Noncommutative Choquet theory

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joint work with Matthew Kennedy

An operator system is a unital self-adjoint subspace of bounded operators on a Hilbert space H: $1 \in A = A^* \subset \mathcal{B}(H)$.
It has an order and norm structure induced from $\mathcal{B}(H)$.
Moreover $\mathcal{M}_n(A) \subset \mathcal{M}_n(\mathcal{B}(H)) \simeq \mathcal{B}(H^{(n)})$, and this induces a norm and order structure.

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A map $\varphi : A \to \mathcal{B}(K)$ induces maps $\varphi_n : \mathcal{M}_n(A) \to \mathcal{B}(K^{(n)})$ coordinatewise. Say φ is completely positive if φ_n is positive for $n \geq 1$. If φ is unital and completely positive (u.c.p.), then $\|\varphi\|_{cb} = \sup \|\varphi_n\| = 1$.

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(Arveson 1969) Every u.c.p. map $\varphi : A \to \mathcal{B}(K)$ extends to a u.c.p. map of $\mathrm{C}^*(A)$ into $\mathcal{B}(K)$.
(Stinespring 1955) A u.c.p. map φ of a C*-algebra has the form $\varphi(a) = \alpha^* \pi(a) \alpha$ where π is a $*$-repn. and α is an isometry.

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If π is a representation of $\mathrm{C}^*(A)$ such that $\pi|_A$ has a unique u.c.p. extension to $\mathrm{C}^*(A)$, say π has the unique extension property. If π is also irreducible, then π is a boundary representation.

Classical:

$$1 \in A = A^* \subset \mathrm{C}(X) \text{ function system.}$$

$$K = S(A) = \{f : A \to \mathbb{C} : f \geq 0,\ f(1) = 1\} \quad \text{state space.}$$

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NC Theory:

$$1 \in A = A^* \subset \mathcal{B}(H) \text{ operator system}$$

$$\Gamma = S(A) = \coprod_{1 \leq n \leq \kappa} \mathrm{UCP}(A, \mathcal{B}(H_n))$$

where $\dim H_n = n$, and $\kappa \geq \aleph_0$ is a cardinal large enough for all cyclic representations of $\mathrm{C}^*(A)$.

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$$\mathcal{M} = \coprod_{1 \leq n \leq \kappa} \mathcal{M}_n \quad \text{where } \mathcal{M}_n = \mathcal{B}(H_n).$$

Classical: K is convex, weak-$*$ compact.

Γ is nc convex: i.e. closed under direct sums and compressions.

$$x \in \Gamma_n,\ y \in \Gamma_m \implies x \oplus y \in \Gamma_{n+m}$$
$$x \in \Gamma_n,\ \alpha \in \mathcal{M}_{nm} \text{ isometry}, \implies \alpha^* x \alpha \in \Gamma_m.$$

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Equivalently,

$$x_i \in \Gamma_i,\ \alpha_i \in \mathcal{M}_{n_i,n},\ \sum_i \alpha_i^* \alpha_i = 1_n \implies \sum \alpha_i^* x_i \alpha_i \in \Gamma.$$

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Remark: Γ is determined by $\coprod_{n<\infty} \Gamma_n$ but need higher levels.

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$\theta : \Gamma \to \Delta$ is nc affine if

1. $\theta(\Gamma_n) \subset \Delta_n$
2. $\theta(\sum \oplus x_i) = \sum \oplus \theta(x_i)$
3. $\theta(\alpha^* x \alpha) = \alpha^* \theta(x) \alpha$ for α isometry.

$\mathrm{A}(\Gamma)$ is the set of continuous nc affine functions $\theta : \Gamma \to \mathcal{M}$.

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Theorem

$A \simeq \mathrm{A}(\Gamma)$ via $a \to \hat{a}$, $\hat{a}(x) = x(a)$.

Classical: $f \in \mathrm{C}(K)$

nc function: $f : \Gamma \to \mathcal{M}$ is graded, respects $\oplus$, $\mathcal{U}$-equivariant:

1. $f(\Gamma_n) \subset \mathcal{M}_n$
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C*-algebra C, then $C \simeq \mathrm{C}(\mathrm{Rep}(C,H))$ and $C^{**} \simeq \mathrm{B}(\mathrm{Rep}(C,H))$.

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$\mathrm{C}^*_{\max}(A)$ of Kirchberg-Wassermann 1998: universal C*-algebra s.t. every u.c.p. map $x \in \Gamma$ extends to a $*$-repn. δ_x of $\mathrm{C}^*_{\max}(A)$.

Theorem

$\mathrm{C}^*_{\max}(A) \simeq \mathrm{C}(\Gamma)$.

Classical: $x \in K$ has representing measures $\mu \in M(K)_1^+$:

$$\mu(a) = a(x) \quad \text{for} \quad a \in \mathrm{A}(K).$$

and x is the barycenter of μ.

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A representing map for $x \in \Gamma_n$ is $\mu \in \mathrm{UCP}(\mathrm{C}(\Gamma), \mathcal{M}_n(\mathcal{M}))$ such that $\mu|_{\mathrm{A}(\Gamma)} = x$; and x is the barycenter of μ.
By Stinespring, $\mu = \alpha^* \delta_y \alpha$ for $y \in \Gamma_m$ and isometry $\alpha \in \mathcal{M}_{mn}$.
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x has unique representing map iff δ_x is only u.c.p.extension of x.
x is maximal if (y, α) represents $x \implies y = x \oplus z$.

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Proposition

x had unique representing map iff x is maximal.

Theorem (Dritschel-McCullough 2005)

$x \in \Gamma$ has a maximal dilation y.

Classical: Extreme points ∂K of K.

$x \in \Gamma$ is pure if $x = \sum \alpha_i^* x_i \alpha_i \implies \alpha_i^* x_i \alpha_i \in \mathbb{R}x$.
x is extreme if it is pure and maximal (boundary representations).
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NC Krein-Milman theorem inspired by Webster-Winkler 1999.

Theorem

Γ is the closed nc convex hull of $\partial\Gamma$.

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Milman converse.

Theorem

1. If $X \subset \Gamma$ closed
2. $x \in X_n$ and isometry $\alpha \in \mathcal{M}_{mn}$ implies that $\alpha^* x \alpha \in X$
3. and $\overline{\mathrm{ncconv}(X)} = \Gamma$

then $X \supset \partial\Gamma$.

Classical: $f \in \mathrm{C}(K)$ convex.
If $f \in \mathrm{C}(K)$, the convex (lower) envelope is

$$\check{f} = \sup\{a \in \mathrm{A}(K) : a \leq f\} = \bigcap_{a \leq f} \mathrm{Epi}(a).$$

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A multivalued s.a. nc function is upward directed: if $F : \Gamma \to \mathcal{M}_n(\mathcal{M})$, then $F(x) = F(x) + \mathcal{M}_n(\mathcal{M}_p)^+$ for $x \in \Gamma_p$.

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F is nc convex and l.s.c. if Graph(F) is nc convex and closed.
The nc convex envelope of $F : \Gamma \to \mathcal{M}_n(\mathcal{M})$ is defined for $x \in \Gamma_p$ by

$$\overline{F}(x) = \bigcap_{m} \bigcap_{a \le 1_m \otimes F} \{\alpha \in (\mathcal{M}_n(\mathcal{M}_p))_{sa} : a(x) \le 1_m \otimes \alpha\}.$$

$\overline{F}$ is nc convex, l.s.c. and $\overline{F} \le F$.

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Theorem

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This relates the convex envelope to representing maps.

Theorem

If $f \in \mathcal{M}_n(\mathrm{C}(\Gamma))$ and $x \in \Gamma_p$,

$$\overline{f}(x) = \bigcup_{\mu\sim x} [\mu(f), \infty).$$

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Dilation order: $\mu \prec_d \nu$ if

1. (x, α) represents μ
2. (y, β) represents ν, and
3. $\exists_\gamma$ s.t. $x = \gamma^* y \gamma$ and $\beta = \gamma\alpha$.

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Theorem

$\mu(\bar{f}) = \bigcap_{\mu \prec_d \nu} [\nu(f), \infty)$.

This is crucial.

Theorem

$\mu \prec_c \nu$ if and only if $\mu \prec_d \nu$.

Classical: (Choquet 1956) If K is metrizable, each $x \in K$ has a representing measure supported on ∂K.
(Bishop-de Leeuw 1959) Every $x \in K$ has a representing measure pseudo-supported on ∂K, i.e. $\mu(f) = 0$ if f is a Baire function with $f|_{\partial K} = 0$.

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The Baire-Pedersen algebra $\mathfrak{B}(\Gamma)$ is the monotone completion of $\mathrm{C}(\Gamma)$ in $\mathrm{B}(\Gamma)$.

Theorem (nc Bishop-de Leeuw)

If $x \in \Gamma$, then there is a dilation maximal μ representing x.
If $f \in \mathfrak{B}(\Gamma)$ with $f|_{\partial\gamma} = 0$, then $\mu(f) = 0$.

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Theorem (nc Choquet)

If A is separable and $x \in \Gamma$, there is an nc probability measure on $\partial\Gamma$ that represents x.

The end.