Free Complementation of Certain Subalgebras of $L(\mathbb{F}_d)$ via Conditional Transport of Measure

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Banff Oct. 1, 2019

(Tracial) Non-commutative Laws

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Equivalently, an element of $\Sigma_{d,R}$ is a unital, positive, tracial map $\mu : \mathbb{C}\langle X_1, \ldots, X_d \rangle \to \mathbb{C}$ satisfying

$$|\mu(X_{i_1} \ldots X_{i_n})| \leq R^n.$$

This encodes the *non-commutative moments* of some *tuple of non-commutative random variables*.

von Neumann Algebras

Tracial non-commutative laws $\leftrightarrow$ finite W^*-algebras with preferred trace and generators (up to isomorphism).

- $\rightarrow$ GNS construction.
- $\leftarrow$ evaluate moments of your generators.

Question

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- There are *many* non-isomorphic II_1 factors (Murray-von Neumann, McDuff, . . .).
- We don't know whether $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ are isomorphic for $n \neq m$.
- Even after imposing some regularity conditions on the laws (e.g. finite free entropy), we don't necessarily get isomorphic W^*-algebras (example of Nate Brown of a semicircular perturbation of generators of a property (T) factor).

Random Matrices and Free Probability

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Does $\lambda_{X^{(N)}}$ converge in probability to some $\mu \in \Sigma_{d,R}$?

Random Matrices and Free Probability

Example

Let $X^{(N)}$ be Gaussian, with probability density $\sim \exp(-N^2 \sum_j \tau_N(x_j^2))$.

Then $\lambda_{X^{(N)}}$ converges in probability to the law of $(S_1, \dots, S_d)$, where are *freely independent semicirculars*,

that is, S_j has semicircular spectral density $(1/2\pi)\sqrt{4 - x^2}\, dx$ on $[-2, 2]$ and $W^*(S_1, \dots, S_d) = W^*(S_1) * \dots * W^*(S_d) \cong L(\mathbb{F}_d)$.

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Theorem (Voiculescu 1998)

If $X_1^{(N)}$, ..., $X_d^{(N)}$ are independent random matrices (bounded in operator norm), their distribution is unitarily invariant, and the spectral distribution of each $X_j^{(N)}$ converges, then the NC law of $X_1^{(N)}$, ..., $X_d^{(N)}$ converges and they become freely independent in the limit.

Convex and Semi-concave Potentials

Generalizing the Gaussian case, we can consider the random matrix density $\exp(-N^2 V^{(N)}(x))$, where $V^{(N)}(x)$ defined by adding (and/or multiplying!) traces of non-commutative polynomials.

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Theorem (J. 2018, cf. Guionnet & Maurel-Segala 2006, Guionnet & Shlyakhtenko 2009, Guionnet & Shlyakhtenko & Dabrowski 2016)

Let $0 < c < C$. Suppose that $V^{(N)} : M_N(\mathbb{C})_{sa}^d \to \mathbb{R}$ satisfies that $V^{(N)}(x) - (c/2)\|x\|_2^2$ is convex and $V^{(N)}(x) - (C/2)\|x\|_2^2$ is semi-concave. Suppose that $DV^{(N)}$ is well-approximated by trace polynomials $(*)$. Then the NC law of $X^{(N)}$ converge in probability to some non-commutative law, called a free Gibbs law for $V^{(N)}$.

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- Trace polynomials are functions like $(x_1, \dots, x_n) \mapsto x_1 + \tau(x_2)x_1x_2 + 3\tau(x_2x_3)1 - \tau(x_1x_3x_2)\tau(x_3)x_3x_2$.
- We want the approximation to occur uniformly on each operator norm ball, with the error measured in $\|\cdot\|_2$ with respect to τ_N.

Examples

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This theorem covers the following cases:

- If $V^{(N)}$ is a small perturbation of the quadratic $\|x\|_2^2$ by some trace polynomial or analytic function.
- This includes generators of q-Gaussian algebras for q small (Dabrowski 2010, Guionnet & Shlyakhtenko 2014).
- Given free semicirculars $(S_1, \ldots, S_d)$ and self-adjoint NC polynomials p_1, ..., p_d, the law of $S + \epsilon p(S)$ will be such such a free Gibbs law for ϵ small enough (depending on the first and second derivatives of p).

Transport to Free Semicircular Family

Theorem (J. 2019, cf. Guionnet & Shlyakhtenko 2014, Guionnet & Shlyakhtenko & Dabrowski 2016)

The associated von Neumann algebra $W^*(X_1, \dots, X_d)$ is isomorphic to $L(\mathbb{F}_d)$ (the Gaussian case).

- Classically, if a measure μ has a smooth enough density, you can construct a function f by solving some PDE, such that $f_* \mu =$ Gaussian (see e.g. Otto-Villani 2000).

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- Do this for each $\mu^{(N)}$ and get some $f^{(N)}$.
- Argue that $f^{(N)}$ is well-approximated by trace polynomials and has a well-defined large-N limit f (in *some* appropriate space of functions).
- Same for inverse function of $f^{(N)}$.
- Then $(S_1, \dots, S_d) := f(X_1, \dots, X_d)$ are free semi-circular generators, so $W^*(X) \cong L(\mathbb{F}_d)$.

Triangular Transport

Theorem (J. 2019)

There is an isomorphism $\phi : W^*(X_1, \ldots, X_d) \to W^*(S_1, \ldots, S_d) \cong L(\mathbb{F}_d)$ *such that*

$$\phi(W^*(X_1, \ldots, X_k)) = W^*(S_1, \ldots, S_k) \text{ for each } k = 1, \ldots, d.$$

In particular, $W^*(X_1)$ is conjugate to the generator MASA in $L(\mathbb{F}_d)$. So for instance, it is maximal abelian, maximal amenable (due to Popa 1983), freely complemented, etc.

Application and Remarks

This result applies to all the examples listed earlier. In particular, if $(S_1, \ldots, S_d)$ are semicircular, then $S_1 + \epsilon p(S)$ generates a freely complemented MASA for ϵ small enough (p self-adjoint).

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Question (Hayes, Peterson-Thom, Popa)

If $N \subseteq L(\mathbb{F}_d)$ is maximal amenable, then is $L^2(L(\mathbb{F}_d)) \ominus L^2(N)$ a coarse N-N-bimodule? Of course, this would be true if it is freely complemented.

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Question (Popa and others?)

What W^*-algebras can embed into $L(\mathbb{F}_d)$? Does $L(\mathbb{F}_d)$ contain any II_1 factors not isomorphic to $\mathcal{R}$ or $L(\mathbb{F}_t)$ (interpolated free group factors)?

Ideas of Proof

By iteration, the previous theorem can be reduced to the following:

Theorem

Let $V^{(N)}(x, y)$ be a sequence of nice convex potentials as above with $x \in M_N(\mathbb{C})_{sa}^d$ and $y \in M_N(\mathbb{C})_{sa}^{d'}$. Let $W^*(X, Y)$ be the corresponding W^*-algebra of the limiting free Gibbs law. Then $W^*(X, Y) \cong W^*(S) * W^*(Y)$.

Ideas of Proof

- Let $(X^{(N)}, Y^{(N)})$ be the corresponding random variables.
- $X^{(N)}$ has a nice conditional probability distribution given $Y^{(N)} = y$, denoted by $\mu_y^{(N)}$. It is given by $V^{(N)}(\cdot, y)$.

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- Construct $f^{(N)}(x, y)$ such that $f^{(N)}(\cdot, y)$ pushes forward $\mu_y^{(N)}$ to Gaussian.
- Patching together the fibers, $(f^{(N)}(X^{(N)}, Y^{(N)}), Y^{(N)})$ has the same law as $(S^{(N)}, Y^{(N)})$, where $S^{(N)}$ is an independent Gaussian.

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- Show that $f^{(N)}(x, y)$ is a nice function of (x, y) jointly, is well-approximated by trace polynomials, has a large N limit f.

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- Show that $f^{(N)}(x, y)$ is a nice function of (x, y) jointly, is well-approximated by trace polynomials, has a large N limit f.
- In the large N limit, $S^{(N)}$ and $Y^{(N)}$ become freely independent.
- So $W^*(X, Y) = W^*(f(X, Y), Y) \cong W^*(S, Y) = W^*(S) * W^*(Y)$.

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- To get convergence of this iteration scheme, I use some dimension-independent regularity of the solutions to the PDE that relies on the convexity and semi-concavity of $V^{(N)}$.
- Finally, to understand the large N limit, we need an appropriate space of functions ...

A Generalized Functional Calculus

Consider functions $(\mathcal{R}^\omega)_{sa}^d \to L^2(\mathcal{R}^\omega)$ that are bounded on operator norm balls, equipped with the family of seminorms

$$\|f\|_{u,R} = \sup_{\|x\|_\infty \leq R} \|f(x)\|_2.$$

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This is a Fréchet space.

Every trace polynomial f in d-variables defines such a function. Take the closure of these functions in the above Fréchet space and call it $\overline{\mathrm{TrP}}_d^1$.

The L^2-continuous Functional Calculus

Lemma

It makes sense to evaluate $f \in \overline{\mathrm{TrP}}_d^1$ on a self-adjoint tuple in $(\mathcal{M}, \tau)$, provided $\mathcal{M}$ embeds into $\mathcal{R}^\omega$. This evaluation produces an element of $L^2(\mathcal{M}, \tau)$, and it is independent of the choice of embedding.

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Lemma

It makes sense to evaluate $f \in \overline{\mathrm{TrP}}_d^1$ *on a self-adjoint tuple in* $(\mathcal{M}, \tau)$*, provided* $\mathcal{M}$ *embeds into* $\mathcal{R}^\omega$*. This evaluation produces an element of* $L^2(\mathcal{M}, \tau)$*, and it is independent of the choice of embedding.*

Lemma

If $\mathcal{M} = W^*(X_1, \dots, X_d)$*, then every element of* $\mathcal{M}$ *can be realized as* $f(X_1, \dots, X_d)$ *for such an* f *(not unique). We can arrange that* f *is uniformly bounded in operator norm, and uniformly continuous in* $\|\cdot\|_2$*.*

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Lemma

If $\mathcal{M} = W^*(X_1, \dots, X_d)$, then every element of $\mathcal{M}$ can be realized as $f(X_1, \dots, X_d)$ for such an f (not unique). We can arrange that f is uniformly bounded in operator norm, and uniformly continuous in $\|\cdot\|_2$.

Note: This f makes sense to evaluate on any tuple of self-adjoints in $\mathcal{R}^\omega$, not just the original $(X_1, \dots, X_d)$ or those coming from $\mathcal{M}$. In particular, we can still evaluate f on perturbations of X by something outside of $\mathcal{M}$, or on tuples of matrices.

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Properties:

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- These functions are closed under (the large N limit) of convolution with the Gaussian density.
- They are closed under certain algebraic operations.

Role in the Proof

The transport maps in the theorems above are tuples of functions in this space, which are in fact Lipschitz in $\|\cdot\|_2$.

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The large-N limit of functions on matrices is captured by the notion of asymptotic approximation: If $f^{(N)}$ is a function on $M_N(\mathbb{C})_{sa}^d$ and $f \in \overline{\mathrm{TrP}}_m^1$, we say that $f^{(N)} \rightsquigarrow f$ if

$$\forall R > 0, \quad \lim_{N \to \infty} \sup_{\substack{x \in M_N(\mathbb{C})_{sa}^d \\ \|x\|_\infty \leq R}} \|f^{(N)}(x) - f(x)\|_2 = 0.$$

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This asymptotic approximation relation respects all the operations on the previous slide. These operations are used to "build" the solutions to some PDE.

Thanks to the organizers for allowing me to give a talk!

Thank you for your attention!