is removed.

$$\mathbb{P}(\mathrm{Posi}_n = k\ell) = \frac{\prod_{c\in\mathcal{Y}^*} \mathrm{hook}_{\mathcal{Y}^*}(c)}{\prod_{c\in\mathcal{Y}} \mathrm{hook}_{\mathcal{Y}}(c)} = \prod_{\substack{c\in\mathcal{Y}^* \text{with } (x\text{-coord of } c) = k\ell \\ \text{or } (y\text{-coord of } c) = (n-k)p}} \frac{\mathrm{hook}_{\mathcal{Y}^*}(c)}{1+\mathrm{hook}_{\mathcal{Y}^*}(c)}.$$

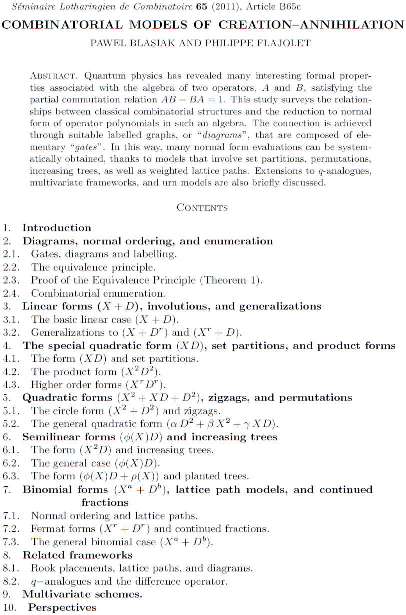
$$\mathbb{P}(\mathrm{Posi}_n = k\ell) \sim \frac{(k/n)^{\delta-1}(1-k/n)^{-\delta}}{\Gamma(\delta)\Gamma(1-\delta)}.$$

=generalized arcsine law on $[0,1]$ with density $\frac{x^{\delta-1}(1-x)^{-\delta}}{\Gamma(\delta)\Gamma(1-\delta)}$. □

This is in sharp contrast with the case of an $n \times n$ square tableau where, for every $t \in (0,1)$, the cell containing the entry tn^2 is asymptotically distributed according to the Wigner semicircle law on its level line (see [PittelRomik07]).

More to dig with combinatorics of differential operators!

$\exp(y\partial_x^2 + xy\partial_y^2)$
(Heisenberg–Weyl algebra)



Séminaire Lotharingien de Combinatoire **65** (2011), Article B65c

COMBINATORIAL MODELS OF CREATION–ANNIHILATION

PAWEL BLASIAK AND PHILIPPE FLAJOLET

Abstract. Quantum physics has revealed many interesting formal properties associated with the algebra of two operators, A and B, satisfying the partial commutation relation $AB - BA = 1$. This study surveys the relationships between classical combinatorial structures and the reduction to normal form of operator polynomials in such an algebra. The connection is achieved through suitable labelled graphs, or "*diagrams*", that are composed of elementary "*gates*". In this way, many normal form evaluations can be systematically obtained, thanks to models that involve set partitions, permutations, increasing trees, as well as weighted lattice paths. Extensions to q-analogues, multivariate frameworks, and urn models are also briefly discussed.

Contents

Coucou aux amateurs de générique de fin