DETERMINATION OF THE SIZE OF AN INCLUSION FROM ONE BOUNDARY MEASUREMENT AT A SPECIFIC MOMENT OF TIME

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Herglotz-Nevanlinna Theory Applied to Passive, Causal and Active Systems
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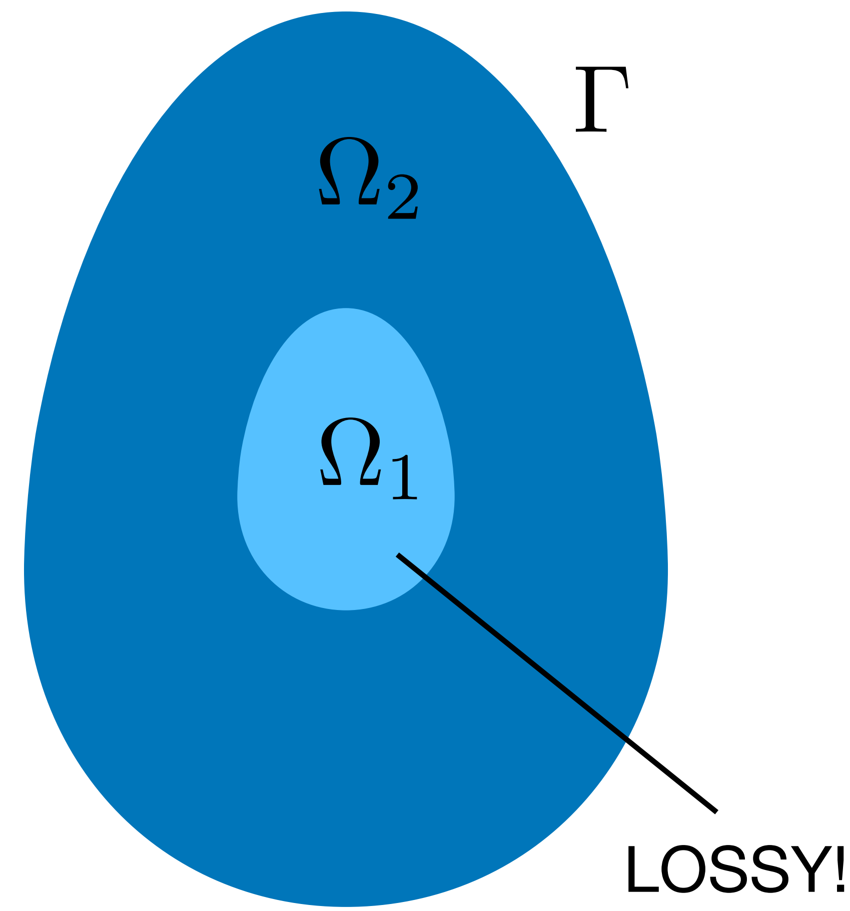
OUTLINE

- Formulation of the problem in the frequency domain
- Analytic method
- Formulation of the problem in the time domain
- Determination of the volume fraction of the inclusion
- Numerical Results
- Concluding remarks

FORMULATION OF THE PROBLEM

Goal: Determination of the volume fraction of the inclusion $f = |\Omega_1|/|\Omega_1 + \Omega_2|$



$$\mathbf{J} = \mathbf{L}\,\mathbf{E} \qquad \nabla \cdot \mathbf{J} = 0 \qquad \mathbf{E} = -\nabla V$$

$$\mathbf{L}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{L}_1 + (1 - \chi(\mathbf{x}))\mathbf{L}_2$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega_1 \\ 0 & \mathbf{x} \in \Omega_2 \end{cases}$$

+ BC: $V(\mathbf{x}) = V_0(\mathbf{x})$ or $\mathbf{n}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x}) = q_0(\mathbf{x})$

	$\mathbf{L}$	$\mathbf{J}$	$\mathbf{E}$
Conductivity	$\boldsymbol{\sigma}$	$\mathbf{j}$	$\mathbf{e}$
Dielectrics	$\boldsymbol{\varepsilon}$	$\mathbf{d}$	$\mathbf{e}$
Viscoelasticity	$\mathbf{c}\boldsymbol{\mu}$	$\boldsymbol{\tau}$	$\boldsymbol{\gamma}$

FORMULATION OF THE PROBLEM

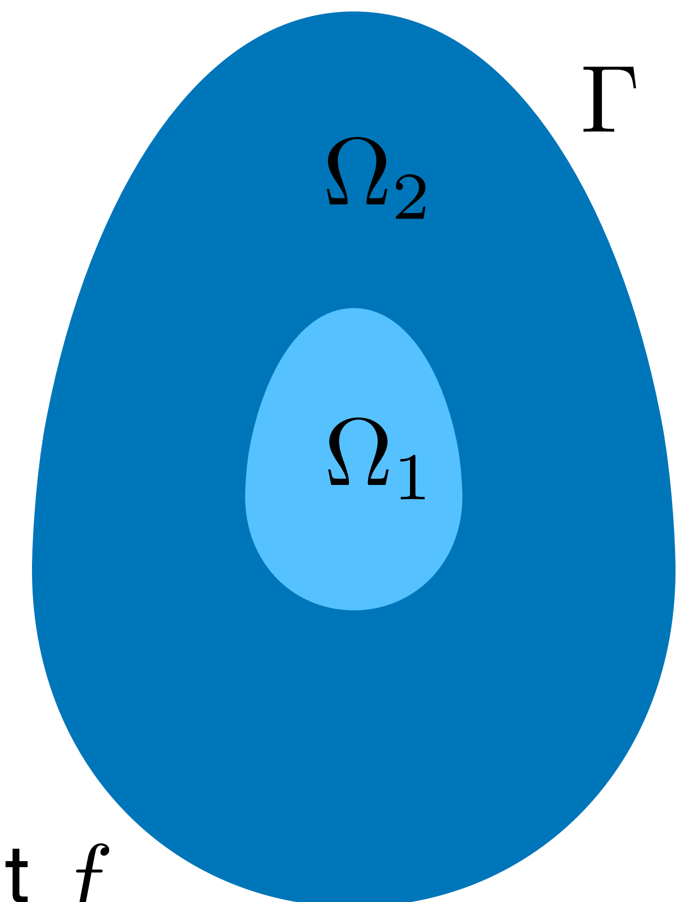
Goal: Determination of the volume fraction of the inclusion $f = |\Omega_1|/|\Omega_1 + \Omega_2|$

EX. $V(\mathbf{x}) = V_0(\mathbf{x})$ on Γ

Measure $q(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x})$

Then the Dirichlet-to-Neumann map (DtN)

$\mathrm{DtN} : V(\mathbf{x}) \mapsto q(\mathbf{x})$ gives information about f



	$\mathbf{L}$	$\mathbf{J}$	$\mathbf{E}$
Conductivity	$\widehat{\boldsymbol{\sigma}}$	$\widehat{\mathbf{j}}$	$\widehat{\mathbf{e}}$
Dielectrics	$\widehat{\boldsymbol{\varepsilon}}$	$\widehat{\mathbf{d}}$	$\widehat{\mathbf{e}}$
Viscoelasticity	$\mathrm{c}\widehat{\boldsymbol{\mu}}$	$\widehat{\boldsymbol{\tau}}$	$\widehat{\boldsymbol{\gamma}}$

BOUNDARY CONDITIONS

Special Dirichlet BC: $V(\mathbf{x}) = -\mathbf{E}_0 \cdot \mathbf{x}$ on Γ

Measure $q(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x})$ on Γ, then:

$$\mathbf{J}_0 = \langle \mathbf{J}(\mathbf{x}) \rangle = \frac{1}{|\Omega|} \int_\Omega \mathbf{J}(\mathbf{x}) \mathrm{d}\mathbf{x} = -\frac{1}{|\Omega|} \int_\Gamma q(\mathbf{x}) \mathbf{x} \mathrm{d}\mathbf{x}$$ known

$\mathbf{E}_0 = \langle \mathbf{E}(\mathbf{x}) \rangle$ $\Rightarrow$ $\mathbf{J}_0 = \mathbf{L}^* \mathbf{E}_0$ ($\mathbf{L}^*$ depends on f)

ANALOGY WITH THE THEORY OF COMPOSITES!

ANALYTICITY OF $\mathbf{L}^*$

[Bergman (1978), Milton (1981), Golden and Papanicolaou (1983)]

Hyp: the two materials are isotropic: λ_1 and λ_2

Then $\mathbf{L}^*(\lambda_1, \lambda_2)$ is an analytic function of λ_1 and λ_2 whenever

$$s = \frac{\lambda_2}{\lambda_2 - \lambda_1} \notin [0, 1)$$

$$\mathbf{L}^* = \lambda_2 \left(\mathbf{I} - \int_0^1 \frac{\mathrm{d}\boldsymbol{\eta}(y)}{s - y} \right)$$

For rational functions:

$$\mathbf{L}^* = \lambda_2 \left(\mathbf{I} - \sum_{i=1}^{m} \frac{\mathbf{B}_i}{s - s_i} \right)$$

$$0 \leq s_0 \leq s_1 \leq \cdots \leq s_m < 1 \qquad \mathbf{B}_i \geq 0 \quad \text{for all } i$$

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SUM RULES:

- Positive semi-definiteness: $$\sum_{i=1}^{m} \frac{\mathbf{B}_i}{1 - s_i} \le \mathbf{I}$$
- Volume fraction: $$\sum_{i=1}^{m} \mathbf{B}_i = f\mathbf{I}$$

THE PROBLEM IN THE TIME DOMAIN

$$\mathbf{J}_0 = \mathbf{L}^* \mathbf{E}_0 \qquad \mathbf{L}^* = \lambda_2 \left(\mathbf{I} - \sum_{i=1}^{m} \frac{\mathbf{B}_i}{s - s_i} \right) \qquad s = \frac{\lambda_2}{\lambda_2 - \lambda_1}$$

$$\Rightarrow \quad \mathbf{J}_0 = \lambda_2 \mathbf{E}_0 - \lambda_2 \sum_{i=1}^{m} \frac{\mathbf{B}_i}{s - s_i} \mathbf{E}_0$$

$$\mathbf{J}_0(t) = (\lambda_2 \star \mathbf{E}_0)(t) - \sum_{i=1}^{m} \mathbf{B}_i \left(\mathcal{L}^{-1} \left[\frac{\lambda_2}{s - s_i} \right] \star \mathbf{E}_0 \right)(t)$$

$\star$ = convolution in time

THE PROBLEM IN THE TIME DOMAIN

$$\mathbf{J}_0 = \mathbf{L}^*\mathbf{E}_0 \qquad \mathbf{L}^* = \lambda_2\left(\mathbf{I} - \sum_{i=1}^{m}\frac{\mathbf{B}_i}{s-s_i}\right) \qquad s = \frac{\lambda_2}{\lambda_2 - \lambda_1}$$

$$\Rightarrow \quad \mathbf{J}_0 = \lambda_2\mathbf{E}_0 - \lambda_2\sum_{i=1}^{m}\frac{\mathbf{B}_i}{s-s_i}\mathbf{E}_0$$

$$\mathbf{J}_0(t) = (\lambda_2 \star \mathbf{E}_0)(t) - \boxed{\sum_{i=1}^{m}\mathbf{B}_i}\boxed{\left(\mathcal{L}^{-1}\left[\frac{\lambda_2}{s-s_i}\right]\star\mathbf{E}_0\right)(t)}$$

$\star$ = convolution in time

Constant term $\approx -c\mathbf{I}$
for a specific moment of time $t = t_0$

Sum rule: $\boxed{\sum_{i=1}^{m}\mathbf{B}_i} = f\mathbf{I} \Rightarrow \quad \mathbf{J}_0(t_0) \approx (\lambda_2 \star \mathbf{E}_0)(t_0) + cf\mathbf{I}$

TIME-HARMONIC LOADING

Focusing only on the first components:

$$E_0(t) = \mathrm{Re}\left[\sum_{n=1}^{N} \alpha_n \mathrm{e}^{-\mathrm{i}\omega_n t}\right]$$

Therefore, if at $t = 0$

$$\mathrm{Re}\left[\sum_{n=1}^{N} \frac{\alpha_n L_2(\omega_n)}{s(\omega_n) - s_i}\right] \approx -c \qquad \text{for all } s_i \in [0, 1)$$

then

$$J_0(0) \approx \mathrm{Re}\left[\sum_{n=1}^{N} \alpha_n L_2(\omega_n)\right] + c\, f$$

TIME-HARMONIC LOADING

$$\mathrm{Re}\left[\sum_{n=1}^{N}\frac{\alpha_n L_2(\omega_n)}{s(\omega_n)-s_i}\right]\approx -c \qquad \text{for all } s_i\in[0,1)$$

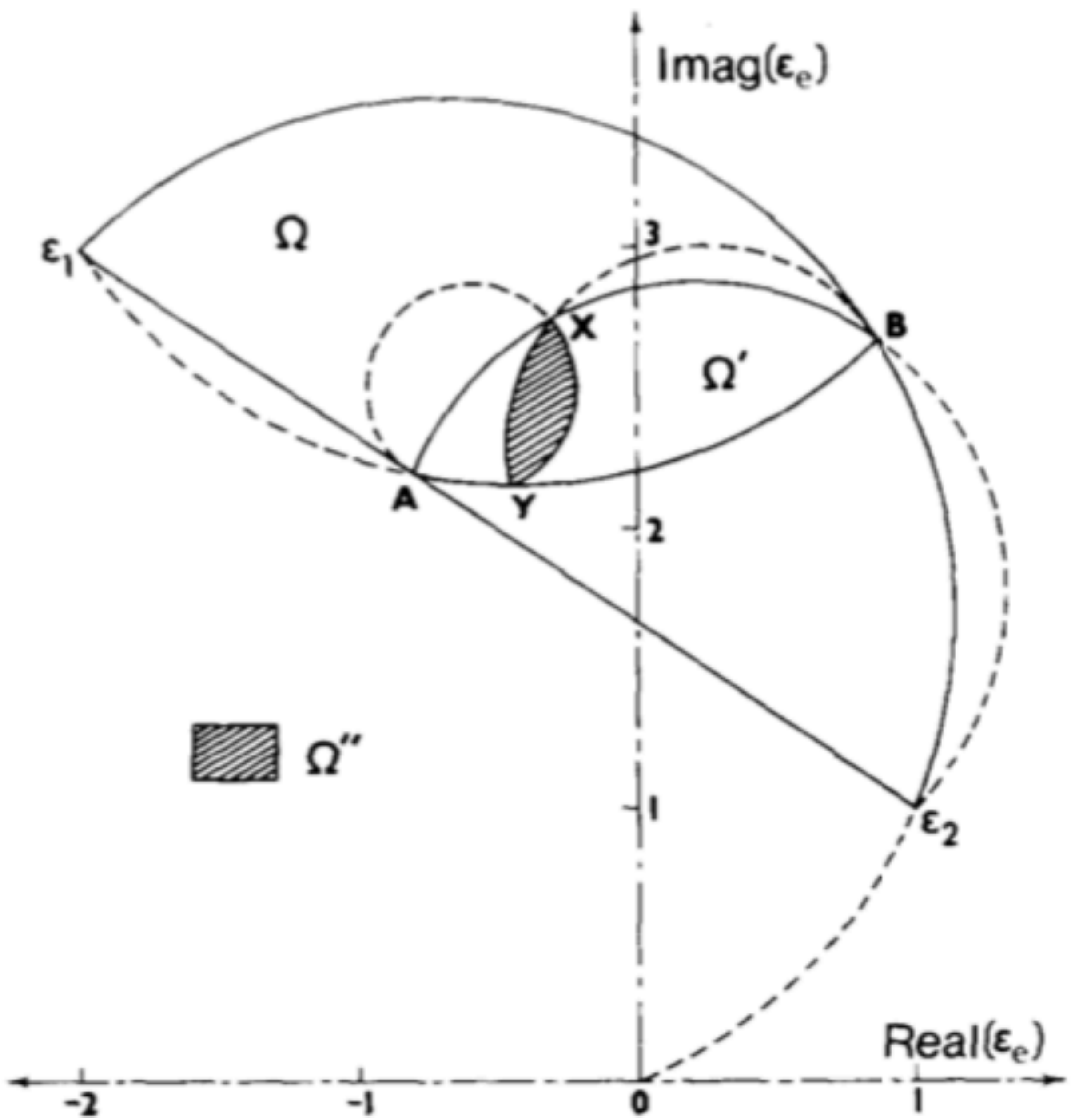
$$g(p)=\frac{1}{2}\sum_{n=1}^{N}\frac{\alpha_n L_2(\omega_n)}{s(\omega_n)-p}+\frac{1}{2}\sum_{n=1}^{N}\frac{\widehat{\alpha_n}\,\widehat{L_2(\omega_n)}}{\widehat{s(\omega_n)}-p}+c$$

$$g(p)\approx 0 \qquad \text{for } p\in[0,1)$$

$$g(p)=c\prod_{l=1}^{N}\frac{\beta(\omega_l)-p}{s(\omega_l)-p}\prod_{m=1}^{N}\frac{\widehat{\beta(\omega_m)}-p}{\widehat{s(\omega_m)}-p}$$

Place the zeros around the interval [0,1) to force the function to be zero on such an interval

FURTHER REMARKS



[Milton (1981)]

NON-TIME-HARMONIC LOADING

Focusing only on the first components:

$$E_0(t) = \sum_{n=1}^{N} \alpha_n H(t - t_n)$$

Therefore, if at $t = t_0$

$$\sum_{n=1}^{N} \alpha_n \left(\mathcal{L}^{-1} \left[\frac{\lambda_2}{s - s_i} \right] \star H(t_0 - t_n) \right) (t_0) \approx -c \quad \text{for all } s_i \in [0, 1)$$

then

$$J_0(t_0) \approx \sum_{n=1}^{N} \alpha_n \left(\lambda_2 \star H(t_0 - t_n) \right) (t_0) + cf$$

WHAT ABOUT OTHER MOMENTS OF TIME?

$$J_0(t) = (\lambda_2 \star E_0)(t) - \sum_{i=1}^{m} B_i \left(\mathcal{L}^{-1} \left[\frac{\lambda_2}{s - s_i} \right] \star E_0 \right)(t)$$

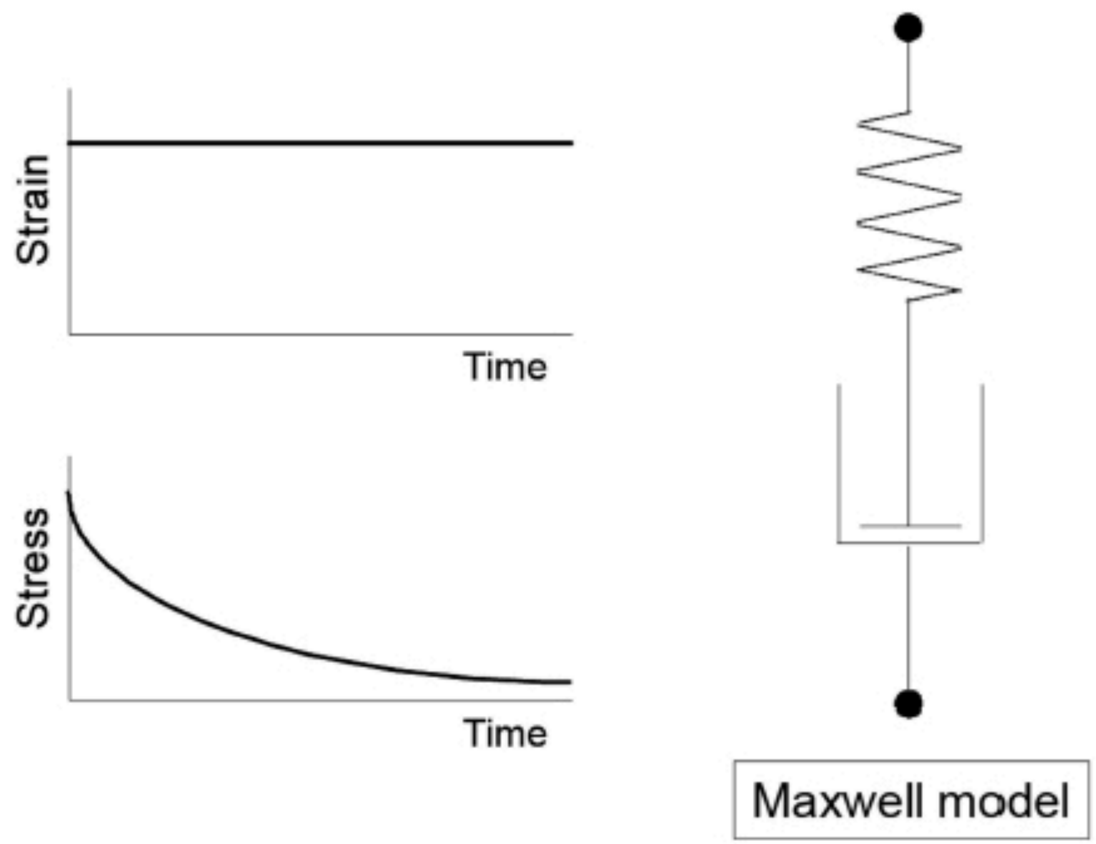
SUM RULES: $$\sum_{i=1}^{M} \frac{B_i}{1 - s_i} \leq 1\,, \qquad \sum_{i=1}^{M} B_i = f$$

BOUNDS ON THE RESPONSE OF THE BODY!

MATERIAL MODELS

Phase 1: Maxwell material

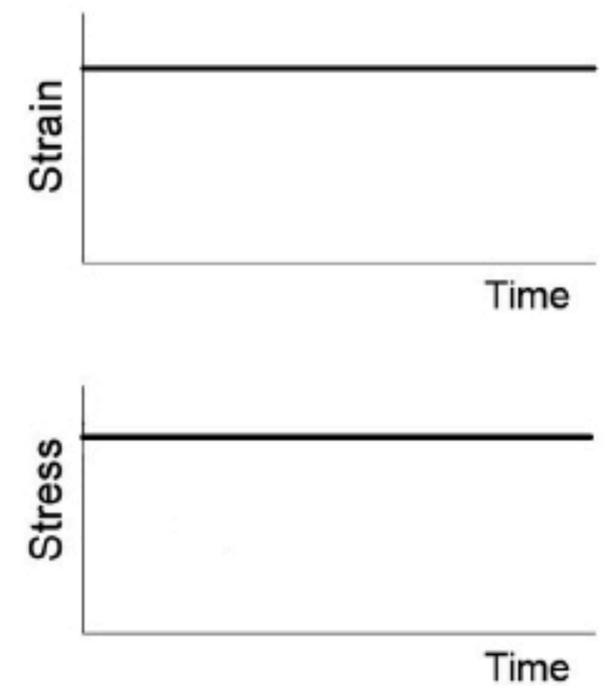
Stress-relaxation test



$$\lambda_1(t) = G_M \exp[-G_M t/\eta_M]$$

Phase 2: Elastic material

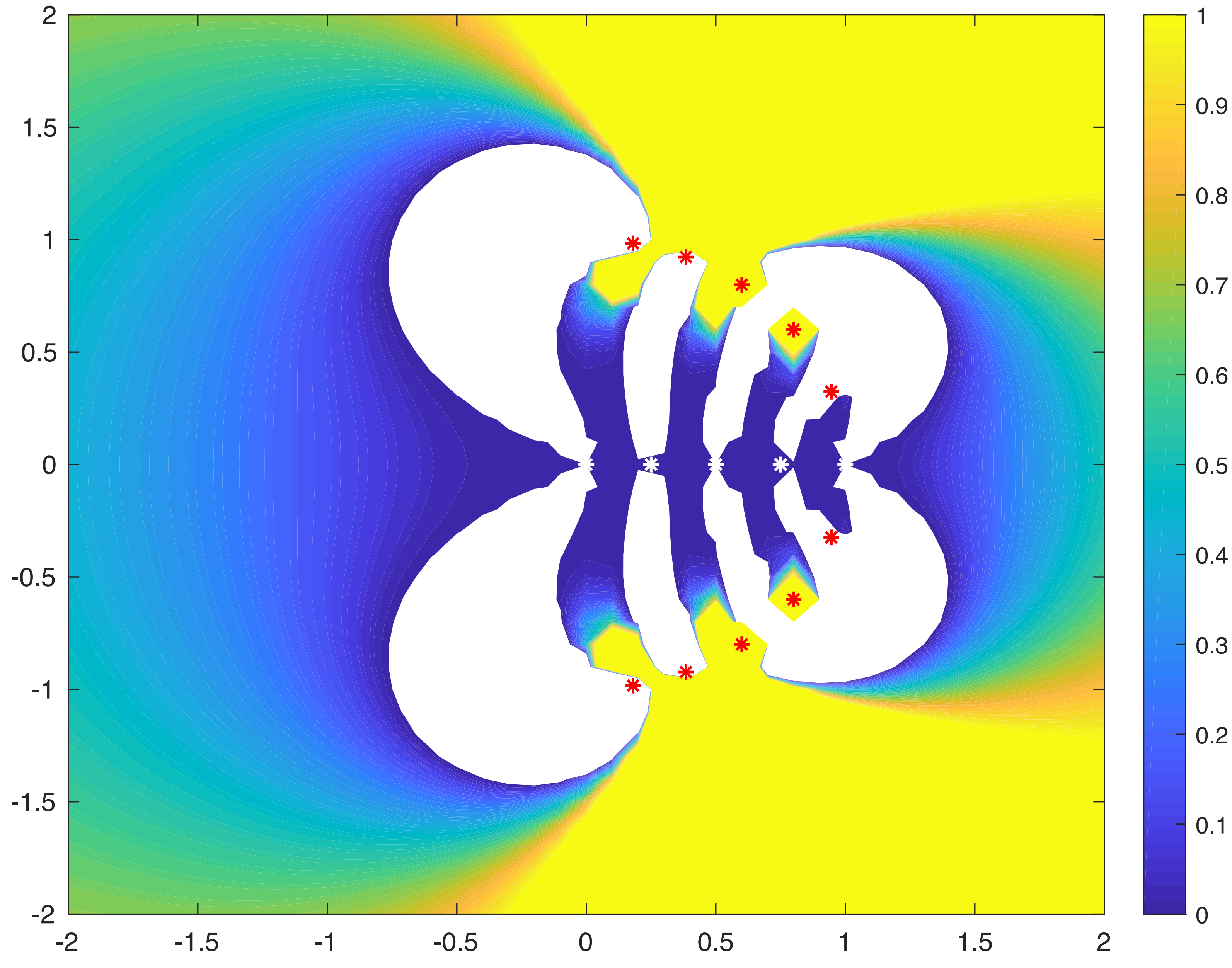
Stress-relaxation test



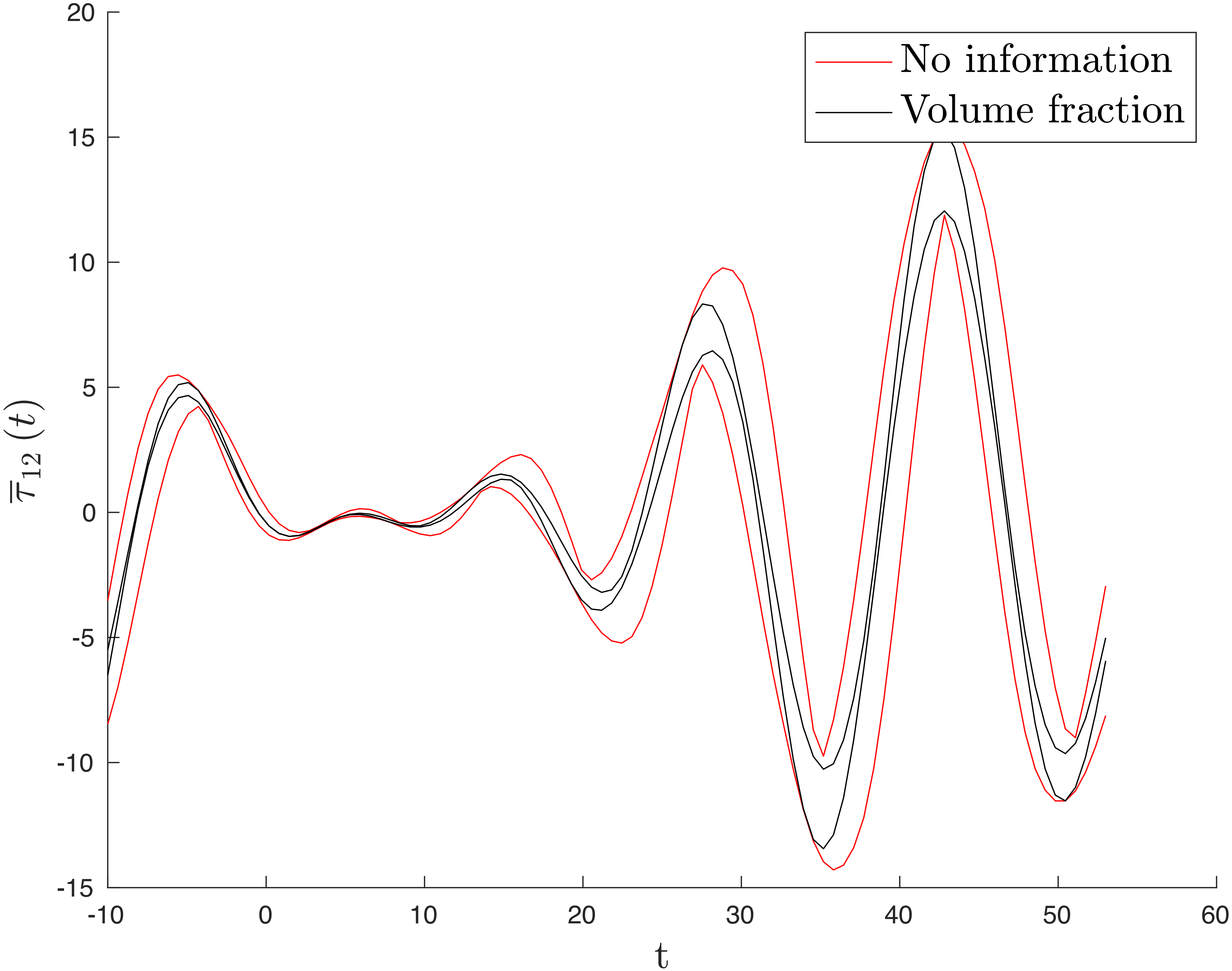
$$\lambda_2(t) = G_2$$

TIME-HARMONIC LOADING

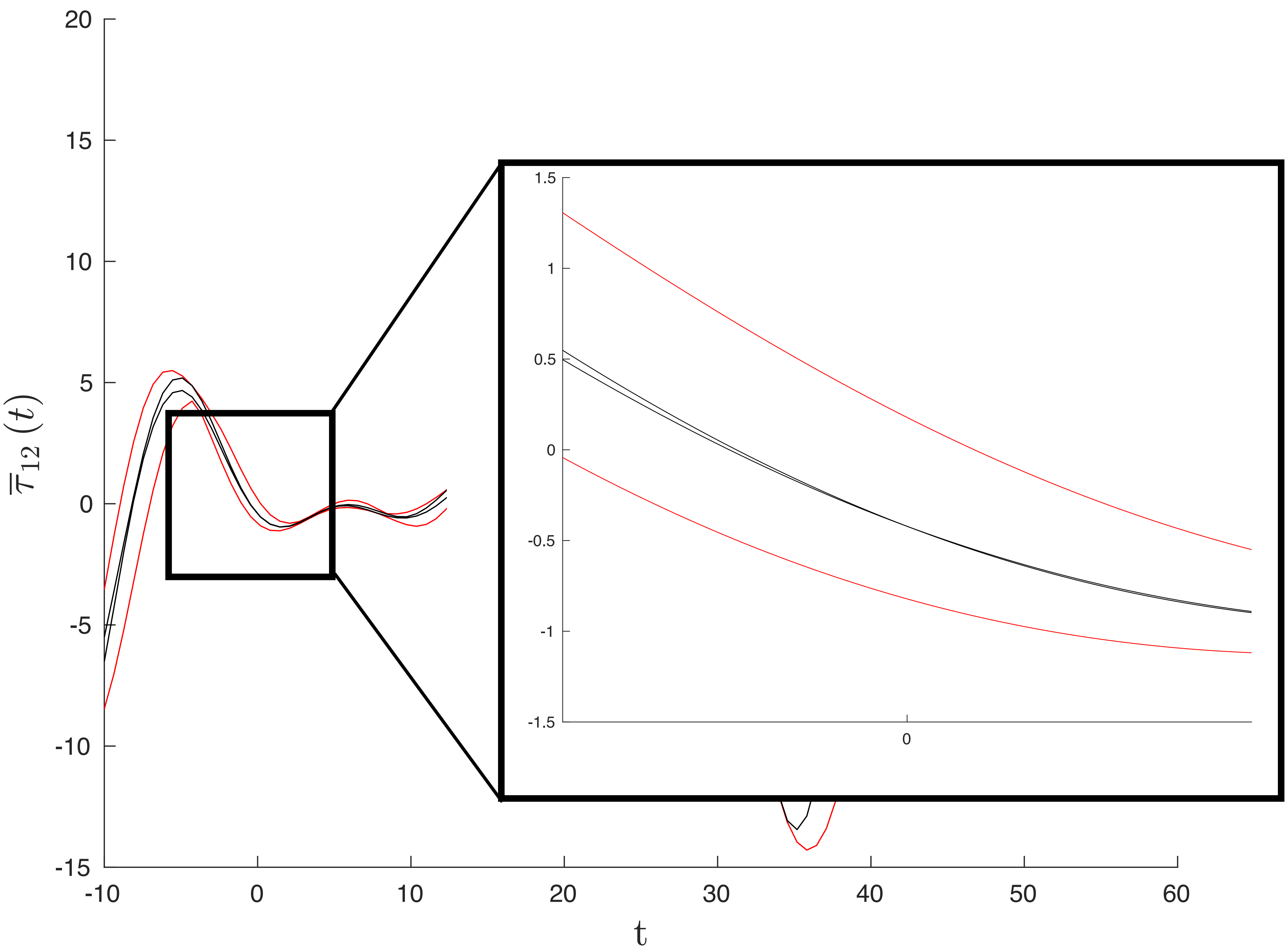
$$g(p) = c \prod_{l=1}^{N} \frac{\beta(\omega_l) - p}{s(\omega_l) - p} \prod_{m=1}^{N} \frac{\widehat{\beta(\omega_m)} - p}{\widehat{s(\omega_m)} - p} \qquad g(p) \approx 0 \quad \text{for } p \in [0,1)$$



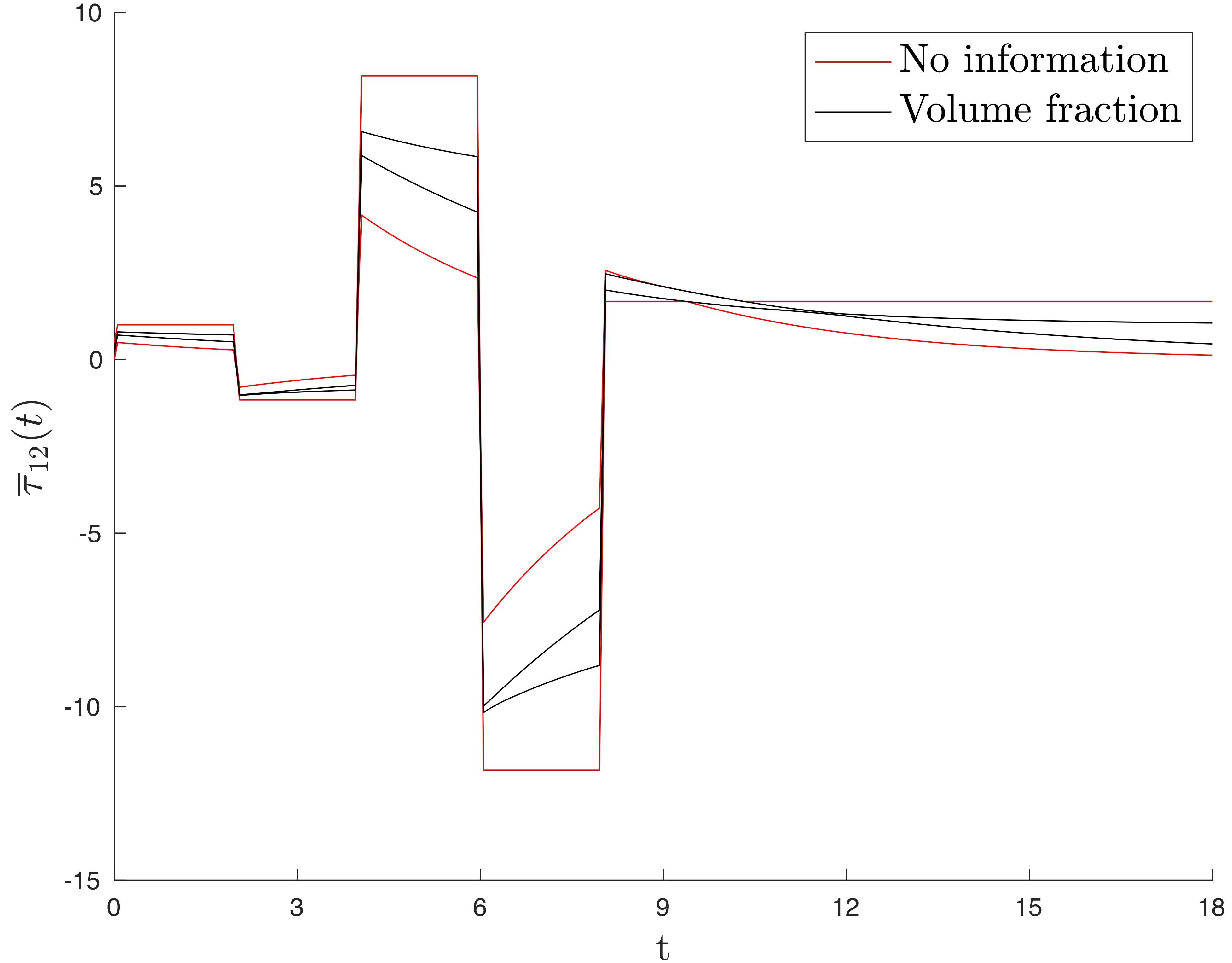
TIME-HARMONIC LOADING



TIME-HARMONIC LOADING



HEAVISIDE-TYPE LOADING



CONCLUSIONS

Analogy between the problem of finding the DtN map for an inhomogeneous body and the problem of finding the effective tensor of a composite

↓

Analytic method

↓

Sum rules in terms of the volume fraction

↓

One measurement of the response of the body at a specific time determines the volume fraction of the inclusion

Thank you for your attention!

O. Mattei, G.W. Milton, 2016. *Bounds for the response of viscoelastic composites under antiplane loadings*. In *Extending the Theory of Composites to Other Areas of Science*, Edited by G.W. Milton, Milton and Patton Publishing (produced by BookBaby.com).

O. Mattei, G.W. Milton. *Determination of the volume fraction of an inclusion by boundary measurements in time*. In preparation.

OPTIMIZATION PROBLEM

$$\tau^0_{12}(t) = \mu_2\gamma_0 \left(1 - \sum_{i=1}^{m} B^{(i)}_{11} \mathcal{L}^{-1}\left[\frac{\mu_2}{\frac{\mu_2}{\mu_2-\mu_1(p)} - s_i}\right](t)\right)$$

SUM RULES:

$$\sum_{i=1}^{m} \frac{B^{(i)}_{11}}{1-s_i} \leq 1 \qquad \sum_{i=1}^{m} B^{(i)}_{11} = f$$

LINEAR PROGRAMMING THEORY:

$$B^{(0)}_{11} = f$$

or

$$B^{(0)}_{11} = \frac{(1-s_0)(s_1-1+f)}{s_1-s_0}$$

$$B^{(1)}_{11} = \frac{(1-s_1)(1-f+s_1)}{s_1-s_0}$$

ANALOGY WITH THE THEORY OF COMPOSITES

- UNIT CELL PROBLEM: [Milton, 2016]

$$\mathbf{J} = \mathbf{L}\,\mathbf{E} \qquad \nabla \cdot \mathbf{J} = 0 \qquad \mathbf{E} = -\nabla V \qquad V(\mathbf{x}) = -\mathbf{E}_0 \cdot \mathbf{x}$$

$$\mathbf{J}_0 + \mathbf{J}' = \mathbf{L}(\mathbf{E}_0 + \mathbf{E}') \qquad \Rightarrow \qquad \mathbf{J}_0 = \mathbf{L}^*\mathbf{E}_0$$

where: $\mathbf{L} : \mathcal{H} \to \mathcal{H} \qquad \mathbf{J}_0, \mathbf{E}_0 \in \mathcal{U} \qquad \mathbf{J}' \in \mathcal{J} \qquad \mathbf{E}' \in \mathcal{E}$

$\mathcal{H}$ = space of square integrable fields = $\mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}$

$\mathcal{U}$ = subspace of constant vector fields

$\mathcal{E}$ = subspace of the gradients of periodic potentials

$\mathcal{J}$ = subspace of divergence-free vector fields with zero average on the unit cell

ANALOGY WITH THE THEORY OF COMPOSITES

- BODY WITH INCLUSION PROBLEM: [Milton, 2016]

$$\mathbf{J} = \mathbf{L}\,\mathbf{E} \qquad \nabla\cdot\mathbf{J} = 0 \qquad \mathbf{E} = -\nabla V \qquad V(\mathbf{x}) = V_0(\mathbf{x})$$

$$\mathbf{J}_0 + \mathbf{J}' = \mathbf{L}(\mathbf{E}_0 + \mathbf{E}') \qquad \Rightarrow \qquad \mathbf{J}_0 = \mathbf{L}^*\mathbf{E}_0$$

where: $\mathbf{L} : \mathcal{H} \to \mathcal{H} \qquad \mathbf{J}_0, \mathbf{E}_0 \in \mathcal{U} \qquad \mathbf{J}' \in \mathcal{J} \qquad \mathbf{E}' \in \mathcal{E}$

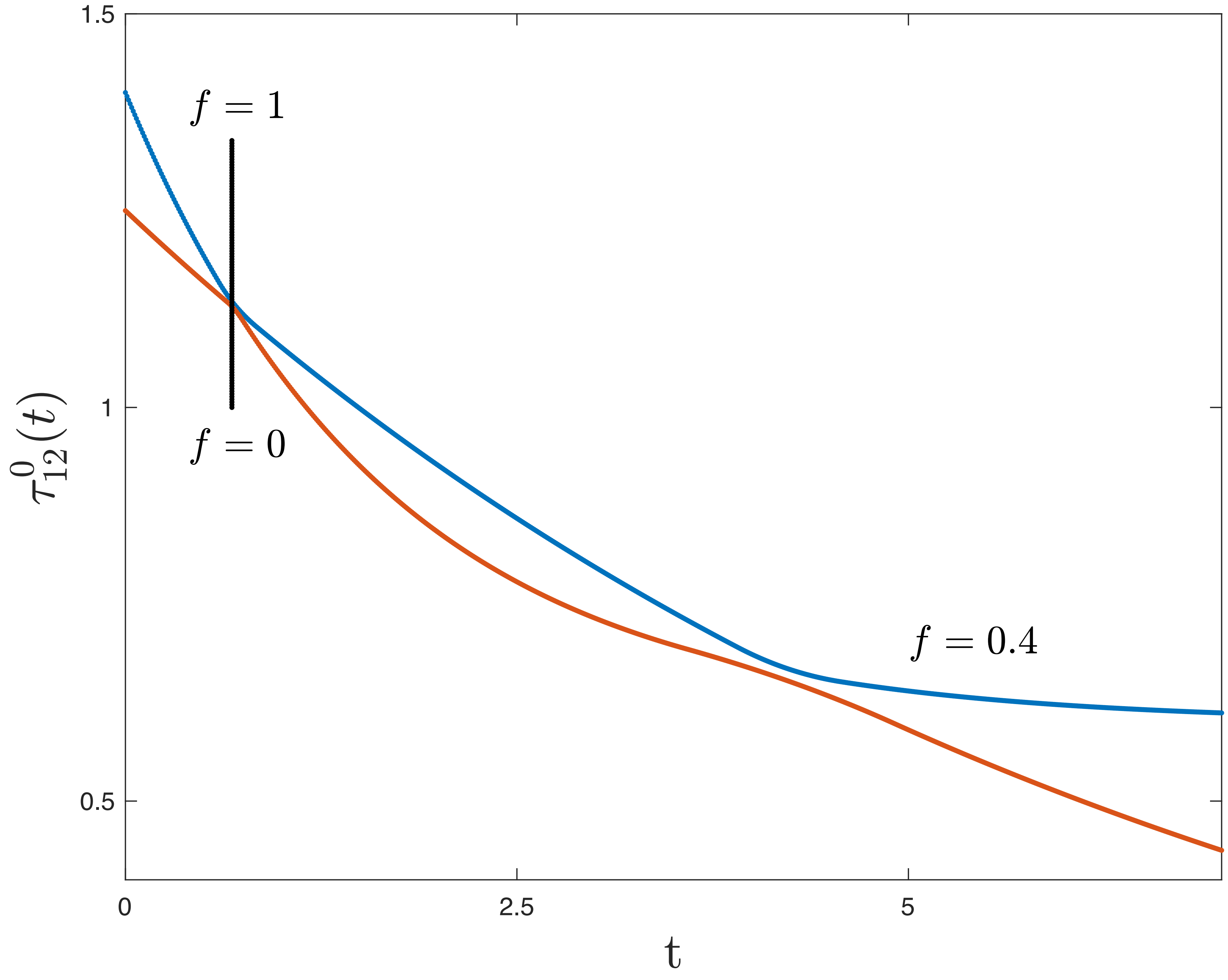
$\mathcal{H}$ = space of square integrable fields = $\mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}$

$\mathcal{U}$ = subspace of the fields solutions of the homogeneous prob.

$\mathcal{E}$ = subspace of the gradients of periodic potentials with zero potential at the boundary

$\mathcal{J}$ = subspace of divergence-free fields with zero flux at the boundary

BOUNDS FOR THE WELL-ORDERED CASE



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