Homogeneous Herglotz class versus homogeneous Herglotz-Agler class

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Overview

(1) Bessmertnyĭ long-resolvent realizations for rational matrix functions

(2) Zoo of metrically-constrained classes of matrix-valued functions

- Schur class over $\mathbb{D}^d$: $\mathcal{S}_d(\mathbb{C}^n)$
- Schur-Agler class over $\mathbb{D}^d$: $\mathcal{SA}_d(\mathbb{C}^n)$
- Herglotz class over Π^d: $\mathcal{H}_d(\mathbb{C}^n)$
- Herglotz-Agler class over Π^d: $\mathcal{HA}_d(\mathbb{C}^n)$
- subclass of rational functions in class $\mathcal{X}(\mathbb{C}^n)$: $\mathcal{X}^{\rm rat}(\mathbb{C})^n$
- homogeneous subclass of class $\mathcal{X}(\mathbb{C}^n)$: $\mathcal{X}^{\rm hom}(\mathbb{C}^n)$

1. Bessmertnyĭ realizations for general $n \times n$-matrix rational functions in d variables

Theorem (Bessmertnyĭ 1982)

(1) Any rational $n \times n$ matrix-valued function in d complex variables $F(z) = F(z_1, \dots, z_d)$ can be represented (realized) as

$F(z) = L_{11}(z) - L_{12}(z)L_{22}(z)^{-1}L_{21}(z)$, $z = (z_1, \dots, z_d) \in \mathbb{C}^d$

where

$L(z) = L_0 + z_1 L_1 + \cdots + z_d L_d = \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ is a matrix pencil

i.e., $F(z)$ $=$ Schur complement of a matrix pencil

(2) If $F(z)$ is homogeneous ($F(\lambda z) = \lambda F(z)$ for all $\lambda \in \mathbb{C}$), then necessarily $L_0 = 0$ (so also $L(\lambda z) = \lambda L(z)$).

Special cases of Bessmertnyĭ representation for the single-variable case $d = 1$

- Transfer-function realization : $L(z) = \begin{bmatrix} D & C \\ B & A - zI \end{bmatrix} \Rightarrow$
  $F(z) = D + C(zI - A)^{-1}B$
  System matrix appearing in control theory (Rosenbrock):
  $\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} L(z) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$
  Such representations exist only for proper $F(z)$

  Good uniqueness properties: two controllable & observable realizations for the same F are similar —not true for general long-resolvent representations
- Descriptor realization : $L(z) = \begin{bmatrix} D & C \\ B & E - zI \end{bmatrix} \Rightarrow$
  $F(z) = D + C(zE - A)^{-1}B$
  (in fact a given $F(z)$ has a realization with $D = 0$)
  Reasonbly good uniqueness properties worked out recently
- Conclusion: The long-resolvent representation = multivariable version of descriptor realizations

Special cases of Bessmertnyĭ representations with $d > 1$

- Fornasini-Marchesini realizations:
  $L(z) = \begin{bmatrix} D & C \\ z_1 B_1 + \cdots + z_d B_d & z_1 A_1 + \cdots + z_d A_d - I \end{bmatrix} \Rightarrow$
  $F(z) = D + C(I - z_1 A_1 - \cdots - z_d A_d)^{-1}(z_1 B_1 + \cdots + z_d B_d)$
  (natural for function theory on the ball)
- Givone-Roesser realizations: $L(z) = \begin{bmatrix} D & \mathbf{P}(z)B \\ C & \mathbf{P}(z)A - I \end{bmatrix}$ where
  $\begin{bmatrix} D & B \\ C & A \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix}$, $\mathbf{P}(z) = z_1 \mathbf{P}_1 + \cdots + z_d \mathbf{P}_d$
  where $\mathbf{P}_k^2 = \mathbf{P}_k$, $\mathbf{P}_k \mathbf{P}_j = 0$ for $k \neq j$, $\mathbf{P}_1 + \cdots + \mathbf{P}_d = I \Rightarrow$
  $F(z) = D + C(I - \mathbf{P}(z)A)^{-1}\mathbf{P}(z)B$
  (natural for function theory on the polydisk)

The zoo of function classes: Schur class over $\mathbb{D}$

Define $\mathcal{S}_d(\mathbb{C}^n)$ = functions $S\colon \mathbb{D}^d \underset{\text{holo}}{\to} \mathcal{L}(\mathbb{C}^n)$ with $\|S(z)\| \le 1$ for $z \in \mathbb{D}^d$. For $d = 1$ we have

Theorem

Given $S\colon \mathbb{D} \to \mathcal{L}(\mathbb{C}^n)$ TFAE:

(1) $S \in \mathcal{S}_1(\mathbb{C}^n)$

(2) $K_S(z,w) = \frac{I - S(z)S(w)^*}{1 - z\overline{w}}$ is a positive kernel on $\mathbb{D}$:
$\sum_{i,j=1}^N u_i^* K_S(z_i, z_j) u_j \ge 0$ for all u_i 's in $\mathbb{C}^n$, z_i 's in $\mathbb{C}^n$, $N \in \mathbb{N}$

(3) $\exists$ contractive $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathbb{C}^n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathbb{C}^n \end{smallmatrix}\right]$
($\mathcal{X}$ = a Hilbert space) so that $S(z) = D + zC(I - zA)^{-1}B$

The rational Schur class $\mathcal{S}_d^{\rm rat}(\mathbb{C}^n)$ over $\mathbb{D}^d$: the $d = 1$ case

Define: $\mathcal{S}_d^{\rm rat}(\mathbb{C}^n)$ = functions $S \colon \mathbb{D}^d \underset{\rm rat}{\to} \mathcal{L}(\mathbb{C}^n)$ so that $\|S(z)\| \le 1$ for $z \in \mathbb{D}^d$

Theorem

Given $S \colon \mathbb{D} \to \mathcal{L}(\mathbb{C}^n)$ TFAE:

(1) $S = P^{-1}Q \in \mathcal{S}_1^{\rm rat}(\mathbb{C}^n)$

(2) $\exists$ matrix polynomials G_j in $\mathbb{C}^{n \times K_j}[z]$ $(j = 1, 2)$ so that $P(z)P(w)^* - Q(z)Q(w)^* = (1 - z\overline{w})G_1(z)G_1(w)^* + G_2(z)G_2(w)^*$

(3) $\exists$ contractive $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right]$ (i.e., $\mathcal{X} = \mathbb{C}^K$ finite-dimensional) so that $S(z) = D + zC(I - zA)^{-1}B$

The rational inner Schur class $\mathcal{IS}_d^{\rm rat}(\mathbb{C}^n)$ over $\mathbb{D}^d$: $d = 1$

Define: $\mathcal{IS}_d^{\rm rat}(\mathbb{C}^n)$ = functions $S\colon \mathbb{D}^d \underset{\rm rat}{\to} \mathcal{L}(\mathbb{C}^n)$ so that $\|S(z)\| \le 1$ for $z \in \mathbb{D}^d$ and $S(1/z^*)S(z) = I_n$ where $(1/z^*) = (1/\overline{z_1}, \dots, 1/\overline{z_d})$ if $z = (z_1, \dots, z_d)$

Theorem

Given $S(z) = P(z)^{-1}Q(z)\colon \mathbb{D} \underset{\rm rat}{\to} \mathcal{L}(\mathbb{C}^n)$ where Q, P = matrix polynomials with $P(z)$ invertible for $z \in \mathbb{D}$, TFAE:

(1) $S \in \mathcal{IS}_1^{\rm rat}(\mathbb{C}^n)$

(2) $\exists\, K \in \mathbb{N}$ so that
$P(z)P(w)^* - Q(z)Q(w)^* = (1 - z\overline{w})G(z)G(w)^*$ with
$G \in \mathbb{C}^{n \times K}[z]$ a polynomial

(3) $\exists$ unitary $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right]$ with
$S(z) = P(z)^{-1}Q(z) = D + zC(I - zA)^{-1}B$

The case $d > 1$

Theorem

Given $S\colon \mathbb{D}^d \underset{\text{holo}}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $S \in \mathcal{S}_d(\mathbb{C}^n)$

(2a) $\frac{I_n - S(z)S(w)^*}{\Pi_{1 \le k \le d}(1 - z_k \overline{w_k})}$ = positive kernel

(2b) For each $p, q \in \{1, \dots, d\}$ $\exists$ positive kernels $K^I_{p,q}$ and $K^{II}_{p,q}$ on $\mathbb{D}^d$ so that

$I_n - S(z)S(w)^* =$
$(\Pi_{k\colon\, k \neq p}(1 - z_k \overline{w_k}))K^I_{pq}(z, w) + (\Pi_{k\colon\, k \neq q}(1 - z_k \overline{w_k}))K^{II}_{pq}(z, w)$
(Grinshpan–Kaliuzhnyi-Verbovetskyi–Vinnikov–Woerdeman 2009)

(3) Realization formula ?

The Schur-Agler classes $\mathcal{SA}_d(\mathbb{C}^n)$

Define: $\mathcal{SA}_d(\mathbb{C}^n)$ = functions $S\colon \mathbb{D}_d \underset{\text{hol}}{\to} \mathcal{L}(\mathbb{C}^n)$ so that $\|S(T_1, \dots, T_d)\| \le 1$ for all commuting operator tuples $(T_1, \dots, T_d)$ with $\|T_j\| < 1$ for each $j = 1, \dots, d$

Theorem (Agler 1990)

Given $S\colon \mathbb{D}^d \underset{\text{holo}}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $S \in \mathcal{SA}_d(\mathbb{C}^n)$

(2) $\exists$ positive kernels K_j on $\mathbb{D}^d$ so that
$I_n - S(z)S(w)^* = \sum_{j=1}^d (1 - z_j \overline{w_j}) K_j(z, w)$

(3) $\exists$ unitary/contractive $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathbb{C}^n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathbb{C}^n \end{smallmatrix}\right]$ and spectral resolution $\mathbf{P}(z) = z_1\mathbf{P}_1 + \cdots + z_d\mathbf{P}_d$ on $\mathcal{X}$ so that
$S(z) = D + C(I - \mathbf{P}(z)A)^{-1}\mathbf{P}(z)B$

Comparison of $\mathcal{S}_d(\mathbb{C}^n)$ vs $\mathcal{S}\mathcal{A}_d(\mathbb{C}^n)$

Note: In particular, can take $(T_1, \dots, T_d) = (z_1, \dots, z_d) \in \mathbb{D}^d$ in definition of Schur-Agler class $\Rightarrow$ $\mathcal{S}\mathcal{A}_d(\mathbb{C}^n) \subset \mathcal{S}_d(\mathcal{C}^n)$

Corollary of GK-VVW result above: $\mathcal{S}\mathcal{A}_2(\mathbb{C}^n) = \mathcal{S}_2(\mathbb{C}^n)$
(but usually (and correctly) attributed to Andô)

For $d > 2$ known that $\mathcal{S}\mathcal{A}_d(\mathbb{C}^n) \underset{\neq}{\subset} \mathcal{S}_d(\mathbb{C}^n)$

(examples due to Crabb-Davie, Holbrook, Varopoulos)

The rational Schur-Agler class

Define: $S \in \mathcal{SA}_d^{\mathrm{rat}}(\mathbb{C}^n)$ = rational matrix functions $S \colon \mathbb{D}^n \underset{\mathrm{rat}}{\to} \mathcal{L}(\mathbb{C}^n)$ such that $\|S(T)\| \leq 1$ for all commuting tuples $T = (T_1, \dots, T_d)$ of Hilbert space operators with $\|T_j\| < 1$

Define: $S \in \mathcal{SA}_d^{o,\mathrm{rat}}(\mathbb{C}^n)$ = functions in $\mathcal{SA}_d^{\mathrm{rat}}(\mathbb{C}^n))$ with $\|S(T)\| \leq \rho < 1$ for all commuting operator tuples $T = (T_1, \dots, T_d)$ with $\|T_j\| < 1$ for each $j = 1, \dots, d$ for some fixed $\rho < 1$

Results for $\mathcal{SA}_d^{\rm rat}(\mathbb{C}^n)$

Theorem

Given $S = P^{-1}Q \colon \mathbb{D}^d \underset{\rm rat}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $S = P^{-1}Q \in \mathcal{SA}_d^{\rm rat}(\mathbb{C}^n)$

(2) $\exists$ polynomials $G_j \in \mathbb{C}^{n \times K_j}[z_1, \dots, z_d]$ $(0 \le j \le d)$ so that

$$P(z)P(w)^* - Q(z)Q(w)^* = \sum_{j=1}^d (1 - z_j \overline{w_j}) G_j(z) G_j(w)^* + G_0(z) G_0(w)^*$$

Assume that $S = P^{-1}Q \in \mathcal{SA}_d^{o,\rm rat}(\mathbb{C}^n)$ Then

(3) $\exists$ contractive $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right]$ and a spectral resolution $\mathbf{P}(z) = z_1 \mathbf{P}_1 + \cdots + z_d \mathbf{P}_d$ so that

$$F(z) = D + C(I - \mathbf{P}(z)A)^{-1}\mathbf{P}(z)B$$

Conversely, (3) $\Rightarrow S \in \mathcal{SA}_d^{\rm rat}(\mathbb{C}^n)$

Grinspan–Kaliuzhnyi-Verbovetskyi–Vinnikov–Woerdeman

Inner rational Schur class $\mathcal{ISA}_d^{\rm rat}(\mathbb{C}^n)$

Define: $\mathcal{ISA}_d^{\rm rat}(\mathbb{C}^n)$ = functions S in $\mathcal{SA}_d^{\rm rat}$ such that $S(1/\overline{z})^* S(z) = I_n$ where $1/\overline{z} = (1/\overline{z_1}, \dots, 1\overline{z_d})$ if $z = (z_1, \dots, z_d)$

Th (B.–Kaliuzhnyi-Verbovetskyi $\leftarrow$ Agler, Knese, CW)

Given $S = P^{-1}Q \colon \mathbb{D}^d \underset{\rm rat}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $S = P^{-1}Q \in \mathcal{ISA}_d^{\rm rat}(\mathcal{L}(\mathbb{C}^n)$

(2) $\exists$ $N_j \in \mathbb{N}$ and G_j matrix polynomials in $\mathbb{C}^{n \times N_j}[z_1, \dots, z_d]$ $(j = 1, \dots, d)$ so that

$P(z)P(w)^* - Q(z)Q(w)^* = \sum_{j=1}^d (1 - z_j \overline{w_j}) G_j(z) G_j(w)^*$

(3) $\exists$ unitary $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \colon \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathbb{C}^K \\ \mathbb{C}^n \end{smallmatrix}\right]$ and a spectral resolution $\mathbf{P}(z) = z_1 \mathbf{P}_1 + \cdots + z_d \mathbf{P}_d$ so that $S(z) = D + C(I - \mathbf{P}(z)A)^{-1}\mathbf{P}(z)B$

Inner Schur class versus inner Schur-Agler class

Note: $\mathcal{SA}_d(\mathbb{C}^n) \subset \mathcal{S}_d(\mathbb{C}^n) \Rightarrow \mathcal{ISA}_d^{\rm rat}(\mathbb{C}^n) \subset \mathcal{IS}_d^{\rm rat}(\mathbb{C}^n)$

Result of GK-VVW: This last inclusion is strict:

$\mathcal{ISA}_d^{\rm rat}(\mathbb{C}^n) \underset{\neq}{\subset} \mathcal{IS}_d^{\rm rat}(\mathbb{C}^n)$

Herglotz classes over the poly-right half-plane

Define: $\mathcal{H}_d(\mathbb{C}^n)$ = functions $H\colon \Pi^d \to \mathcal{L}(\mathbb{C}^n)$ such that $\Re H(s) \succeq 0$ for $s = (s_1, \dots, s_d) \in \Pi^d$ (Π = open right half plane)

Define: $\mathcal{HA}_d(\mathbb{C}^n)$ = functions $H\colon \Pi^d \to \mathcal{L}(\mathbb{C}^n)$ so that $\Re H(T_1, \dots, T_d) \succeq 0$ whenever $T = (T_1, \dots, T_d)$ is a commutative operator tuple with $\Re T_j \succ 0$ for each $j = 1, \dots, d$

Define: $\mathcal{H}_d^{\rm rat}(\mathbb{C}^n)$ = rational functions in $\mathcal{H}_d(\mathbb{C}^n)$

Define: $\mathcal{HA}_d^{\rm rat}(\mathbb{C}^n)$ = rational functions in $\mathcal{HA}_d(\mathbb{C}^n)$

Double Cayley transform

Recall Cayley transform:

$z \in \mathbb{D} \mapsto s = \frac{1+z}{1-z} \in \Pi$ with inverse $s \in \Pi \mapsto z = \frac{s-1}{s+1} \in \mathbb{D}$

Given $H \colon \Pi^d \to \mathcal{L}(\mathbb{C}^n,$ define double Cayley transform $\mathcal{C}(H) \colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U}^n)$ of H by

$$\mathcal{C}(H)(z) = \left(H(\tfrac{1+z_1}{1-z_1}, \cdots, \tfrac{1+z_d}{1-z_d}) - I_n \right) \left(H(\tfrac{1+z_1}{1-z_1}, \cdots, \tfrac{1+z_d}{1-z_d}) + I_n \right)^{-1}$$

Given $S \colon \mathbb{D}^d \to \mathcal{L}(\mathbb{C}^n)$, then

$$\mathcal{C}^{-1}(S)(s) = \left(I_n + S(\tfrac{s_1-1}{s_1+1}, \cdots \tfrac{s_d-1}{s_d+1}) \right) \left(I_n - S(\tfrac{s_1-1}{s_1+1}, \cdots \tfrac{s_d-1}{s_d+1}) \right)^{-1}$$

Then $\mathcal{C} \colon \begin{cases} \mathcal{H}_d(\mathbb{C}^n) \to \mathcal{S}_d(\mathbb{C}^n) \\ \mathcal{HA}_d(\mathbb{C}^n) \to \mathcal{SA}_d(\mathbb{C}^n) \\ \mathcal{H}_d^{\rm rat}(\mathbb{C}^n) \to \mathcal{S}_d^{\rm rat}(\mathbb{C}^n) \\ \mathcal{HA}_d^{\rm rat}(\mathbb{C}^n) \to \mathcal{SA}_d^{\rm rat}(\mathbb{C}^n) \end{cases}$

and $\mathcal{C}^{-1}$ the reverse

Cayley-inner Herglotz/Herglotz-Agler class

Define: $\mathcal{CIH}_d^{\rm rat}(\mathbb{C}^n)$ = functions in $H \in \mathcal{H}_d(^{\rm rat}\mathbb{C}^n)$ such that $H(-\overline{s}) + H(s) = 0$, where $-\overline{s} = (-\overline{s_1}, \dots, -\overline{s_d})$ if $s = (s_1, \dots, s_d)$

Define: $\mathcal{CIHA}_d^{\rm rat}(\mathbb{C}^n)$ = functions in $\mathcal{HA}_d^{\rm n}(\mathbb{C}^n)$ such that $H(-\overline{s}) + H(s) = 0$

Then also

$$\mathcal{C}\colon \begin{cases} \mathcal{CIH}_d^{\rm rat}(\mathbb{C}^n) \to \mathcal{IS}_d^{\rm rat}(\mathbb{C}^n) \\ \mathcal{CIHA}_d^{\rm rat}(\mathbb{C}^n) \to \mathcal{ISA}_d^{\rm rat}(\mathbb{C}^n) \end{cases}$$

and $\mathcal{C}^{-1}$ the reverse

Schur results ⇒ Herglotz results via Cayley transform

By using double Cayley transform to reduce results concerning Herglotz classes to results concerning Schur classes, we arrive at

Theorem

Given $H\colon \Pi \to \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{H}_1(\mathbb{C}^n)$

(2) $K_H^{\mathcal{H}}(s,t) = \frac{H(s)+H(t)^*}{s+\bar{t}} =$ positive kernel over Π^d

(3) H has a unbounded Bessmertnyĭ long-resolvent representation

$H(s) = L_{11}(s) - L_{12}(s) L_{22}(s)^{-1} L_{21}(s)$

where $L(s) = L_0 + sL_1 = \begin{bmatrix} L_{11}(s) & L_{12}(s) \\ L_{21}(s) & L_{22}(s) \end{bmatrix}$ with $L_0 = -L_0^*$ and $L_1 = L_1^* \succeq 0$

Results for $\mathcal{H}_1^{\mathrm{rat}}(\mathbb{C}^n)$

Theorem

Given $H\colon \Pi \underset{\mathrm{rat}}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{H}_1^{\mathrm{rat}}(\mathbb{C}^n)$

(2) $\exists$ rational $n \times K_j$ matrix G_j $(j = 0, 1)$ so that
$H(s) + H(t)^* = (s + \overline{t}) G_1(s) G_1(t)^* + G_0(s) G_0(t)^*$

(3) Realization formula? (should not be hard: analogue of contractive realization for the Schur case)

Results for $\mathcal{H}_d(\mathbb{C}^n)$

Theorem

Given $H\colon \Pi^d \to \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{H}_d(\mathbb{C}^n)$

(2) For each $1 \le p < q \le d$ $\exists$ positive kernels $K^I_{p,q}$, $K^{II}_{p,q}$ on Π_d so that

$H(s) + H(t)^* =$
$(\prod_{k\colon k \neq p}(s_k + \overline{t_k}))K^I_{p,q}(s,t) + (\prod_{k\colon k \neq q}(s_k + \overline{t_k}))K^{II}_{p,q}(s,t)$

(3) Realization formula?

Characterization of $\mathcal{HA}_d(\mathbb{C}^n)$

Theorem

Given $H\colon \Pi^d \to \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{H}_d(\mathbb{C}^n)$

(2) $\exists$ positive kernels K_j $(1 \le j \le d)$ on Π^d so that
$H(s) + H(r)^* = \sum_{j=1}^d (s_j + \overline{t_j}) K_j(s,t)$

(3) H has a unbounded Bessmertnyĭ long-resolvent representation
$H(s) = L_{11}(s) - L_{12}(s) L_{22}(s)^{-1} L_{21}(s)$

where $L(s) = L_0 + s_1 L_1 + \cdots + s_d L_d = \begin{bmatrix} L_{11}(s) & L_{12}(s) \\ L_{21}(s) & L_{22}(s) \end{bmatrix}$ with
$L_0 = -L_0^*$ and $L_j = L_j^* \succeq 0$ for $1 \le j \le d$

Caveat: Additional technicalities due to possibly unbounded Hilbert space operators with delicate domain issues

B.–Kaliuzhnyi-Verbovetskyi (also Agler–Tully-Doyle–Young)
Connections with Staffans-Weiss theory of well-posed linear systems

Theorem

Given $H = P^{-1}Q \colon \Pi^d \underset{\text{rat}}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{H}_d^{\text{rat}}(\mathbb{C}^n)$

(2) Conjecture? For each choice of $1 \le p < q \le d$ $\exists$ rational matrix functions $G^I_{p,q}$, $G^{II}_{p,q}$, G_0 so that

$H(s) + H(t)^* = (\prod_{k\colon k \neq p}(s_k + \overline{t_k}))G^I_{p,q}(s)G^I_{p,q}(t)^* + (\prod_{k\colon k \neq q}(s_k + \overline{t_k}))G^{II}_{p,q}(s)G^{II}_{p,q}(t)^* + G_0(s)G_0(t)^*$

(3) Realization formula? (Analogue of GK-VVW partial result on existence of contractive realizations for the Schur case?)

Cayley-inner rational Herglotz-Agler class

Theorem

Given $H \colon \Pi^d \to \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{CIHA}_d^{\mathrm{rat}}(\mathbb{C}^n)$

(2) $\exists\ N_j \in \mathbb{N}$ and rational $G_j \in \mathbb{C}^{n \times N_j}(s_1, \dots, s_d)$ so that
$H(s) + H(t)^* = \sum_{j=1}^d (s_j + \overline{t_j}) G_j(s) G_j(t)^*$

(3) H has a finite-dimensional Bessmertnyĭ realization
$H(s) = L_{11}(s) + L_{12}(s) L_{22}(s)^{-1} L_{21}(s)$
with $L(s) = L_0 + L_1 s_1 + \cdots + L_d s_d = \begin{bmatrix} L_{11}(s) & L_{12}(s) \\ L_{21}(s) & L_{22}(s) \end{bmatrix}$
with matrices $L_0, \dots, L_d$ of size $(n+K) \times (n+K)$ such that
$L_0 = -L_0^*$, $L_j = L_j^* \succeq 0$ for $j = 1, \dots, d$

Homogeneous Herglotz classes

Define: $\mathcal{CIH}_d^{\text{hom, rat}}(\mathbb{C}^n)$ = functions H in $\mathcal{CIH}_d^{\text{rat}}(\mathbb{C}^n)$ which are homogeneous: $H(\lambda s) = \lambda H(s)$ for $\lambda \in \mathbb{C}$, $s \in \mathbb{C}^d$

Define: $\mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}^n)$ = functions H in $\mathcal{CIHA}_d^{\text{rat}}(\mathbb{C}^n)$ which are homogeneous

Fake-homogeneous Schur/Schur-Agler classes

Define: $\mathcal{IS}_d^{\text{hom}}(\mathbb{C}^n)$ = functions S in $\mathcal{IS}_d(\mathbb{C}^n)$ such that $H = \mathcal{C}^{-1}(S)$ is in $\mathcal{CIH}_d^{\text{hom}}(\mathbb{C}^n)$

Define: $\mathcal{ISA}_d^{\text{hom, rat}}(\mathbb{C}^n)$ = functions S in $\mathcal{ISA}_d(\mathbb{C}^n)$ such that $H = \mathcal{C}^{-1}(S)$ is in $\mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}^n)$

By definition, $\mathcal{C}: \begin{cases} \mathcal{CIH}_d^{\text{hom, rat}}(\mathbb{C}^n) \to \mathcal{IS}_d^{\text{hom, rat}}(\mathbb{C}^n) \\ \mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}^n) \to \mathcal{ISA}_d^{\text{hom, rat}}(\mathbb{C}^n) \end{cases}$

and $\mathcal{C}^{-1}$ the reverse

Relation between Herglotz homogeneous class and Schur fake-homogeneous class

Theorem (Kaliuzhnyi-Verbovetskyi)

Given $S\colon \mathbb{D}^d \underset{\text{rat}}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $S \in \mathcal{ISA}_d^{\text{hom, rat}}(\mathbb{C}^n)$

(2) S has a finite-dimensional Givone-Roesser realization $S(z) = D + C(I - \mathbf{P}(z)A)^{-1}\mathbf{P}(z)B$ (with $\mathbf{P}(z) = z_1\mathbf{P}_1 + \cdots + z_d\mathbf{P}_d$ a spectral resolution) such that the system matrix $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is self-adjoint and unitary:
$U = U^* = U^{-1}$

Characterization of $\mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}^n)$

Theorem

Given $H \colon \Pi^d \underset{\text{rat}}{\to} \mathcal{L}(\mathbb{C}^n)$, TFAE:

(1) $H \in \mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}^n)$

(2) $\exists$ rational $(n \times K_j)$ matrix functions G_j satisfying $G_j(\lambda z) = G_j(z)$ for $\lambda \in \mathbb{C}$ so that $H(s) = \sum_{j=1}^d s_j G_j(s) G_j(t)^*$ for all $s, t \in \Pi^d$

(3) $H(s) = L_{11}(s) + L_{12}(s) L_{22}(s)^{-1} L_{21}(s)$ with $L(s) = L_1 s_1 + \cdots + L_d s_d$ a homogeneous Bessmertnyĭ matrix pencil $(L_0 = 0)$ with $L_j = L_j^* \succeq 0$ for $j = 1, \ldots, d$

(Corollary of general Bessmertnyĭ result: in general $H(s) = L_{11}(s) + L_{12}(s) L_{22}(s)^{-1} L_{21}(s)$ homogeneous $\Rightarrow L_0 = 0$

$\mathcal{CIH}_d^{\text{hom, rat}}(\mathbb{C}^n)$ versus $\mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}^n)$?

Summary

Known: $\mathcal{IS}_d^{\text{rat}}(\mathbb{C}) \subsetneq \mathcal{ISA}_d^{\text{rat}}(\mathbb{C})$ (GK-VVW 2014)

Application of double Cayley transform $\mathcal{C}$ $\Rightarrow$

$\mathcal{CIHA}_d^{\text{rat}}(\mathbb{C}) \subsetneq \mathcal{CIH}_d^{\text{rat}}(\mathbb{C})$

By definition, $\mathcal{CIHA}_d^{\text{hom, rat}}(\mathbb{C}) \subset \mathcal{CIH}_d^{\text{hom, rat}}(\mathbb{C})$

Open question: Does above hold with $\subsetneq$ or with $=$?

Difficulty: GK-VVW give us examples of functions S in the crack $\mathcal{IS}_d^{\text{rat}}(\mathbb{C}) \setminus \mathcal{ISA}_d^{\text{rat}}(\mathbb{C})$

It remains to find a such an example S (or to show that no such example exists) such that $H = \mathcal{C}^{-1}(S)$ is homogeneous?

Summary continued

Tool for Schur setting: Rudin representation for a multivariable inner function S in $\mathcal{IS}_d^{\rm rat}(\mathbb{C})$ in terms of $\mathbb{D}^d$ stable polynomial denominator

Difficulty for Herglotz setting: Apparently there is no such convenient canonical form for elements H of $\mathcal{CIH}_d^{\rm rat}(\mathbb{C})$

Possible new approach: Characterize $\mathcal{CIH}_d^{\rm hom,\ rat}(\mathbb{C})$ in terms of representation in terms of Koranyi-Pukánszky measure?

Thanks for your attention!