Women in Analysis (WoAN) - A Research Collaboration Conference for Women
Report on Workshop 19w5082

September 27, 2019

1 Short Overview

Following the successful models of the mathematical communities WIN, WINASc, WiSh, WhAM!, WIT, the one-week collaboration workshop 19w5082 co-organized by Donatella Danielli (Purdue University) and Irina Mitrea (Temple University), was the first international activity of the newly founded Women in Analysis (WoAN) research group. The workshop hosted the following collaborative research teams:

1. **Complex Analysis**
2. **Free Boundary Problems**
3. **Geometric Analysis**
4. **Harmonic Analysis**
5. **Inverse Scattering Theory**
6. **Nonlinear Dispersive Equations**

Each team was led by internationally recognized women experts in these fields. Scientific activities at the workshop included introductory lectures and discussions, collaborative research time, a poster session for junior participants, and wrap-up sessions in which teams reported on their progress. The workshop schedule also included a professional development session. Below we will elaborate on the scientific content of the workshop and the progress registered by the various collaboration teams.

2 Introductory Lectures/Discussions

The goal of these colloquium style lectures and discussions were to introduce *all* workshop participants to the history and general developments in each of the emphasis areas.

- **Complex Analysis**. One of the main themes discussed in the Complex Analysis group was the Hartog's triangle in complex Euclidean space and its corresponding formulation in complex projective space. The Hartog's triangle in $\mathbb{C}^2$ is defined by:

$$H =: \big\{(z, w) \in \mathbb{C}^2 : \ |z| < |w| < 1\big\}.$$

  and is an important example in Several Complex Variables (SCV) as it provides many interesting phenomena in SCV which do not exist in one complex variable. It is the first example of a pseudoconvex domain which does not admit a Stein neighborhood basis. At the same time, H is rectifiable but not Lipschitz. It is also a non-tangentially accessible domain, a recent subject of intense research by many leading harmonic analysts. The specific problems on the Hartog's triangle in Complex Euclidean space discussed during the program were:

1. the density and extension problems in the Sobolev spaces for the Hartog's triangle in complex Euclidean space;
2. boundary integral representation formulas for functions holomorphic in the interior of the Hartog's triangle that satisfy suitable regularity up to the boundary, and related questions concerning the notion of Shilov boundary; theory of holomorphic Hardy spaces, Szegö projection, etc.

The presence of even just a single non-Lipschitz boundary point makes the study of any aspect of boundary behavior of holomorphic functions much more involved than the analysis of their interior behavior, and it accounts for the minimal progress to date in the existing literature for item 2 above.

- **Free Boundary Problems.** Reaction-diffusion systems with strong interaction terms appear in many multi-species physical problems as well as in population dynamics. The qualitative properties of the solutions and their limiting profiles in different regimes have been at the center of the community's attention in recent years. A prototypical example is the system of equations

$$\begin{cases} -\Delta u + a_1 u = b_1 |u|^{p+q-2} u + cp|u|^{p-2}|v|^q u, \\ -\Delta v + a_2 v = b_2 |v|^{p+q-2} v + cq|u|^p |v|^{q-2} v \end{cases}$$

in a domain $\Omega \subset \mathbb{R}^N$ which appears, for example, when looking for solitary wave solutions for Bose-Einstein condensates of two different hyperfine states which overlap in space. The sign of b_i reflects the interaction of the particles within each single state. If b_i is positive, the self interaction is attractive (focusing problems). The sign of c, on the other hand, reflects the interaction of particles in different states. This interaction is attractive if $c > 0$ and repulsive if $c < 0$. If the condensates repel, they eventually separate spatially giving rise to a free boundary. Similar phenomena occurs for many species systems. As a model problem, we consider the system of stationary equations:

$$\begin{cases} -\Delta u_i = f_i(u_i) - \beta u_i \sum_{j \neq i} g_{ij}(u_j) \\ u_i > 0 \ . \end{cases}$$

The cases $g_{ij}(s) = \beta_{ij} s$ (Lotka-Volterra competitive interactions) and $g_{ij}(s) = \beta_{ij} s^2$ (gradient system for Gross-Pitaevskii energies) are of particular interest in the applications to population dynamics and theoretical physics respectively.

The introductory lecture discussed recent advances in the analysis of phase separation phenomena arising in competition-diffusion system. Indeed, phase separation has been described in the recent literature, both physical and mathematical. Relevant connections have been established with optimal partition problems involving spectral functionals. The classification of entire solutions and the geometric aspects of phase separation are of fundamental importance as well. The lecture focused on the most recent developments of the theory in connection with problems featuring:

1. Competition-diffusion problems with fractional laplacians.
2. Competition-diffusion problems with non local interactions.
3. Spiralling solutions in the non symmetrical case.

- **Geometric Analysis**. The field of geometric flows has been thriving in the past few decades because of its powerful applications to topology, geometry, analysis, and general relativity. In many applications, it is important to understand how the flow could continue after a singular time, by a better understanding of singular formation, which is the focus of our discussion during the workshop.

Let (M, g_0) by a compact Riemannian manifold without boundary. A solution to the Ricci flow is a family of metrics $\{g(\cdot,t)\}$ on M satisfying the deformation $\frac{\partial g}{\partial t} = -2\text{Ric}$ where Ric is the Ricci curvature of $g(\cdot,t)$ with $g(\cdot,0) = g_0$. We say that T is a singular time if there is a sequence of points $p_k \in M$ and a sequence $t_k \to T$ such that

$$Q_k = |\text{Rm}|(p_k, t_k) = \max_{M\times[0,t_k]} |\text{Rm}| \to +\infty \quad \text{as } k \to \infty.$$

A singular model is the limiting metric $g_\infty = \lim_{k\to+\infty} g_k$ where $g_k(\cdot,t) = Q_k g(\cdot, t_k + tQ_k^{-1})$ are the appropriate rescaling metrics corresponding to $\{p_k\}$. If the blow-up rate of Q_k is sublinear in $T - t_k$, which in particular implies it must be linear in $T - t_k$, the singularity is called Type I. All other singularities are called Type II.

We now turn to the mean curvature flow, which is an extrinsic geometric flow that deforms hypersurfaces in $\mathbb{R}^{n+1}$. Let M be a complete hypersurface in $\mathbb{R}^{n+1}$ and let $F(x,t) : M \times [0,\epsilon) \to \mathbb{R}^{n+1}$ be a family of immersions parametrized by t with $F(\cdot,0) = M$. The mean curvature flow is a solution F whose speed of deformations is given by the mean curvature vector at each instant time, that is, $\frac{\partial F}{\partial t} = -H\nu$, where ν is the outward unit normal to $M_t = F(\cdot,t)$ and the mean curvature is $H = -\text{div}_{M_t}\nu$. If M is compact, the mean curvature flow must stop at a finite time by avoidance principle. One can similarly define the singular models and types as for the Ricci flow.

There are many similarities between the two flows, they are both gradient flows and in both flows the monotonicity formula has been discovered, in the mean curvature flow by Huisken and in the Ricci flow by Perelman. Those monotonicity formulas play important role in singularity analysis in both flows. Three main topics that the Geometric Analysis groups is interested in pursuing are (1) Ricci flow solutions with degenerate neck-pinches, (2) Stability of cylindrical solutions to the Ricci flow, and (3) Stability of translating solutions to the mean curvature flow.

- **Harmonic Analysis.** Pattern identification in sets has long been a focal point of interest in geometry, combinatorics and number theory. No doubt the source of inspiration lies in the deceptively simple statements and the visual appeal of these problems. A few prototypical examples discussed in the introductory part of the workshop where:

  1. *A set occupying a positive proportion of the natural numbers contains arbitrarily long arithmetic progressions (AP-s).* This affirmative answer by Szemerédi to the famous Erdös-Turán conjecture is one of the masterpieces of modern mathematics [24, 14], and a trendsetter in this field. More generally, given a set $A \subseteq \mathbb{Z}^d$ with positive density, i.e.,

  $$\limsup_{N\to\infty} \frac{\#\left(A \cap [-N,N]^d\right)}{\#\left([-N,N]^d\right)} = d(A) > 0, \qquad \text{where } \#(A) = \text{cardinality of } A,$$

  and any finite configuration $S \subseteq \mathbb{Z}^d$ (say a polytope), A has infinitely many $\mathbb{Z}$-affine copies of S.

  2. Contrast this with a problem in the Euclidean setting. A set $S \subset \mathbb{R}$ is said to be *universal* if every set of positive Lebesgue measure contains an affine copy of S. A classical theorem of Steinhaus shows that all finite sets are universal. A famous question of Erdös [12] asks: *does there exist an infinite universal set*? Despite its superficial analogy with Szemerédi-type questions in $\mathbb{Z}^d$, this problem remains unsolved. All results to date merely establish that certain infinite structures are non-universal [13, 6, 16]. In particular, we do not even know if $\{2^{-n} : n \geq 1\}$ is universal.

  3. Alternatively, one could ask for a set containing *all sufficiently large copies* instead of *infinitely many affine copies.* A result of Bourgain [5] says that if $A \subseteq \mathbb{R}^d$ has positive

upper density, i.e.,

$$\limsup_{R\to\infty} \frac{|A \cap B_R|}{|B_R|} = \delta(A) > 0, \quad \text{where } |\cdot| = \text{Lebesgue measure}, \; B_R = \{x \in \mathbb{R}^d : |x| < R\},$$

and $S \subseteq \mathbb{R}^d$ is any set of d points spanning a $(d-1)$-dimensional hyperplane (e.g. a line in $\mathbb{R}^2$ or a triangle in $\mathbb{R}^3$), then there exists ℓ_0 such that A contains an isometric copy of ℓS for every $\ell > \ell_0$. The corresponding statement when $\#(S) > d$ is not known, though there are some partial results [25]. For instance, we do not know if a set in $\mathbb{R}^3$ of positive upper density contains all sufficiently large regular tetrahedra.

These problems share the common feature that they aim to identify patterns in thick sets. There is now an immense variety of results in this genre, asserting existence or avoidance of configurations under assumptions on size, often stated in terms of measure, dimension or density. While this body of work has contributed significantly to our understanding, a complete picture is yet to emerge. Not surprisingly, such questions are nontrivial when posed for a thin set whose content is insignificant when measured on some of these scales.

- **Inverse Scattering Theory**. The introductory lecture/discussion was concerned with inverse scattering, i.e., inverse problems for linear hyperbolic partial differential equations which model sound, electromagnetic or elastic waves. We discussed in particular setups where a collection (array) of sensors probes a heterogeneous medium with signals and measures the resulting wave. The goal of the inverse scattering problem is to process these measurements in order to determine the heterogeneous medium, the so-called reflectivity function.

  The introductory lecture considered such a problem, for the case of broadband probing signals and time resolved array measurements, at regular time sample intervals T. It discussed a novel reduced order modeling (ROM) strategy, where the reduced order model is a proxy of the wave propagator, which is the operator that takes the wave at a given time t and maps it to the wave at the next time step $t + T$. The ROM has the following important properties:

  1. It is data driven, meaning that it can be obtained just from the array measurements, without any knowledge of the medium.
  2. It is a matrix of size determined by the number of sensors in the array and the duration of the measurements and yet, it fits the array measurements exactly.
  3. The ROM corresponds to a Galerkin projection of the wave propagator operator on an approximation space that is spanned by the wave field at the sample time instants (so called solution snapshots).
  4. The ROM is a matrix with special algebraic structure that allows an efficient (well conditioned, almost linear) inversion procedure for determining the unknown reflectivity of the medium.

  The lecture described the construction of the ROM, proved its main properties and showed how it can be used for solving inverse scattering. The theoretical analysis of the ROM based inversion requires further analysis, which is why it was presented at the beginning of the conference.

- **Nonlinear Dispersive Equations**. The introductory efforts of this group were focused on presenting recent results on the short and long time dynamics of solutions to nonlinear Schrödinger equations (NLS). The Schrödinger equation is arguably the most famous one in the class of dispersive nonlinear equations, and it plays a fundamental role in quantum mechanics.

The team leaders emphasized the striking difference between the behavior of solutions to the NLS when no boundary data are imposed, hence wave solutions can *disperse*[1] without encountering any obstacle, versus when boundary data, such as the periodic ones, are given as a constraint. In this case dispersion does not happen, and to the contrary, the wave solutions may be periodic in time, a situation that happens for example in 2d, when the ratio between the two periods is a rational number. The leaders reported on the most recent advances in the study of periodic solution to the NLS equation, while emphasizing the many different mathematical tools, taken for example from analytic number theory, probability, dynamical systems, symplectic geometry and more, that have been used to make this progress. An idea of how these tools have been used and a list of open questions were also presented during the lecture.

3 Team Progress Reports

A summary description of the specific problems attacked by the research groups participating in the workshop is as follows.

- **Complex Analysis**. With the participation and involvement of the Harmonic Analysis group, led by Almut Burchard and Malabika Pramanik, during the workshop significant progress was made towards the solution of the aforementioned density and extension problems in the Hartog's triangle. Though details are yet to be verified, it seems that recently obtained results in harmonic analysis can be employed to resolve the questions raised in the density and extension problems in the Sobolev spaces for the Hartog's triangle in complex Euclidean space. Group discussions lead by Loredana Lanzani ignited a new and ongoing collaboration involving Anne-Katrine Gallagher; Purvi Gupta; Loredana Lanzani and Liz Vivas. Progress has already been made on several among the questions raised on the boundary integral representation formulas for functions holomorphic in the interior of the Hartog's triangle that satisfy suitable regularity up to the boundary.

  Further activities at the workshop included a follow-up discussion led by Mei-Chi Shaw on the Hartog's triangle in complex projective space, which exhibits distinct features from its Euclidean counterpart due to the presence of positive curvature. In this context several questions were raised pertaining complex foliations, which are equivalent to limit cycles in the real setting; also the question of existence of Levi-flat hyper surfaces without singularities, which arises from complex foliation theory and is of interest to the complex dynamics, topology and geometry communities.

- **Free Boundary Problems**. The following problem has been introduced to the junior team members by Susanna Terracini. A classic free boundary problem arises by considering a model of segregated populations. Suppose that there are N segregated populations occupying a bounded domain Ω. The optimal space occupied by each population is represented by the positivity set of the respective function u_i, for $i \in \{1, 2, \dots, N\}$, and where the u_i minimize the following energy:

$$\mathcal{E} = \min \Big\{ \int_\Omega \sum_{i=1}^{N} |\nabla u_i|^2 : u_i|_{\partial\Omega} = \varphi_{u_i} \text{ and } u_i u_j = 0 \text{ for } i \neq j \Big\},$$

  under a boundary condition induced by given functions φ_{u_i} on $\partial\Omega$. A major problem is to understand the free boundary, which in this case is the boundary between the different

[1] In this case *dispersion* means that the amplitude of the wave tends to zero as the time tends to infinity, while the energy of the system remains constant.

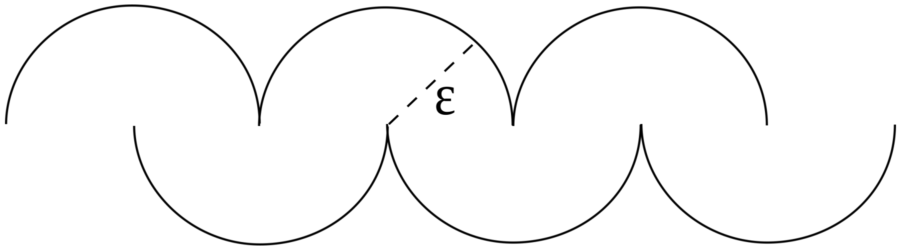

Figure 1: A potential buffer region between two populations that are ε apart.

populations within Ω. To understand the free boundary, it is useful to look at the domain variation formula. For $Y \in C_0^\infty(\Omega)$ let

$$\begin{cases} \dot{\Phi}_t = Y(\Phi_t) \\ \Phi_0(x) = x. \end{cases}$$

be a local variation. Then the domain variation formula is found by computing

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}(\Phi_t(\Omega)).$$

From the expansion of the first variation one may recover a reflection law which describes the symmetry with which two functions u_i and u_j approach their common boundary, and using Almgren's monotonicity formula, further regularity results, including that the boundary is mostly regular outside of a small singular set which is small in the sense of dimension.

A related problem is to understand segregated populations in the case where there is a buffer zone between the populations. In this case one minimizes:

$$\mathcal{E}_\epsilon(u) = \min\left\{ \int_\Omega \sum_{i=1}^N |\nabla u_i|^2 : u_i|_{\partial\Omega} = \varphi_{u_i} \text{ and } \operatorname{dist}(\operatorname{spt}\{u_i\}, \operatorname{spt}\{u_j\}) \geq \varepsilon \text{ for } i \neq j \right\}$$

where spt is the support of the given function.

Progress on this problem has been made in the works of Soave, Tavares, Terracini, and Zilio (2018) and of Caffarelli, Patrizi, and Quitalo (2017). More work is required to get regularity of the free boundary because they need to address shapes like the following: suppose that you have two populations in a cylinder, where the boundary takes a scalloped shape for each population formed by semi-circles of radius ε connected in a line, and offset so that the boundaries are constantly ε apart. While this set does have singular set of small dimension, it is expected that one should be able to show that this arrangement is not a candidate even to be a local minimizer, perhaps by using the domain variation formula. This problem is interesting for fixed $\varepsilon = 1$, or as a variational problem where ϵ is sent to 0, so that the potentially invalid arrangement approaches a minimizer for the classic segregation problem in the limit.

- **Geometric Analysis**. The team reported work in the following directions.

  1. **Ricci flow solutions with degenerate neck-pinches**. For Ricci flow, a family of "dumbbell" initial metrics leads to two drastically different singular models: either spherical or non-degenerate cylindrical singular model, depending on the ratio between the radius of the neck and the radii of two balls of the initial dumbbell manifold. A dumbbell metric at the threshold ratio gives rise to the so-called *peanut solution*, which was shown to exist rigorously by Angenent, Isenberg, Knopf in 2015. The peanut solution gives two different singular models, depending on the choice of the sequence of points

p_k. If $\{p_k\}$ goes to the neck region, the resulting singular model is degenerate cylindrical and if $\{p_k\}$ goes to the ball region, the resulting singular model is the Bryant soliton. Such peanut solution is expected to be an unstable solution to the Ricci flow, so it is in general difficult to construct. Our ultimate goal is to show that the peanut solution is unstable. A closely related counterpart is the following question.

Problem 1. *Show that a doubly warped Berger metric on $S^1 \times S^3$ gives rise to a Ricci flow solution similar to the peanut solution, in the sense that its normalized Ricci flow converges to a normalized peanut solution.*

The group discussed how the peanut solution (with rotational symmetry) is constructed in the original two papers of Angenent, Isenberg, and Knopf. Their approach suggests to first derive a formal expansion of the warping factors of the metrics and then show the formal solution exists.

2. **Stability of cylindrical solutions to the Ricci flow**. As discussed above, the cylindrical solution to the Ricci flow appears naturally as a singular model. It gives rise to the natural question of how "stable" a cylindrical solution is.

   Problem 2. *Let g_0 be the standard cylindrical metric defined on $\mathbb{R} \times S^3$. Show that there is $\epsilon > 0$ so that if g is in an ϵ-neighborhood of g_0 (in a suitable topology), then the Ricci flow solution of g converges to the solution of g_0, in the sense that the normalized solution converges to g_0, after a diffeomorphism change.*

   The group discussed the first step to understand the long time existence of the normalized solution $\tilde{g}_t$. It relies on analyzing the linearized normalized Ricci flow equation and employing the de Turck trick to eliminate the diffeomorphism group. The second step is to provide estimates on $\tilde{g}_t$ and show that up to a diffeomorphisms the flow converges to a normalized cylindrical solution as $t \to \infty$. We plan to understand the work of Schnürer, Schulze, and Simon where they prove stability of Euclidean space, as well as hyperbolic space, under the Ricci flow.

3. **Stability of translating solutions to the mean curvature flow**. Let M be a hypersurface in $\mathbb{R}^{n+1}$. We say that M is a translating solution if $F : M \times [0, \epsilon)$ satisfies $\frac{\partial F}{\partial t} = -w$ for a w is a constant vector in $\mathbb{R}^{n+1}$, where F a family of immersions parametrized by t. Comparing with the mean curvature equation discussed above, a translating solution to the mean curvature flow, sometimes called the translator, must satisfy $H = \langle \nu, w \rangle$. Much progress has been made to classify translators. For surfaces in $\mathbb{R}^3$ that are contained in a slab $(-\pi/2, \pi/2) \times \mathbb{R}^2 \subset \mathbb{R}^3$, there are only three types of translators: Bowl solitons, Grim reaper planes, and the delta-wing solutions.

   Problem 3. *Let M_0 be a graphical surface in $\mathbb{R}^3$ defined on a slab $(-\pi/2, \pi/2) \times \mathbb{R} \subset \mathbb{R}^3$. Suppose that M_0 is asymptotic to the two vertical planes that bound the slab. Then the mean curvature flow M_t converges to either grim reaper plane or the delta-wing solution.*

   The team first discussed the long time existence of M_t. It relies on a general interior gradient estimate of Ecker and Huisken and the gradient estimate toward the boundary in the recent work of Spruck and Xiao. We believe that the pancake solutions of Bourni, Langford, and Tinaglia would give barriers to guarantee that the solution M_t stays in the same slab and converges to a nontrivial solution. Last, to show that the solution converges to a translating solution, the group may employ the techniques in the recent work to Choi, Choi, and Daskalopoulos for the Gauss curvature flow.

- **Harmonic Analysis.** The introductory harmonic analysis discussions provided an overview of results concerning geometric and analytic configurations that exist in sparse sets in Euclidean spaces. Here sparsity implies zero Lebesgue measure and size is phrased in terms of finer notions such as Hausdorff or Fourier dimensions, occasionally with additional structures. The background literature on a few questions of the following flavor were discussed in the group meetings:

1. Does a set in $\mathbb{R}$ with dimension $\alpha < 1$ contain algebraic patterns, such as arithmetic progressions, or solutions of a translation-invariant linear equation? If so, are such patterns abundant in some quantifiable sense?
2. Does a sparse set in $\mathbb{R}^2$ contain geometric configurations such as vertices of a right triangle?

Looking ahead, one can formulate more refined questions. Lower-dimensional surfaces such as curves and hypersurfaces yield a class of thin sets in $\mathbb{R}^d$ for $d \geq 2$ that arise naturally from the differential geometry of Euclidean spaces. The induced surface measure on such sets is rich in geometric and analytic structure. It is the source of a vast literature and the inspiration of the following genre of questions, the study of which is one of the long-term research goals of the harmonic analysis group:

1. For $d \geq 1$ and $0 < \alpha < d$, $\alpha \notin \mathbb{Z}$, do there exist sparse subsets of $\mathbb{R}^d$ with Hausdorff dimension α supporting measures that behave in some quantifiable sense like the induced Lebesgue measure on surfaces in $\mathbb{R}^d$?
2. What properties of fractal sets ensure/prevent analytic and geometric phenomena seen on manifolds?
3. What are the scope and the limitations of Fourier-analytic methods in such problems?

The group also discussed four problems on sumsets and convolutions. The *Minkowski sum* $A + B = \{a + b \mid a \in A, b \in B\}$ arises frequently in the study of convolution operators. It is typically large compared to the individual sets; the general principle is that small sumsets imposes strong geometric and arithmetic constraints on the sets. We propose four problems that seek to quantify these constraints.

For non-empty sets $A, B \subset \mathbb{R}^d$, the *Brunn-Minkowski* inequality $|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$ provides a fundamental lower bound on the volume of the sumset. Equality holds only if the sets A and B are scaled and translated copies of the same convex set K [Henstock-Macbeath 1953, Hadwiger-Ohmann 1956].

1. If the Brunn-Minkowski inequality holds with near-equality for two sets A, B, must they be close to homothetic and convex? (How close?)

Affirmative answers are known in certain cases, for example when A and B are convex, when the two sets are comparable in volume, and in one dimension [Christ, Figalli, Jerison, Maggi, Pratelli and others since 2010]. The general problem, for sets of disparate size, is open.

Much of the recent progress on this problem is motivated by additive combinatorics, where the study of sets with small sumsets has been a core problem for almost 100 years. For finite sets of integers, the cardinality of $A+B$ can be as large as the product $|A| \cdot |B|$; a lower bound is $|A + B| \geq |A| + |B| - 1$. Equality occurs only if A and B are arithmetic progressions with the same increment. Large sets with small sumsets were characterized in terms of generalized arithmetic progressions by Freiman [1973] and Ruzsa [1994].

2. Among subsets $C \subset \mathbb{N}$ of given cardinality, which have the largest number of decompositions as sumsets (modulo translations)?

One may suspect that arithmetic progressions may be the answer also here. there is a surprising connection with *lunar arithmetic*, an exotic algebra on the nonnegative integers [Applegate-LeBrun-Sloane 2011, G. Gross 2019].

One of the functional versions of the Brunn-Minkowski inequality is the *Riesz-Sobolev* inequality, that convolution integrals of the form $\int_{\mathbb{R}^d} f * gh$ can only increase under symmetric decreasing rearrangement of the three functions. The characterization of equality cases is quite complicated and depends on the relative size of the level sets of the three functions [Burchard 1994]. The Brunn-Minkowski inequality is equivalent to the special case of indicator functions of sets that are in a critical size relation. Analogous inequalities hold on the integers [Hardy-Littlewood-Polya 1934] and on the unit circle S^1 [Baernstein 1989]. Equality and near-equality cases on $\mathbb{R}$ and S^1 were classified by Christ [2013] and Christ-Iliopoulou [2018].

3. Are there any other groups that admit rearrangement inequalities of Riesz-Sobolev type? What are the obstructions?

It is a folklore result that Riesz-Sobolev inequalities *cannot* hold on the special orthogonal groups $SO(d)$ for $d > 2$, and perhaps on no other groups. It would be interesting to work out a rigorous proof and identify the precise nature of the geometric obstruction. Part of the question is how to define a suitable rearrangement. One possible approach is to construct the rearrangement in a different space. For example, S^1 has proved useful as a comparison space for compact connected Abelian groups [Kneser 1956, Candela-De Roton 2016, Tau 2018, Christ-Iliopoulou 2018].

We close with an intriguing problem that connects additive combinatorics with the geometry of Banach spaces, due to Oleskiewicz [2016]. To state the question, let us call a subset of a metric space *well-separated*, if any two distinct points in the set have distance at least 1.

4. Let A, B be non-empty finite subsets of a normed vector space. If A and B are well-separated, does $A+B$ contain a well-separated subset C of cardinality $|C| = |A|+|B|-1$?

The answer is known to be positive when the norm is Euclidean [Oleskiewicz 2016], and when one of the sets has only one or two elements. It is open in all other cases, even for the spaces ℓ^p with $p \neq 2$ in dimension two.

- **Inverse Scattering Theory**. The team reported on two research problems:

  1. **Reduced order model**: The ROM based inversion methodology was discussed by our group throughout the week, for both the setup in the lecture as well as for inversion with time harmonic waves and in anisotropic media. The group identified a few analysis problems to work on. In particular, the study of the dependence of the Galerkin projection on the unknown reflectivity was determined to be important and of interest to the group.
  2. **Transmission eigenvalues**: This problem arises in the analysis of scattering operator for inhomogeneous media of compact support. It is a non-selfadjoint and non-linear eigenvalue problem for a set of two elliptic PDEs defined in the support of inhomogeneity and sharing the same Cauchy data on the boundary. Transmission eigenvalues relate to interrogating frequencies for which there is an incident field that does not scatter. They can be determined from scattering data, hence can be used to obtain information about scattering media. Our group was interested in two main open theoretical questions: a) spectral properties of this eigenvalue problem in the case when contrast in the media changes sign up to its boundary, and b) regularity assumptions on the given media for which a transmission eigenvalue is indeed a non-scattering frequency, i.e. understanding when it is possible to extend eigenfunctions corresponding to incident waves outside the support of inhomogeneity as a solution of PDEs governing the background.

- **Nonlinear Dispersive Equations**. During the week spent at Banff the Nonlinear Dispersive PDE group centered their discussion mainly on two projects.

  1. The first project, that involves D. Mendelson, A. Nahmod, N. Pavlovic and G. Staffilani, is related to the active area of research of deriving effective evolution equations from many-body quantum systems. One important and ubiquitous example of a limiting effective equation is the NLS equation mentioned above, which more in details is a scalar dispersive equation which describes in a certain regime the evolution of a system of infinitely many bosons with a two particle interaction. The NLS equation is an important model in its own right, as both a representative example of an infinite dimensional Hamiltonian system, and, in the one dimensional cubic case, an example of an integrable PDE. Recently, our group together with M. Rosenzweig, a PhD student at UT Austin under the supervision of N. Pavlovic, has been able to derive geometric aspects of the Hamiltonian structure of the NLS equation from the many-body quantum model. Moreover, we have been able to connect the integrability of the scalar NLS equation with integrability for a certain system of equations, called the GP hierarchy, which models the interaction between infinitely many quantum particles and arises in the derivation of the NLS equation from the finite particle models.

     While at the workshop at BIRS, the group discussed several possible extensions to this recent work. The first direction, which seems very promising, is related to the connection between the NLS equation and the Vlasov equation. Specifically, in the so-called semi-classical limit, solutions of the NLS equation tend to solutions of the Vlasov equation. A natural question is thus "what happens to the geometric Hamiltonian structure associated to the NLS?". We believe, after some preliminary investigation, that our techniques should enable us to derive a Hamiltonian structure for the Vlasov equation from a *classical* finite particle system.

  2. The second project, which involves M. Czubak, A. Nahmod, G. Staffilani and X. Yu, is based on a question proposed by Luis Vega. Consider the following NLS equation in one spatial dimension:
     $$iu_t + u_{xx} = |u|^8 u. \tag{3.1}$$
     Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a complex-valued function of time and space and the scaling invariant Sobolev norm is $H^{\frac{1}{4}}$. The goal is to study the long time dynamics for this initial value problem, namely global well-posedness (GWP) and scattering when the initial data are taken in $H^{\frac{1}{4}}$.

     It is known that given an initial data u_0 with finite energy, that is $u_0 \in H^1$, due to energy conservation, and the fact that H^1 is a subcritical norm in this case, GWP and scattering is well understood, while at the critical regularity, $u_0 \in \dot{H}^{\frac{1}{4}}$, a similar argument only gives *small* data GWP and scattering results. If one considers intermediate (but still subcritical) regularity, that is $u_0 \in H^s$, $\frac{1}{4} < s < 1$, then one can combine the I-method and the following Morawetz estimate for the solution u of (3.1):
     $$\|u\|^8_{L^8_{t,x}} \lesssim \|u_0\|^6_{L^2_x} \|u\|^2_{L^\infty_t \dot{H}^{\frac{1}{2}}_x} \tag{3.2}$$
     to prove the GWP and scattering for data $u_0 \in H^s$, $s > \frac{8}{11}$. During the discussion at Banff X. Yu noticed that if we replace the Morawetz estimate (3.2) by the one derived in Planchon-Vega [21]:
     $$\|u\|^{12}_{L^{12}_{t,x}} \lesssim \|u_0\|^2_{L^2_x} \|u\|^2_{L^\infty_t \dot{H}^{\frac{1}{2}}_x}, \tag{3.3}$$
     we are able to improve the index for GWP and scattering to $s > \frac{4}{7}$. Note that this index still leaves a gap if one want to reach the critical regularity. Following a suggestion by L.

Vega we would like to consider this GWP and scattering problem from another point of view, which would be new also for other dispersive equations. Instead of concentrating the study of the global dynamics for the NLS equation (3.1) on the analysis of the behavior in time of the solution in terms of Sobolev and L^p norms, we want to analyze the asymptotic behavior in time of the more natural $h(t)$ quantity defined in the proof of Planchon-Vega [21]:

$$h(t) := \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,x)|^2 |u(t,y)|^2 |x-y| \, dxdy. \tag{3.4}$$

The second derivative of $h(t)$ in essence gives the Morawetz estimates, which encodes the fact that some L^p norm is decaying in time, which in turn implies scattering. Therefore this $h(t)$ function is closely linked to the scattering of the NLS equation (3.1), and it should be the right quantity to look at to prove scattering.

4 Poster Session Research Themes

The following scientific themes have been discussed at the poster session.

- **A generalized radial Brèzis-Nirenberg problem**. In a poster presentation by Soledad Benguria, Mathematics Department, University of Wisconsin the following problem was discussed. In an 1983 paper Brèzis and Nirenberg consider

$$-\Delta u = \lambda u + u^p \qquad \text{in} \qquad \Omega, \tag{4.5}$$

where Ω is a bounded, smooth, open subset of $\mathbb{R}^n$, $n \geq 3$, with $u > 0$ in Ω and $u = 0$ in $\partial\Omega$. Here $p = (n+2)/(n-2)$ is the critical Sobolev exponent. They show there are no positive solutions to (4.5) if $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω. And if Ω is star-shaped, there are no solutions if $\lambda \leq 0$. However, the existence of solutions for $0 < \lambda < \lambda_1$ depends on the dimension of the space.

In fact, if $n \geq 4$, then there is a solution $u \in H_0^1(\Omega)$ for all $\lambda \in (0, \lambda_1)$. But if $n = 3$, there is a positive $\lambda_*(\Omega)$ such that (4.5) has no solution if $\lambda \leq \lambda_*$, and (4.5) has a solution if $\lambda \in (\lambda_*, \lambda_1)$. If Ω is a ball, then $\lambda_* = \lambda_1/4$. Many variants of (4.5) have been studied. Among others, the Brèzis-Nirenberg problem in other spaces of constant curvature, such as $\mathbb{S}^n$ and $\mathbb{H}^n$ (see, e.g., [3], [22], [8], [4]).

The problem that Benguria and her collaborators attacked is as follows. Let $R \in (0, \infty)$ and let a be a smooth function such that $a \in C^3[0, R]$; $a(0) = a''(0) = 0$; $a(x) > 0$ for all $x \in (0, R)$; and $\lim_{x\to 0} a(x)/x = 1$. Given $n \in (2, 4)$, the goal is to study the existence of positive solutions $u \in H_0^1(\Omega)$ of

$$-u''(x) - (n-1)\frac{a'(x)}{a(x)}u'(x) = \lambda u(x) + u(x)^p \tag{4.6}$$

with boundary condition $u'(0) = u(R) = 0$. Notice that the radial Brèzis-Nirenberg problem on the Euclidean space corresponds to taking $a(x) = x$; on the hyperbolic space, to taking $a(x) = \sinh(x)$; and on the spherical space, to taking $a(x) = \sin(x)$. Benguria shows that this boundary value problem has a positive solution if $\lambda \in (\mu_1, \lambda_1)$. Here, λ_1 is the first positive eigenvalue of $y'' + \frac{a'}{a}y' + \left(\lambda - \alpha^2 \left(\frac{a'}{a}\right)^2 + \alpha\frac{a''}{a}\right) y = 0$ with boundary conditions $\lim_{x\to 0} y(x)x^{\alpha} = 1$; and μ_1 is the first positive eigenvalue with boundary conditions $\lim_{x\to 0} y(x)x^{-\alpha} = 1$, with $\alpha = (2-n)/2$. She also obtains non-existence and uniqueness results.

- **Aharonov–Bohm operators in planar domains.** The poster by Laura Abatangelo (University of Milan - Bicocca) concerned possible multiple eigenvalues for the so-called Aharonov–Bohm operators. These operators are special as they present a strong singularity at a point (pole), for they cannot be considered small perturbations of the standard Laplacian. More precisely, for $a = (a_1, a_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, we consider the vector potential

$$A_a^\alpha(x) = \alpha\left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\},$$

which generates the Aharonov-Bohm delta-type magnetic field in $\mathbb{R}^2$ with pole a and circulation α; such a field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1, x_2) at the point a, as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to α. So, they are responsible of the so-called Aharonov–Bohm effect: in a quantum mechanics context, a charged particle living in this region is affected by the presence of a magnetic field even if this is zero almost everywhere.

From an analytic point of view, the particle's dynamics is described by solutions to Schrödinger equations, where the (stationary) operators are defined as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u$$

acting on functions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$. As one can easily understand, in the last years, a particular interest has been devoted to the spectrum of the stationary operator defined above. In particular, the spectrum is stable under small movement of the pole. Moreover, small movements of the pole can make a double eigenvalue to be simple in many situations. When the domain posseses strong symmetries, it seems that it can remain simple not only for small movements, but also globally in the domain.

The two main results achieved in this direction are the following

Theorem 4.1 ([2]). *Let $0 \in \Omega$ and $\alpha_0 \in \{\frac{1}{2}\} + \mathbb{Z}$. Let $n_0 \geq 1$ be such that the n_0-th eigenvalue $\lambda := \lambda_{n_0}^{(0,\alpha_0)}$ of $(i\nabla + A_0^{\alpha_0})^2$ with Dirichlet boundary conditions on $\partial\Omega$ has multiplicity two. Let φ_1 and φ_2 be two orthonormal in $L^2(\Omega, \mathbb{C})$ and linearly independent eigenfunctions corresponding to λ. Let c_k, $d_k \in \mathbb{R}$ be the coefficients in the expansions $\varphi_k^{(a,\alpha)}(a + r(\cos t, \sin t)) = e^{i\frac{t}{2}} r^{1/2} \left(c_k \cos\frac{t}{2} + d_k \sin\frac{t}{2}\right) + o(r^{1/2})$. If φ_1 and φ_2 satisfy both the following*

$$(i)\ c_k^2 + d_k^2 \neq 0 \text{ for } k = 1, 2; \qquad (ii)\ \int_\Omega (i\nabla + A_0^{\alpha_0})\varphi_1 \cdot A_0^{\alpha_0}\overline{\varphi_2} \neq 0;$$

$$(iii) \text{ there does not exist } \gamma \in \mathbb{R} \text{ such that } (c_1, d_1) = \gamma(c_2, d_2);$$

then there exists a neighborhood $U \subset \Omega \times \mathbb{R}$ of $(0, \alpha_0)$ such that the set

$$\{(a, \alpha) \in U : (i\nabla + A_a^\alpha)^2 \text{ admits a double eigenvalue close to } \lambda\} = \{(0, \alpha_0)\}.$$

Theorem 4.2 ([1]). *Let $\lambda_1^{(a,\frac{1}{2})}$ be the first eigenvalue on the disk. Then $\lambda_1^{(a,\frac{1}{2})}$ is simple if and only if $a \in (-1, 1) \setminus \{0\}$.*

- **Recovering Riemannian metrics from least-area data**. This poster was presented by Tracey Balehowsky (Postdocoral Researcher at the University of Helsinki) and contained her joint work with Spyros Alexakis and Adrian Nachman (Professors at the University of Toronto). The following question was considered: Given any simple closed curve γ on the boundary of a Riemannian 3-manifold (M, g), suppose the area of the least-area surfaces bounded by γ are known. From this data may we uniquely recover g?

This question can be thought of as an $n-2$ codimensional version of boundary rigidity, wherein one seeks to determine the metric g given knowledge of the geodesic distance $d(x,y)$ between any two points x, y on the boundary of M. In the cases where this is possible, we say the manifold is *boundary rigid.* The problem of boundary rigidity has been solved in 2 dimensions, but remains open in higher dimensions.

The poster summarized some of the history of the problem of boundary rigidity, highlighting the works of Michel [18], Gromov [15], Croke [11], Pestov and Uhlmann [20] (which settled the 2D case), Lassas et. al. [17], Burago and Ivanov [9, 10], and Stefanov et. al. [23]. It also contained a brief description of the obstacles to boundary rigidity as motivation for the analogous obstacles one faces when instead considering least-area data instead of distances.

Next presented were three theorems which gave conditions when one can uniquely recover (up to boundary-fixing diffeomorphisms) the Riemannian metric g, given knowledge of the areas of least-area surfaces circumscribed by simple curves on the boundary of M. The results do not require this area data for all such simple curves on the boundary; rather just certain families of curves.

1. The first theorem addressed the question of what is the least amount of area data possible to achieve global uniqueness. It showed that if the metric was either C^3-close to Euclidean or "straight-thin", knowledge of the areas of least-area surfaces with boundary given by a leaf of a particular 1-parameter family of foliations of the boundary by simple curves was enough to uniquely determine the metric.
2. The second theorem was a global uniqueness result which demonstrated that the curvature conditions of the first theorem could be relaxed if more data was given and an additional foliation structure was assumed. This theorem showed that if the manifold was of the type which "admitted foliations from all directions", knowledge of the areas of the least-area surfaces arising as leaves in the admitted foliations uniquely recovered the metric.
3. The third theorem was a local result which showed that if the boundary ∂M was strictly mean convex at $p \in \partial M$ and one knows the areas of a certain 2-parameter family of least-area surfaces which are near p, then the metric is uniquely determined in a small neighbourhood $V \subset M$ containing p. It was emphasized that a key starting point for all the results presented was that the area data gave information about the Dirichlet-to-Neumann map for the Jacobi operator on the 2-dimensional least-area surfaces, from which curvature information was determined via the result of Nachman [19].

- **The Mixed Boundary Value Problem for the Laplacian in Non-Smooth Domains**. The poster, presented by Katharine Ott from Bates College, summarized results regarding well-posedness of the L^p-mixed boundary value problem in Lipschitz domains for the Laplacian. The theorems presented have appeared in a series of recent papers with coauthors H. Awala, R. Brown, S. Kim, I. Mitrea, and J. Taylor.

  To give a sense of the work presented, consider the case of the L^p-mixed problem for the Laplacian. In this setting, let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, and let $\partial\Omega = D \cup N$, where D is an open subset of the boundary and $D \cap N = \emptyset$. Then the boundary problem is given by

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u\,|_D = f_D \in W^{1,p}(D), \\ \partial_\nu u\,|_N = f_N \in L^p(N), \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases} \tag{4.7}$$

  Above, ∂_ν denotes differentiation in the normal direction. For any function $v : \Omega \to \mathbb{R}$, v^* stands for the non-tangential maximal function.

The first result is well-posedness (meaning existence and uniqueness of solutions) of the L^p-mixed problem for the Laplacian in Lipschitz domains for a range $p \in (1, 1+\varepsilon)$ under a mild assumption on the boundary between D and N (the Dirichlet and Neumann portions of the boundary, respectively). It is important to note that well-posedness in L^2 may actually fail and thus the result for small $p > 1$ is, in a sense, optimal. An important step in the proof of well-posedness of (4.7) (and similarly in the case where the Laplacian is replaced with the Lamé system of elastostatics) is establishing decay of solutions when the boundary data is an atom. This decay is encoded in estimates for the Green function for the mixed problem, which constituted the second theorem of the poster.

The final portion of the poster addressed an alternative approach to studying the L^p-mixed problem for the Laplacian in the special case where $\Omega \subset \mathbb{R}^2$ is the infinite upward sector with vertex at zero and aperture $\theta \in (0, 2\pi)$. Here, we define the left edge of the sector to be D and the right edge of the sector to be N. Under these conditions, we can translate (4.7) into an integral boundary equation. The solvability of this aforementioned equation hinges on whether or not an associated integral operator is invertible on the prescribed function spaces. This approach results in a *sharp* well-posedness results for (4.7).

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