The eight-vertex model via dimers

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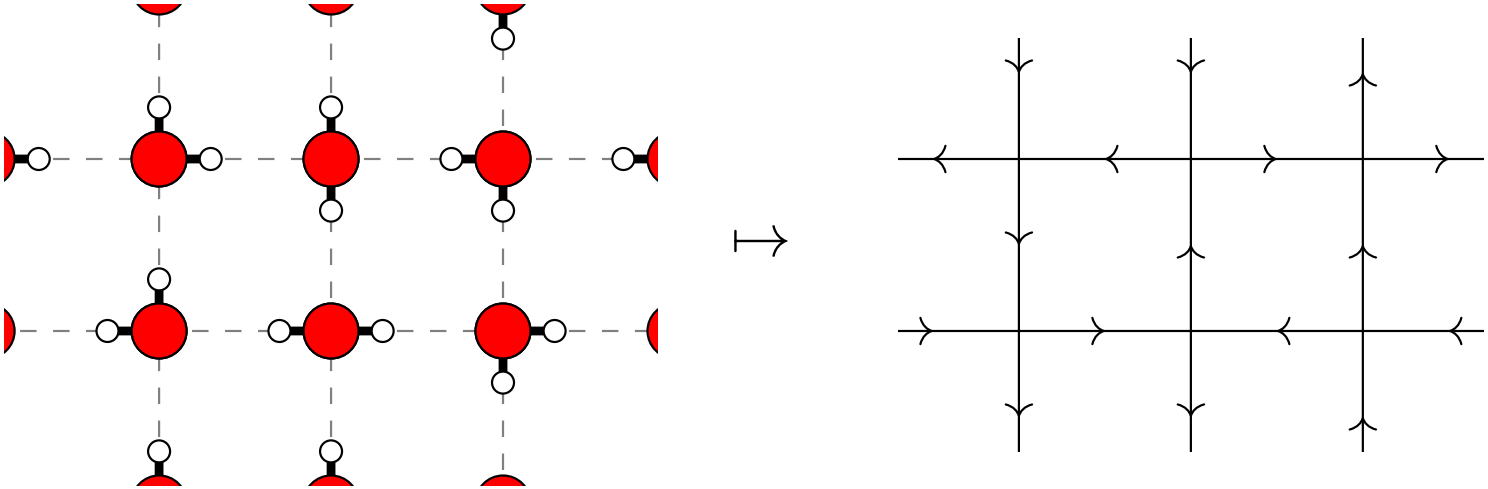
Dimers, Ising Model, and their Interactions

BIRS
22 November 2019

1. The eight-vertex model
2. The free-fermion regime
3. Non-bipartite dimers to bipartite
4. Isoradial Z-invariant case

Six-vertex configurations



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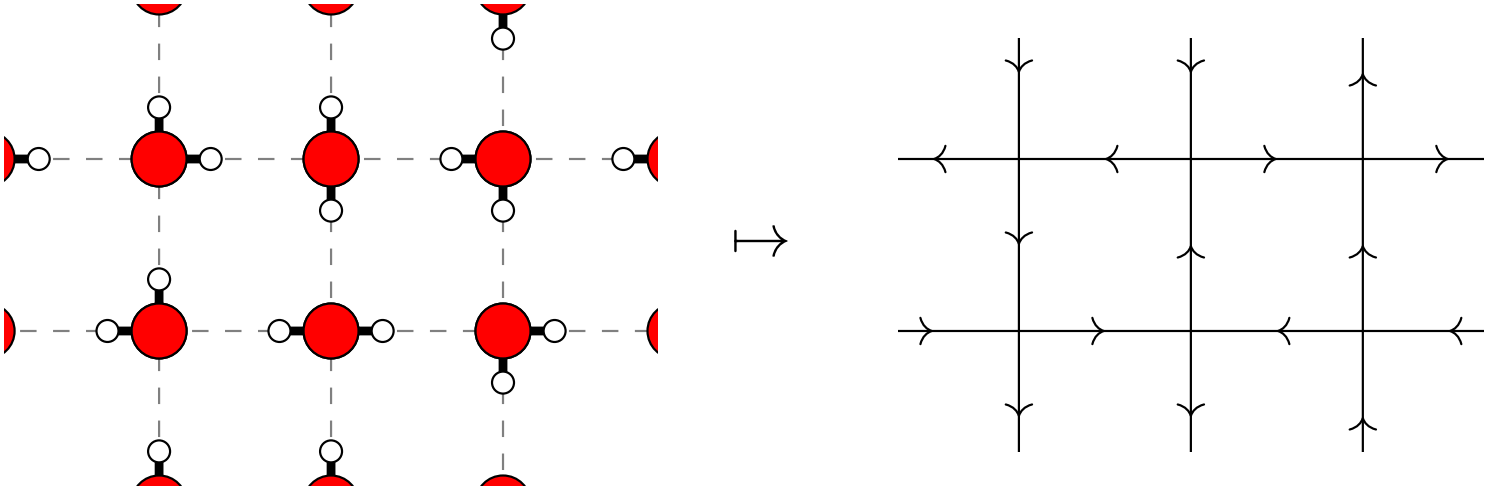
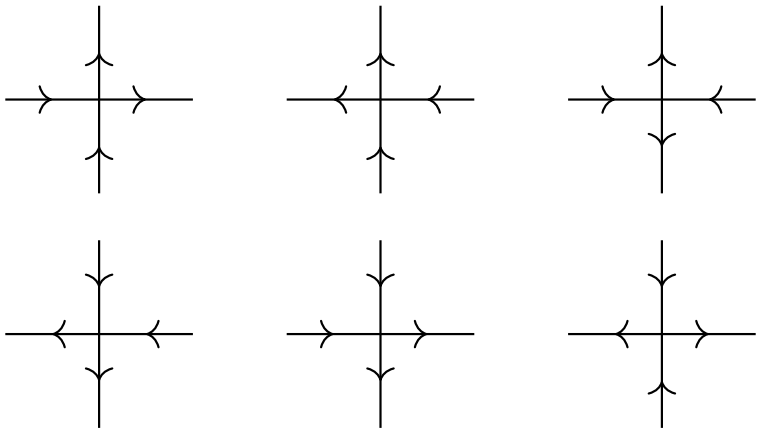
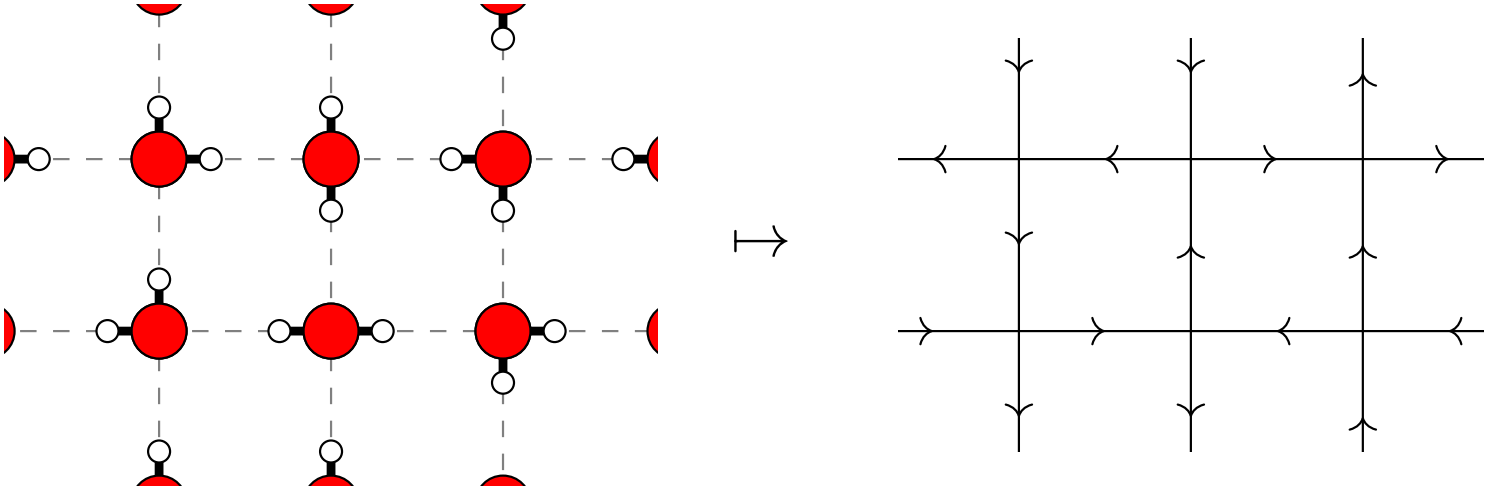


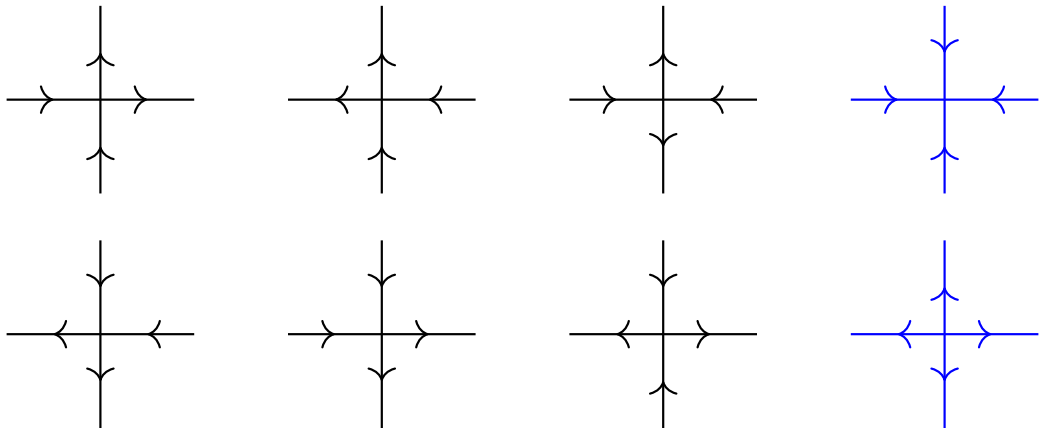
6 local configurations:



Eight-vertex configurations



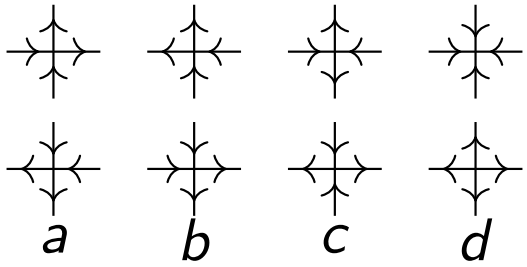
8 local configurations:



...with the introduction of defects.

The (symmetric, classical) eight-vertex model

Fix the local weights $a, b, c, d > 0$:



On a finite $G \subset \mathbb{Z}^2$, an orientation τ satisfying these rules has *weight*

$$w(\tau) = a^{N_a} b^{N_b} c^{N_c} d^{N_d}.$$

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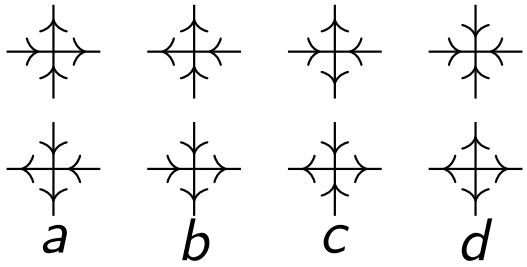

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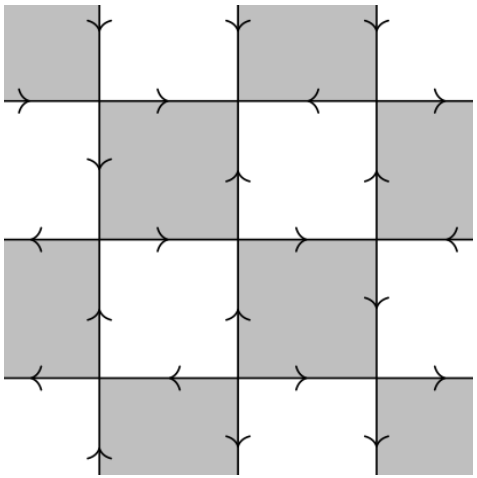

$$w(\tau) = ab^3c^4d.$$

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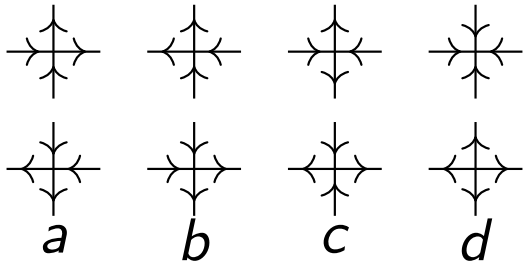

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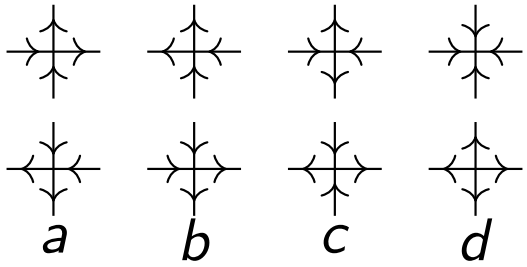

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Related to the XYZ spin chain, Ashkin-Teller model,...

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- Free energy:

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Computed in the 1970s (Baxter; Johnson, Krinsky, McCoy; Takhtadzhan, Faddeev,...) through Bethe Ansatz methods.

Thermodynamic limit

- Correlations: For edges e, e' faraway in G, is there a **correlation length** ξ s.t.

$$\mathrm{Cov}\left(1_{e\uparrow \text{ dans } \tau}, 1_{e'\uparrow \text{ dans } \tau}\right) \sim \exp\left(-\frac{|e-e'|}{\xi}\right)?$$

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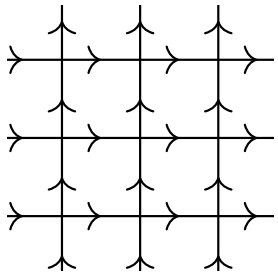
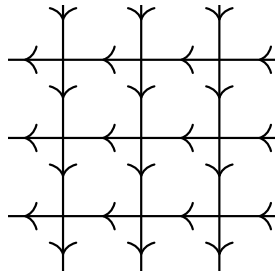
However, in the six-vertex case ($d = 0$), ξ is always "infinite" (power law decay)

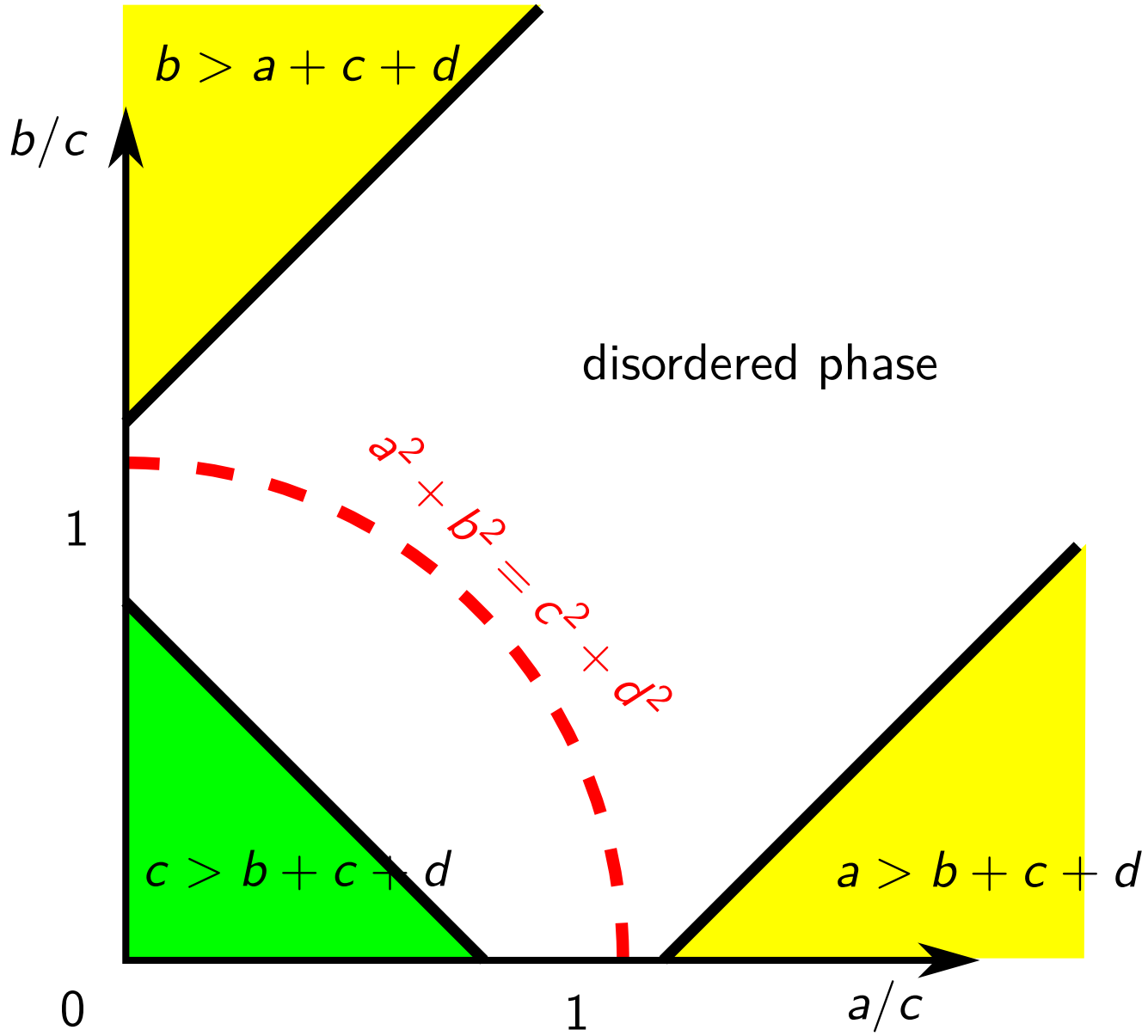
Phase diagram

Predicted in the 1970s-1980s (Baxter, Fan, Lieb, Sutherland, Wu,...):

- If $a \geq b + c + d$, the measure on $\mathbb{Z}^2$ concentrates on  and 

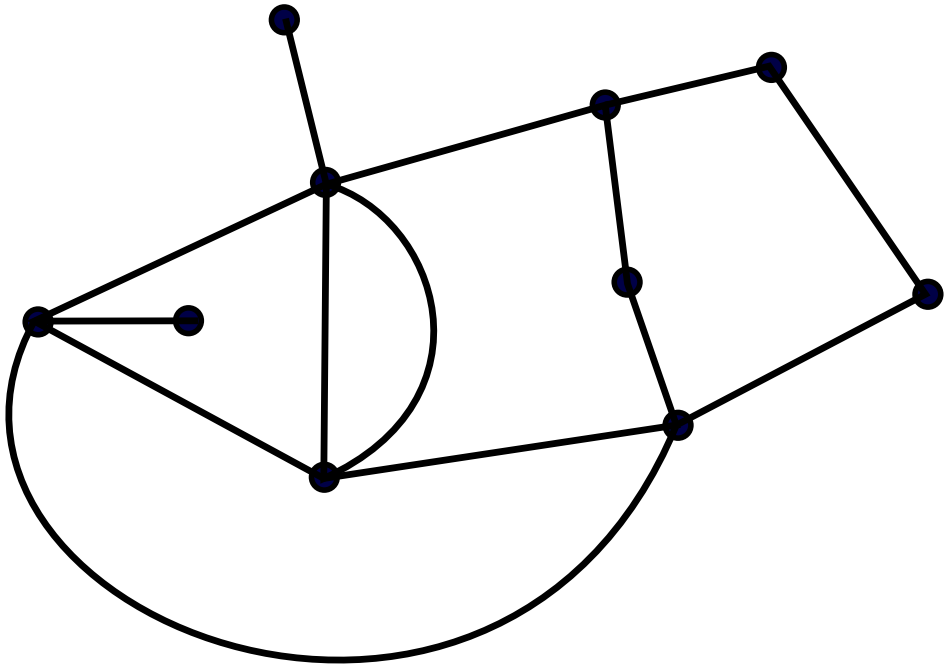
  . This is a *ferroelectric* phase.
- Similar for $b \geq a + c + d$ (*ferroelectric*),
  $c \geq a + b + d$ (*anti-ferroelectric*),
  $d \geq a + b + c$ (*anti-ferroelectric*).
- Otherwise, all configurations appear with positive density (*disordered*).

Phase diagram ($d < c$)



A generalization

G a finite graph on the sphere or the torus.



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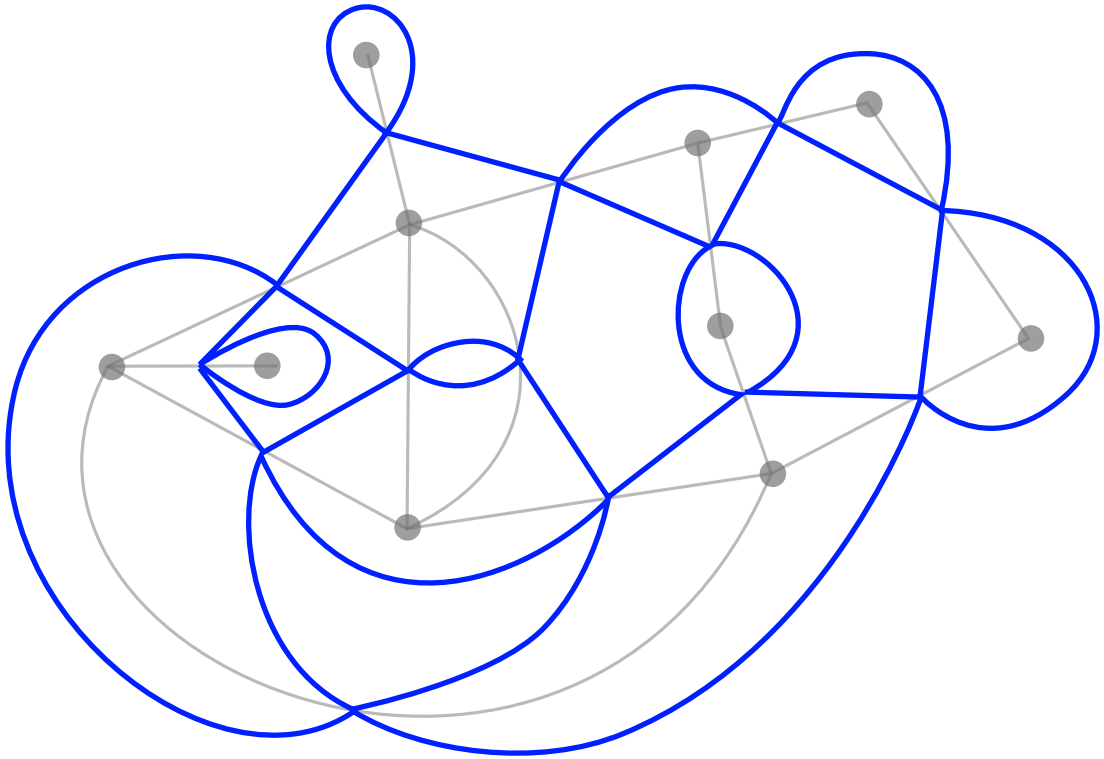

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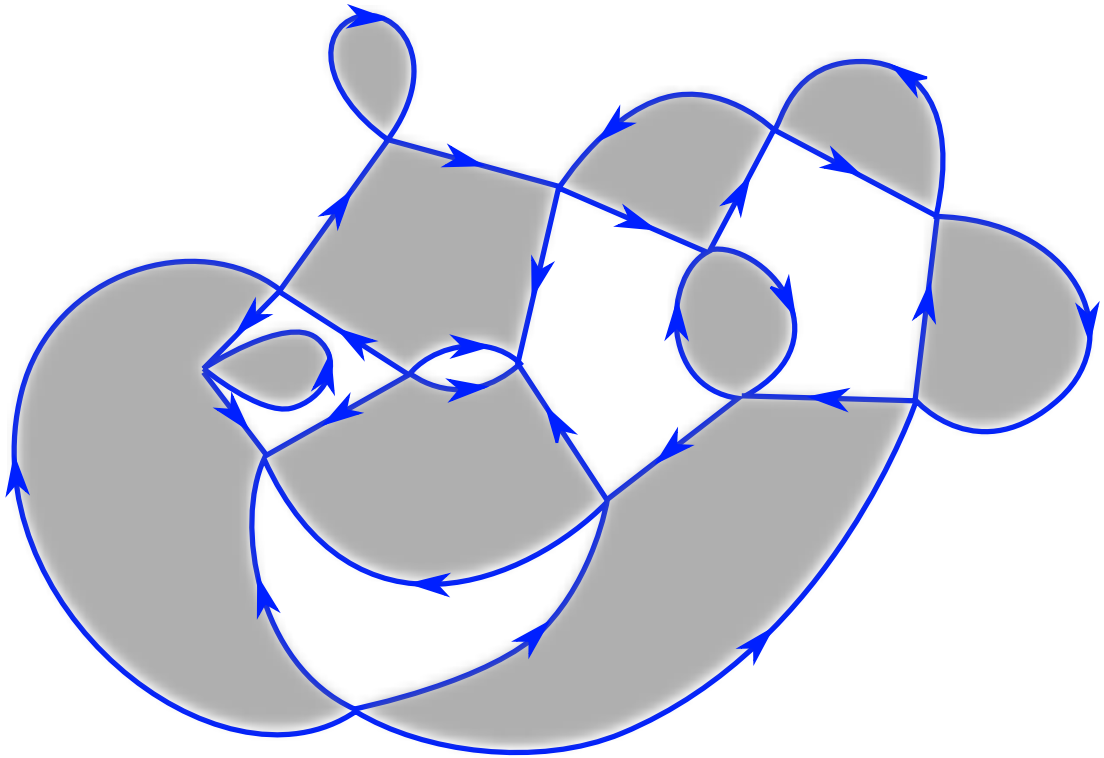

An eight-vertex configuration τ on G_m.

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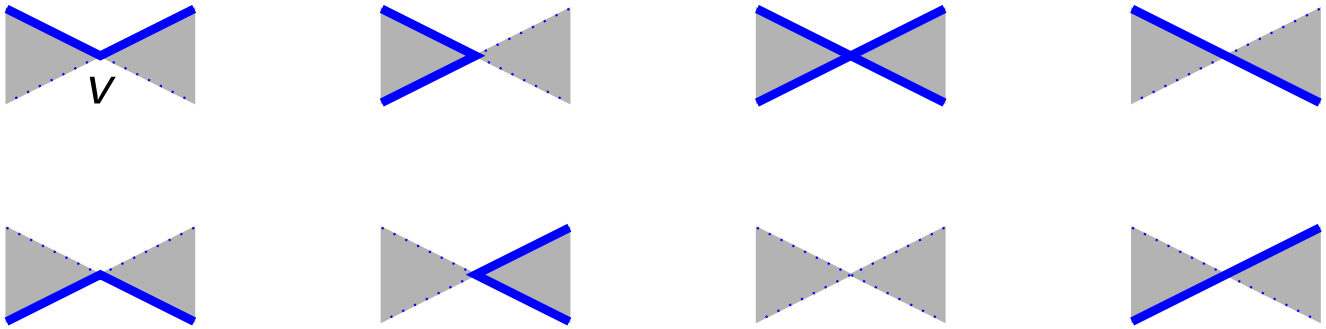
A generalization

Local configurations at a **medial** vertex $v \in V_m$:



$w_v(\tau) =$ a_v b_v c_v d_v

$$w(\tau) = \prod_{v \in V_m} w_v(\tau).$$

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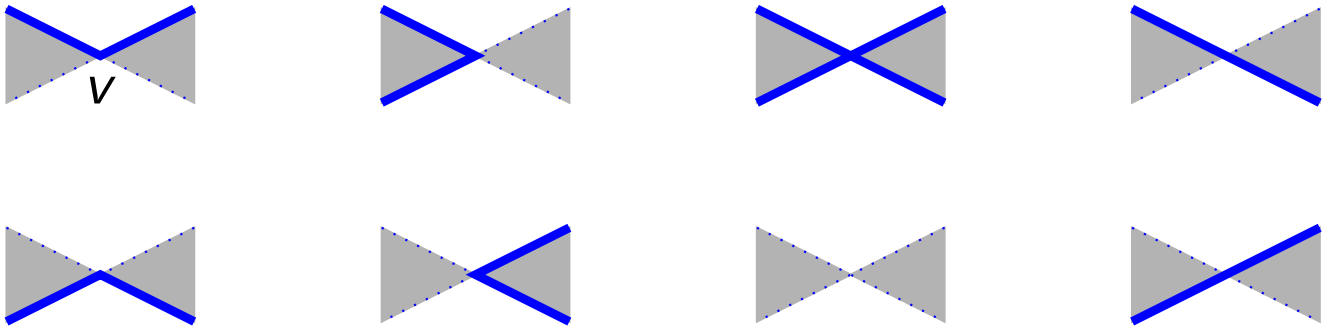

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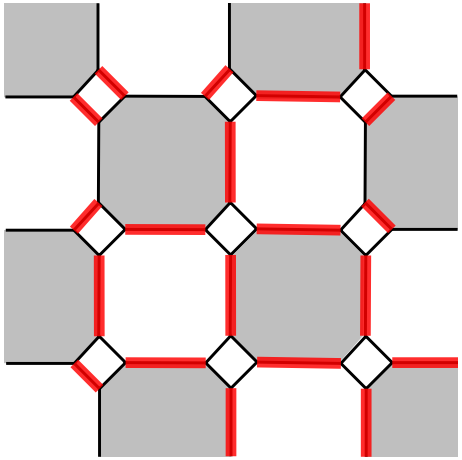
where a, b, c, d are **functions** of the medial vertices.

The free-fermion regime

Decorated graphs

Dimer configuration $m \mapsto$ **six**-vertex configuration τ.



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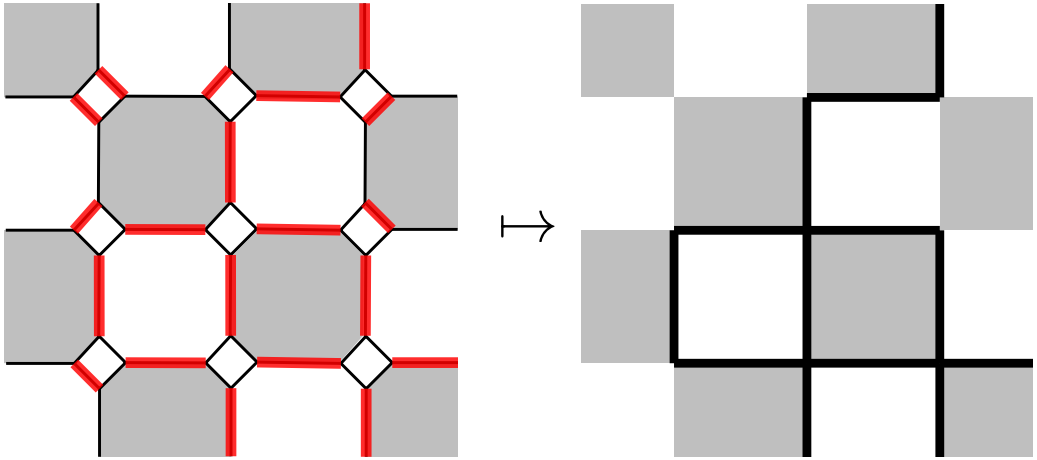

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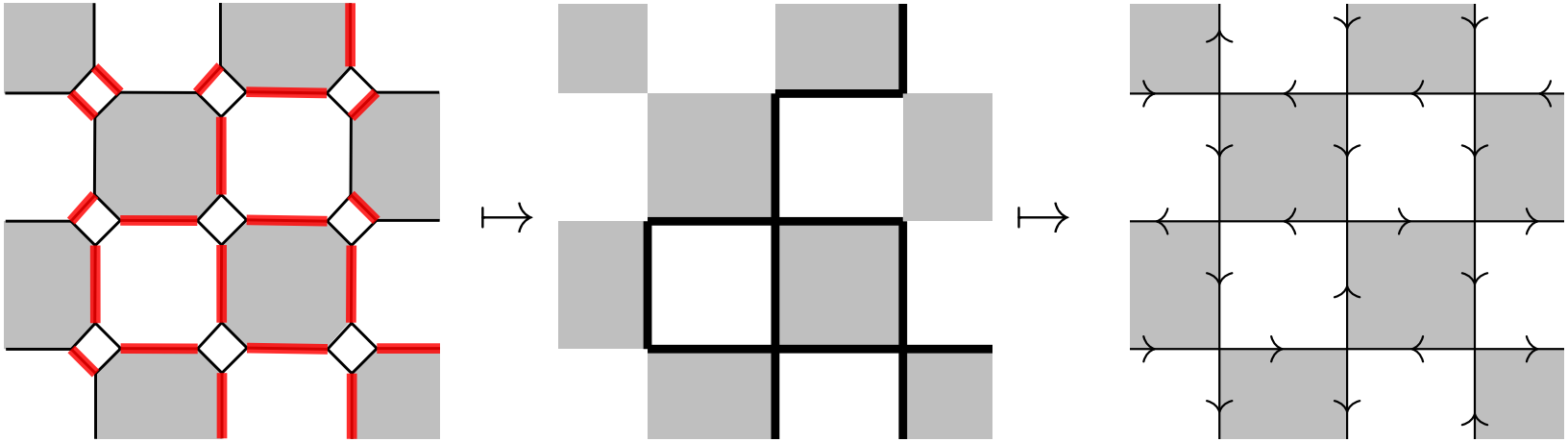

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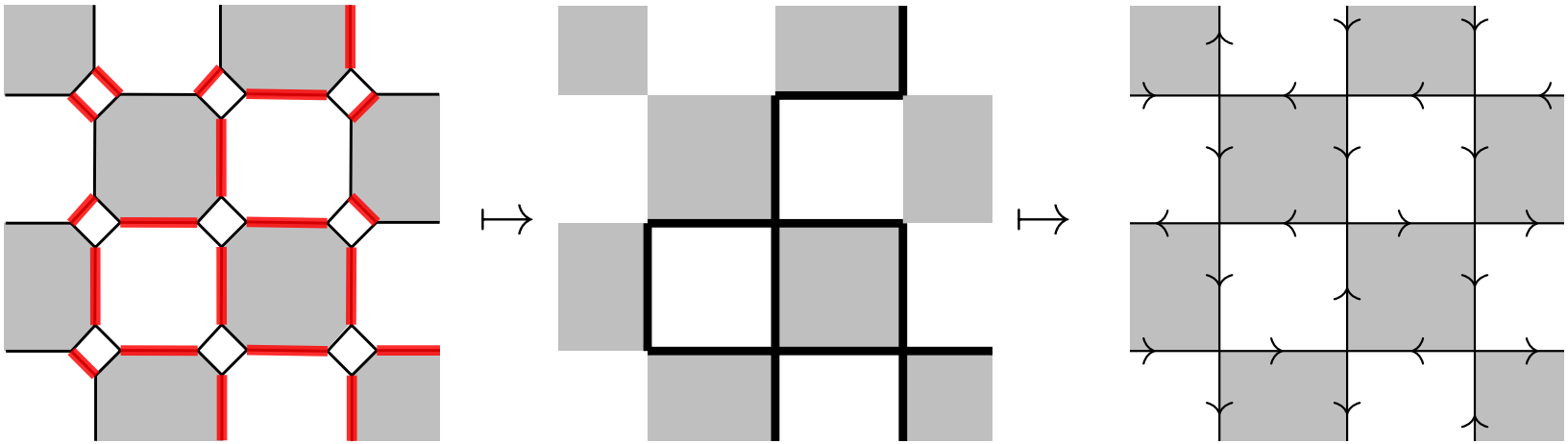

Proposition *[Fan, Lin, Wu 1970s]*

If the **six**-vertex model ($d = 0$) satisfies $a^2 + b^2 = c^2$ (at every vertex), then for some dimer weights,

$$\sum_{m \mapsto \tau} w_{\text{dim}}(m) = w(\tau).$$

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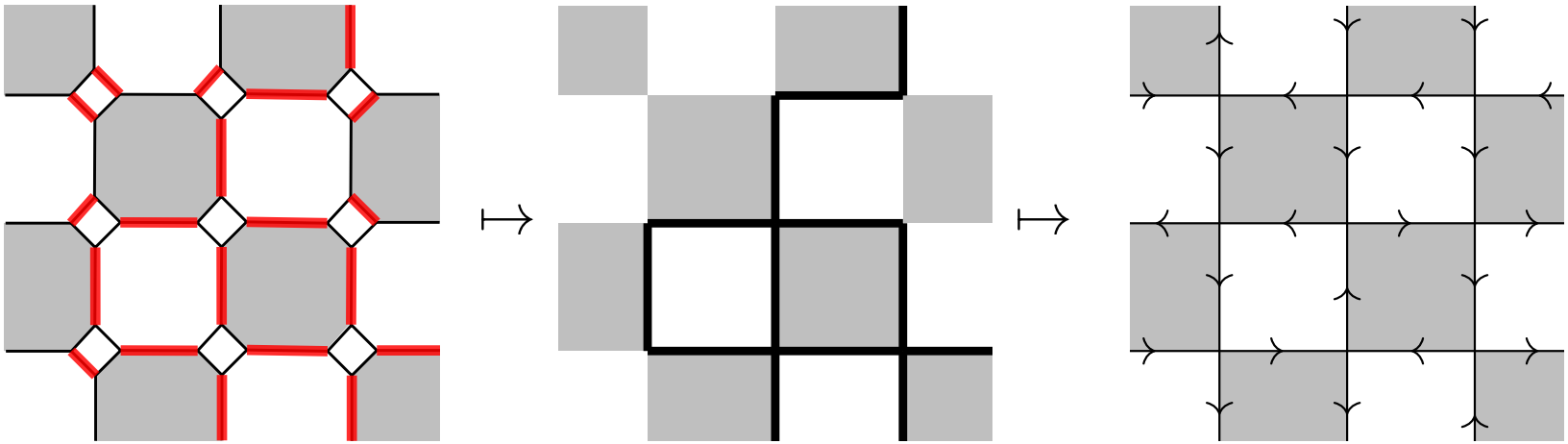

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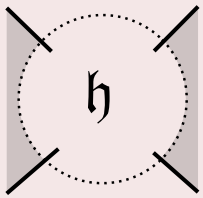
Free energy, correlations, Gibbs measure,...

Existence of decorations

Question

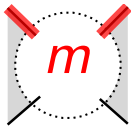
For the **eight**-vertex model, are there decorations $\mathfrak{h}$ s.t.

$$\sum_{m \mapsto \tau} w_{\mathrm{dim}}(m) = w(\tau)?$$



For instance, at a single vertex $v \in V_m$,

$$\sum_{m \text{ s.t. }} w_{\mathrm{dim}}(m) = a_v.$$



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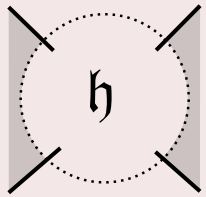


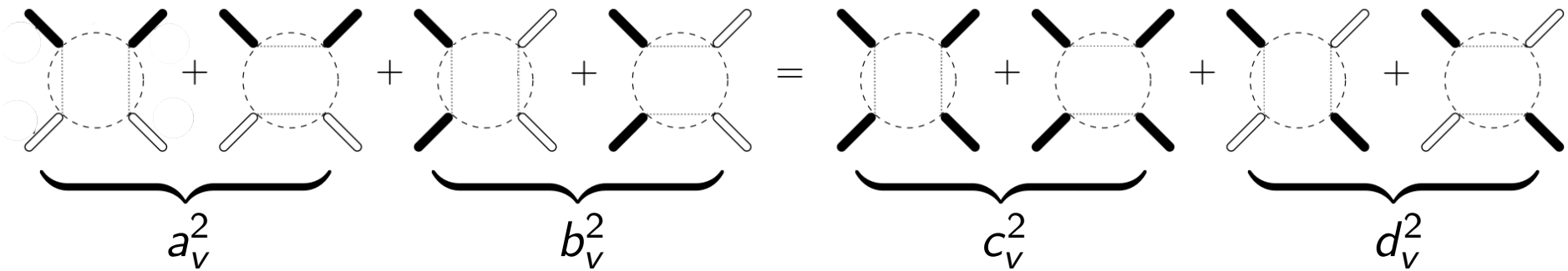
Lemma

If such a **planar** $\mathfrak{h}$ exists, then

$$\forall v \in V_m, \quad a_v^2 + b_v^2 = c_v^2 + d_v^2 \qquad (\text{“free-fermion” regime.})$$

Claim:
If such a **planar** $\mathfrak{h}$ exists, then $\forall v \in V_m, \quad a_v^2 + b_v^2 = c_v^2 + d_v^2$.

Proof:



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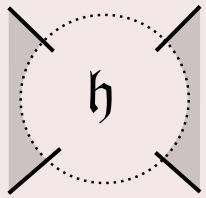

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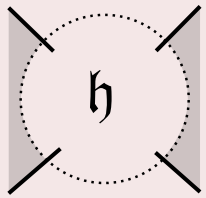

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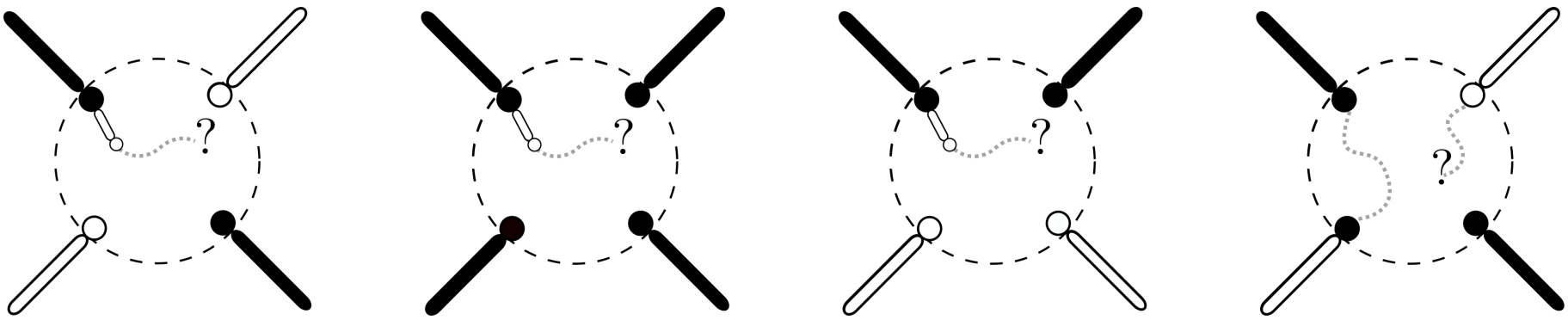
Lemma

If such a **planar bipartite** $\mathfrak{h}$ exists, then

$$\forall v \in V_m, \quad a_v^2 + b_v^2 = c_v^2 + d_v^2 \ \textbf{and}\ a_v b_v c_v d_v = 0.$$

Claim:
If such a **planar bipartite** $\mathfrak{h}$ exists, then one of a_v, b_v, c_v, d_v is zero.

Proof:



Existence of decorations

Proposition *[Hsue, Lin, Wu 1970s]*

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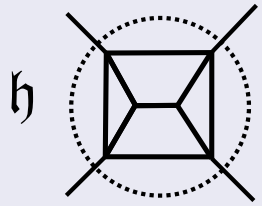

then for some dimer weights on the $\mathfrak{h}$-decorated graph ($\mathfrak{h}$ non-bipartite),

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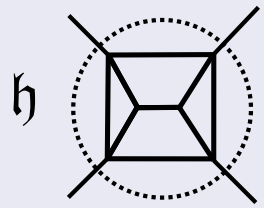

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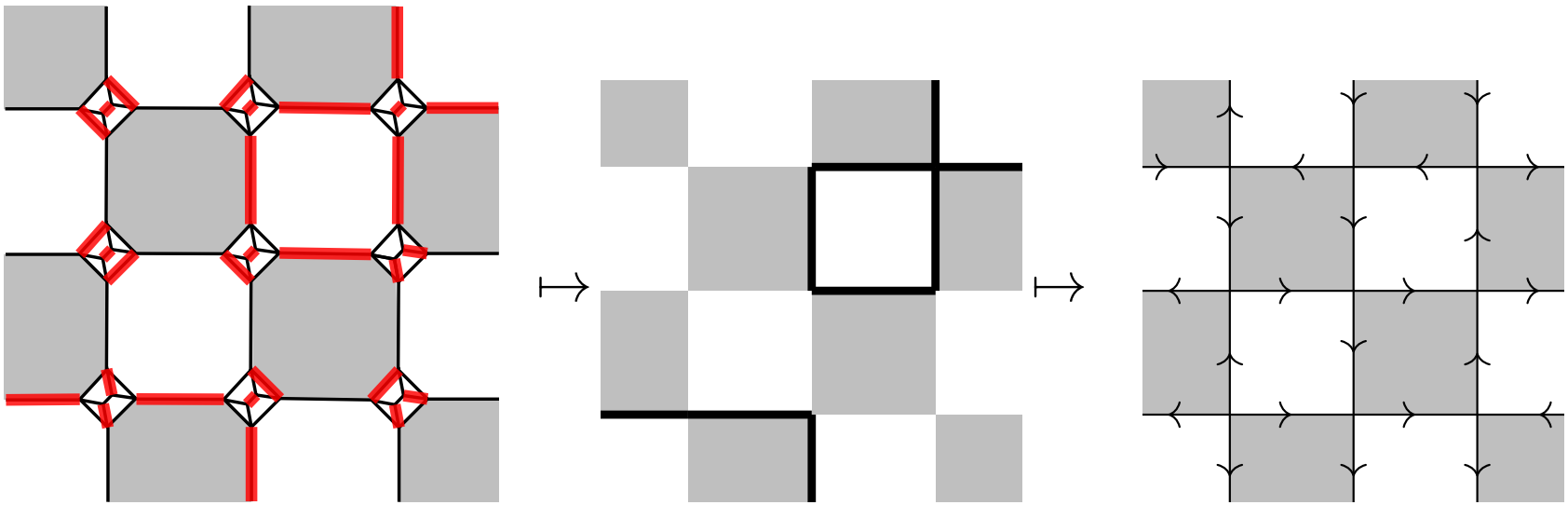
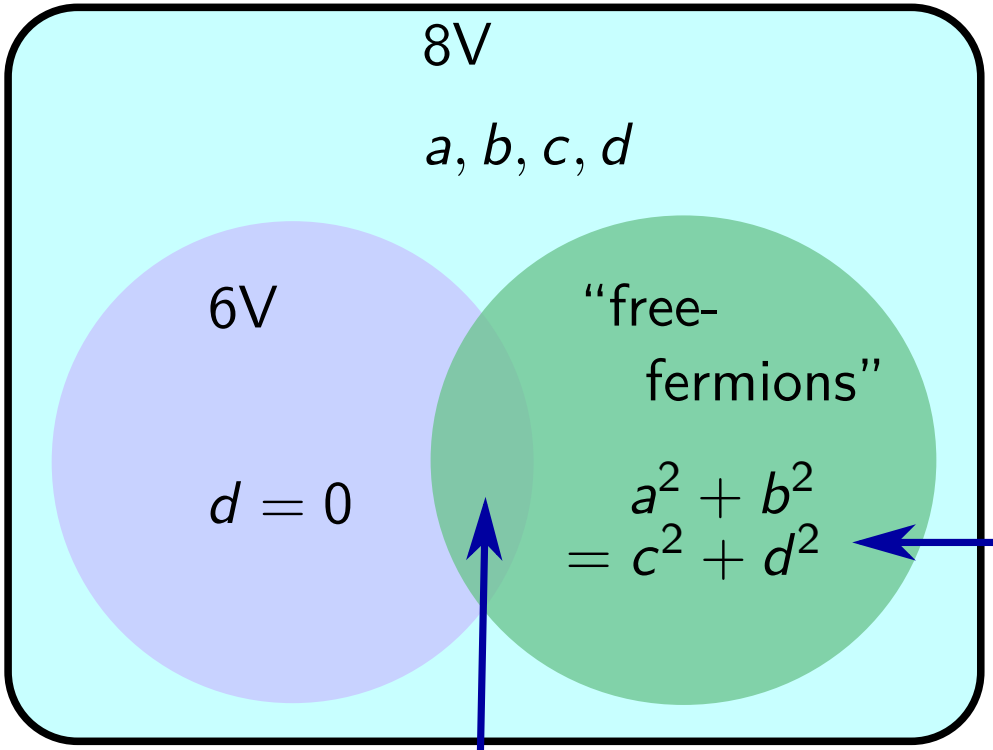
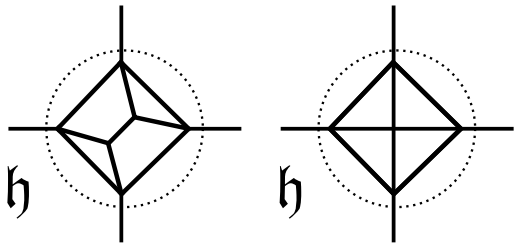


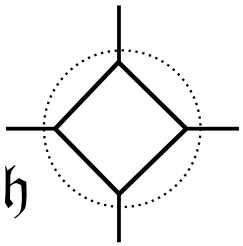
Recap



Non-bipartite dimers



- Free energy



Bipartite dimers
- Free energy
- Correlations
- Ergodic Gibbs Measures...

Characteristic polynomial: examples

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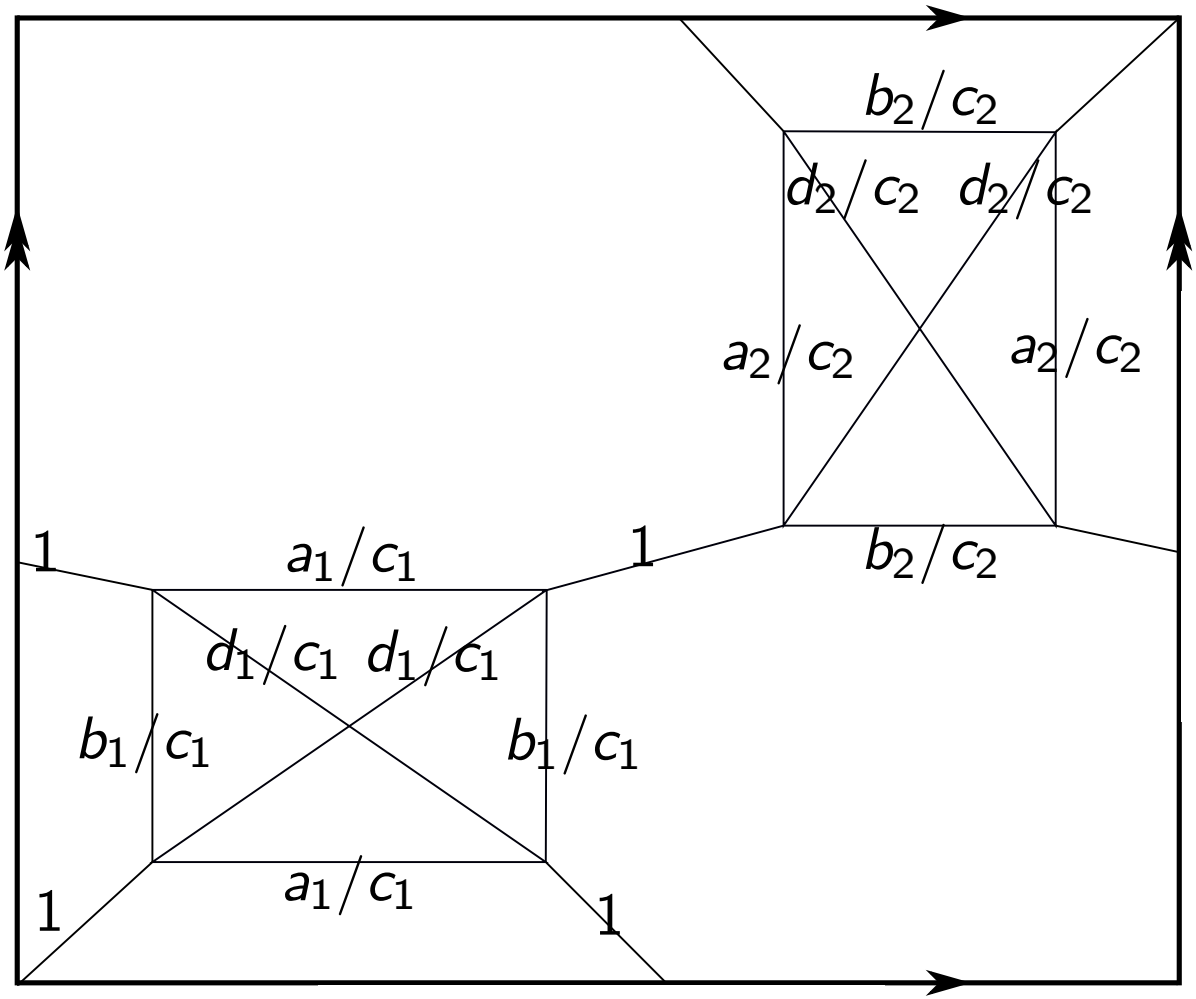

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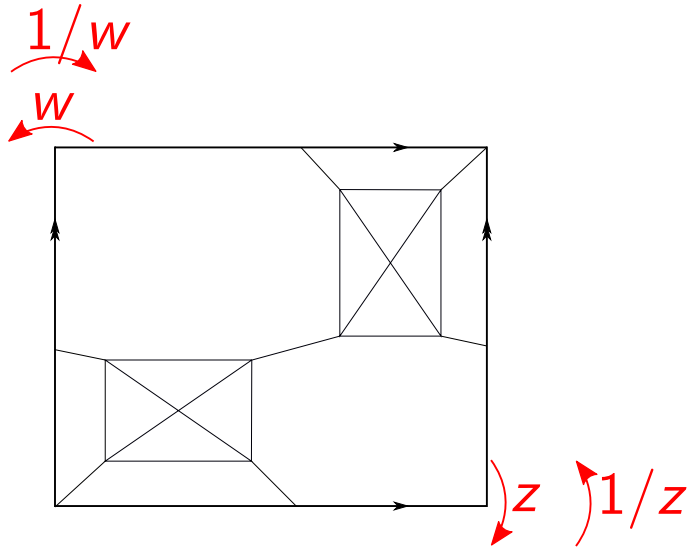

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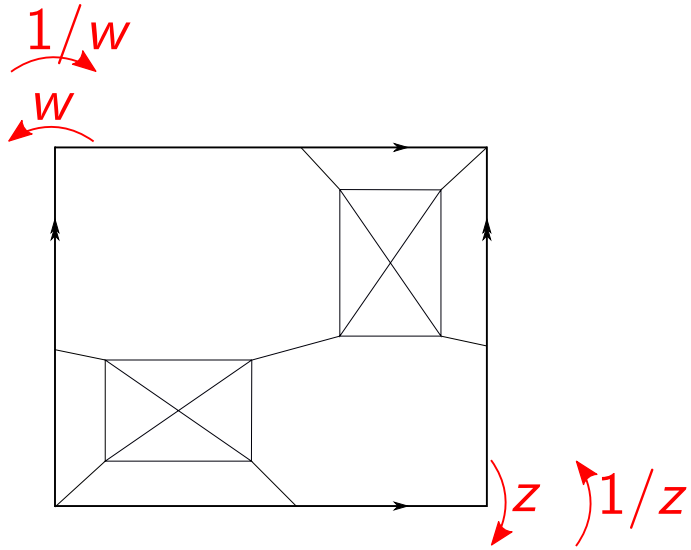

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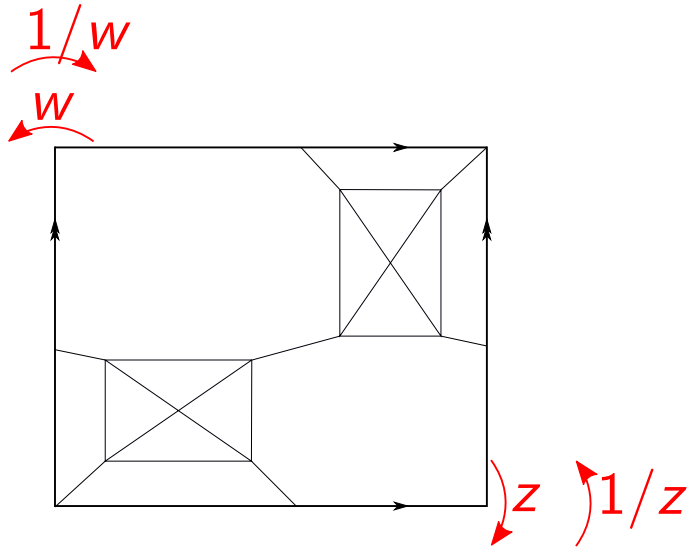

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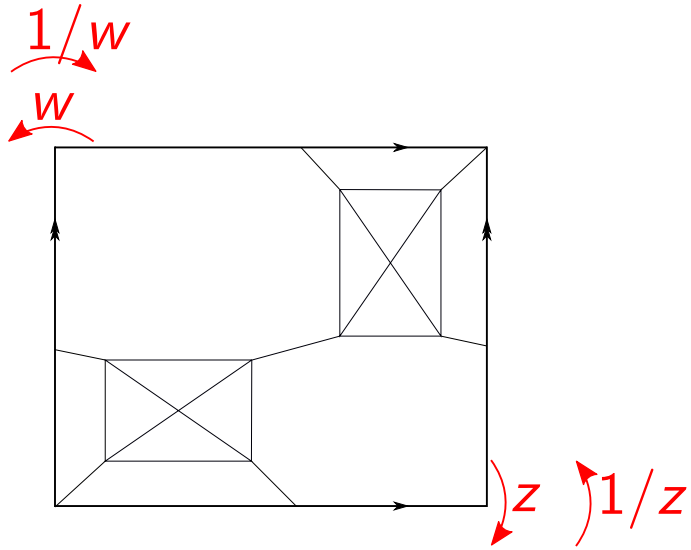

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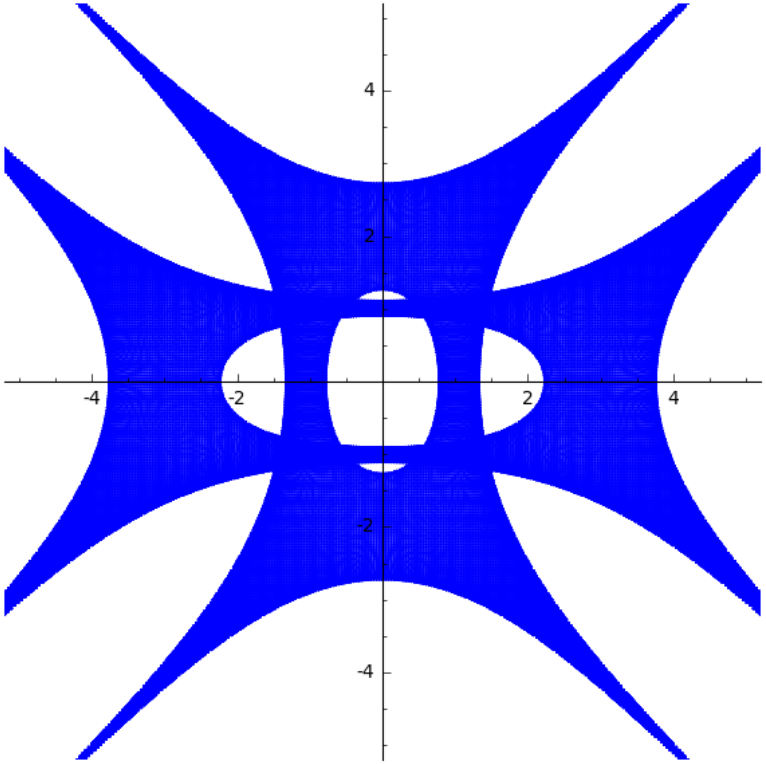
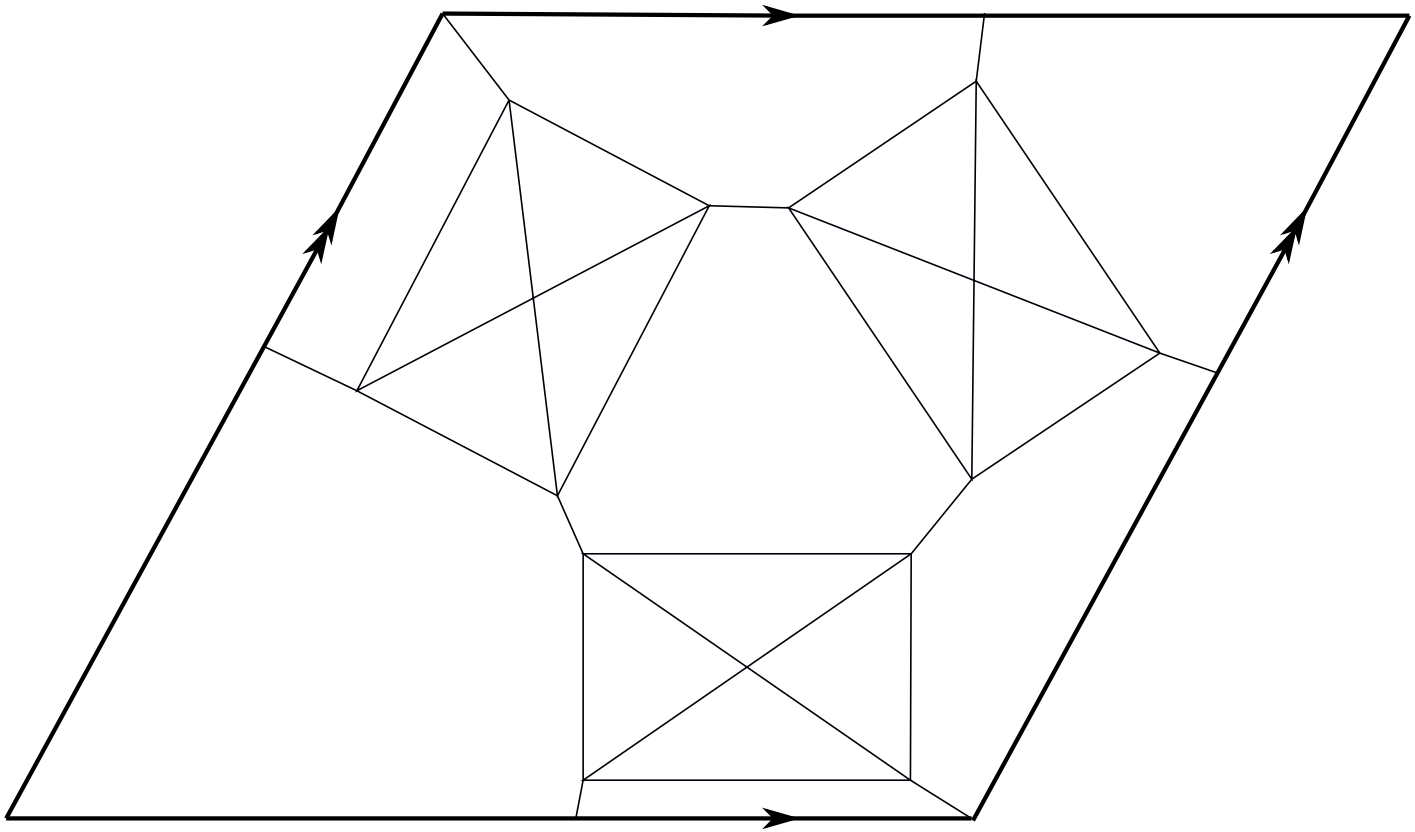


Amoeba: image of the zero locus of P by $(z,w) \mapsto (\log|z|, \log|w|)$.

Characteristic polynomial: examples

The free-fermion 8V model for $G =$ triangular lattice gives dimers:



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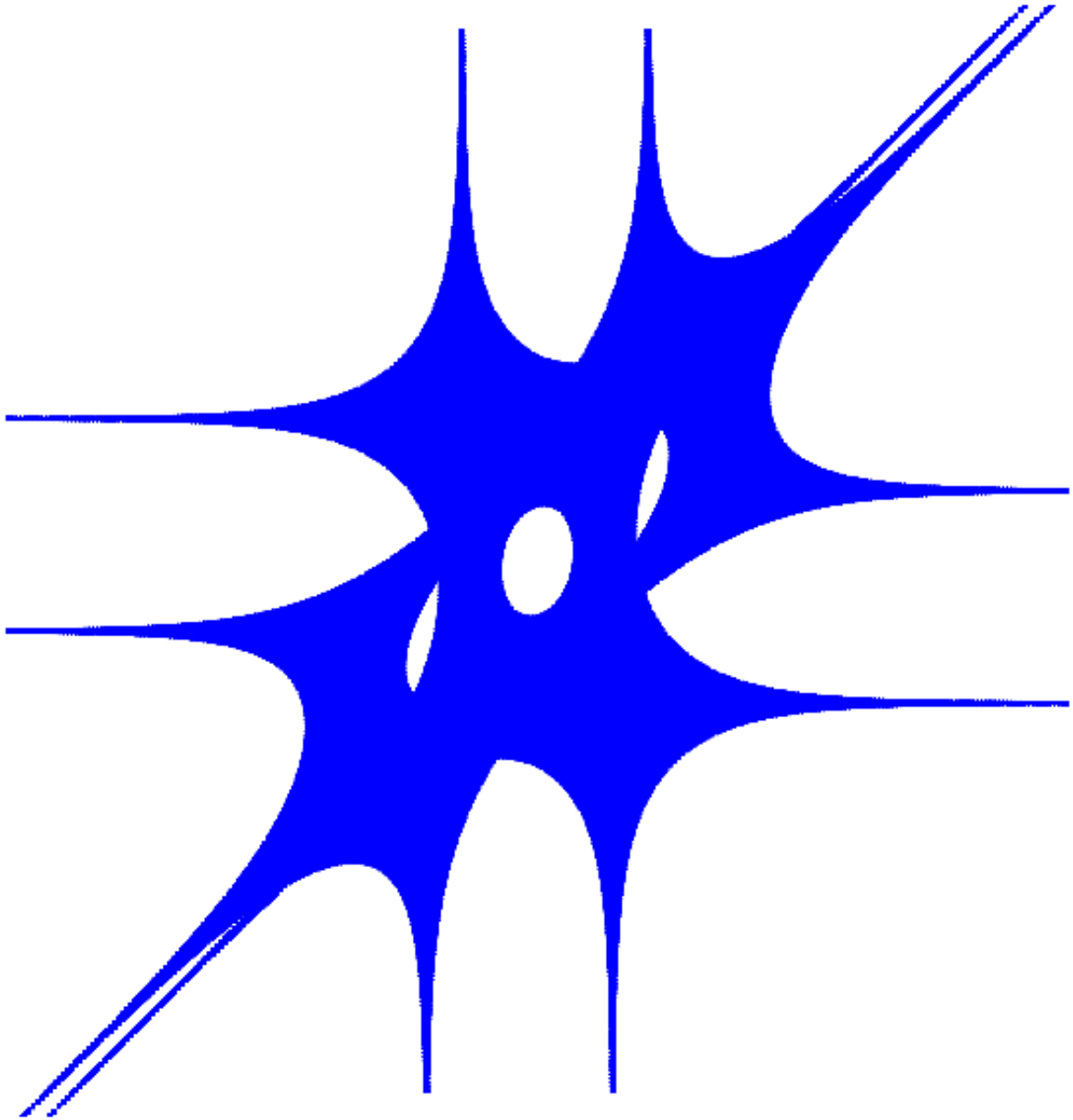

Non-bipartite to bipartite

Non-bipartite to bipartite (1)

Theorem *[M. 2018]*

For a finite graph on the torus equipped with a free-fermion **8V** model (a, b, c, d), let $P(z, w)$ be the characteristic polynomial of the corresponding (non-bipartite) dimers.
Then there exists two free-fermion **6V** models, (a_1, b_1, c_1) et (a_2, b_2, c_2), with characteristic polynomials P_1, P_2 s.t.

$$P(z, w) = P_1(z, w)P_2(z, w).$$

The transformation

$$\left[a : b : c : d\right] = \left[\sin\left(\tfrac{\alpha+\beta}{2}\right) : \cos\left(\tfrac{\alpha+\beta}{2}\right) : \cos\left(\tfrac{\beta-\alpha}{2}\right) : \sin\left(\tfrac{\beta-\alpha}{2}\right)\right]$$

$$\mapsto \left[a_1 : b_1 : c_1\right] = \left[\sin\alpha : \cos\alpha : 1\right],$$

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In these variables, the free-fermion 8V model becomes a 6V one when $\alpha = \beta$.

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In these variables, the free-fermion 8V model becomes a 6V one when $\alpha = \beta$.

Remark: The previous result can be generalized into

$$P_{\alpha,\beta}(z,w)P_{\alpha',\beta'}(z,w) = P_{\alpha,\beta'}(z,w)P_{\alpha',\beta}(z,w).$$

Example

For the initial "classical" 8V model on $\mathbb{Z}^2$, with

$$\left[a : b : c : d\right] = \left[\sin\left(\tfrac{\alpha+\beta}{2}\right) : \cos\left(\tfrac{\alpha+\beta}{2}\right) : \cos\left(\tfrac{\beta-\alpha}{2}\right) : \sin\left(\tfrac{\beta-\alpha}{2}\right)\right],$$

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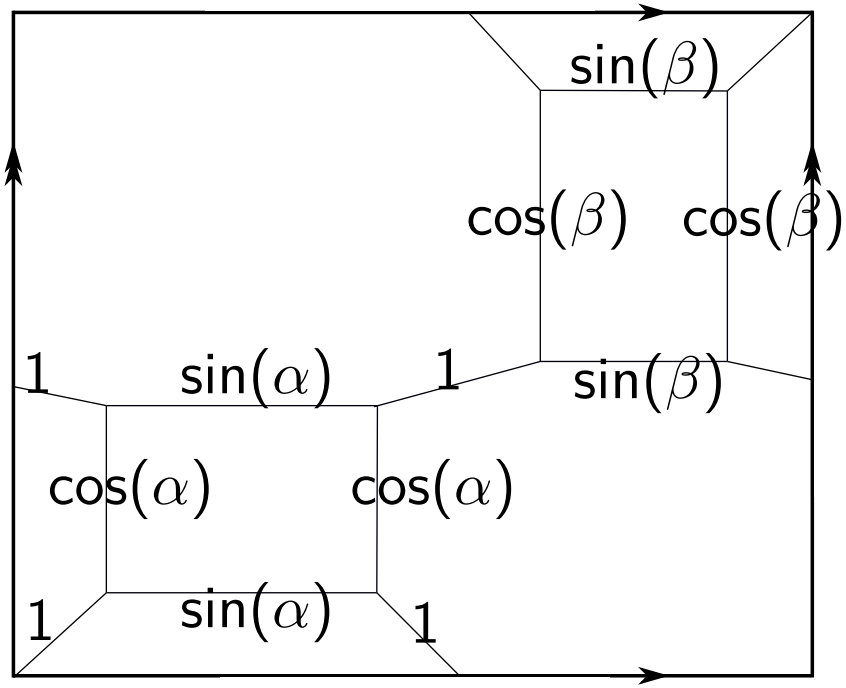

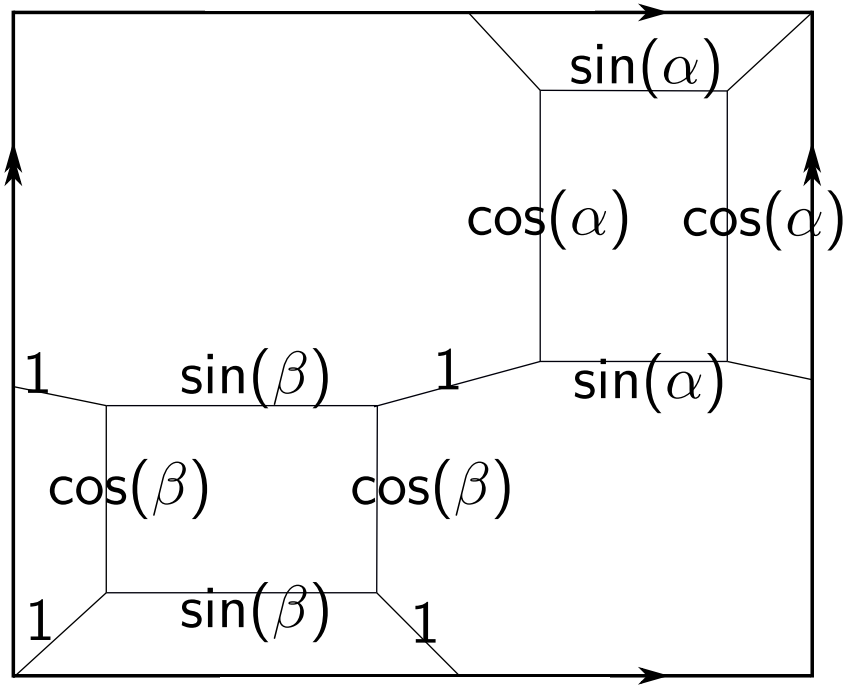

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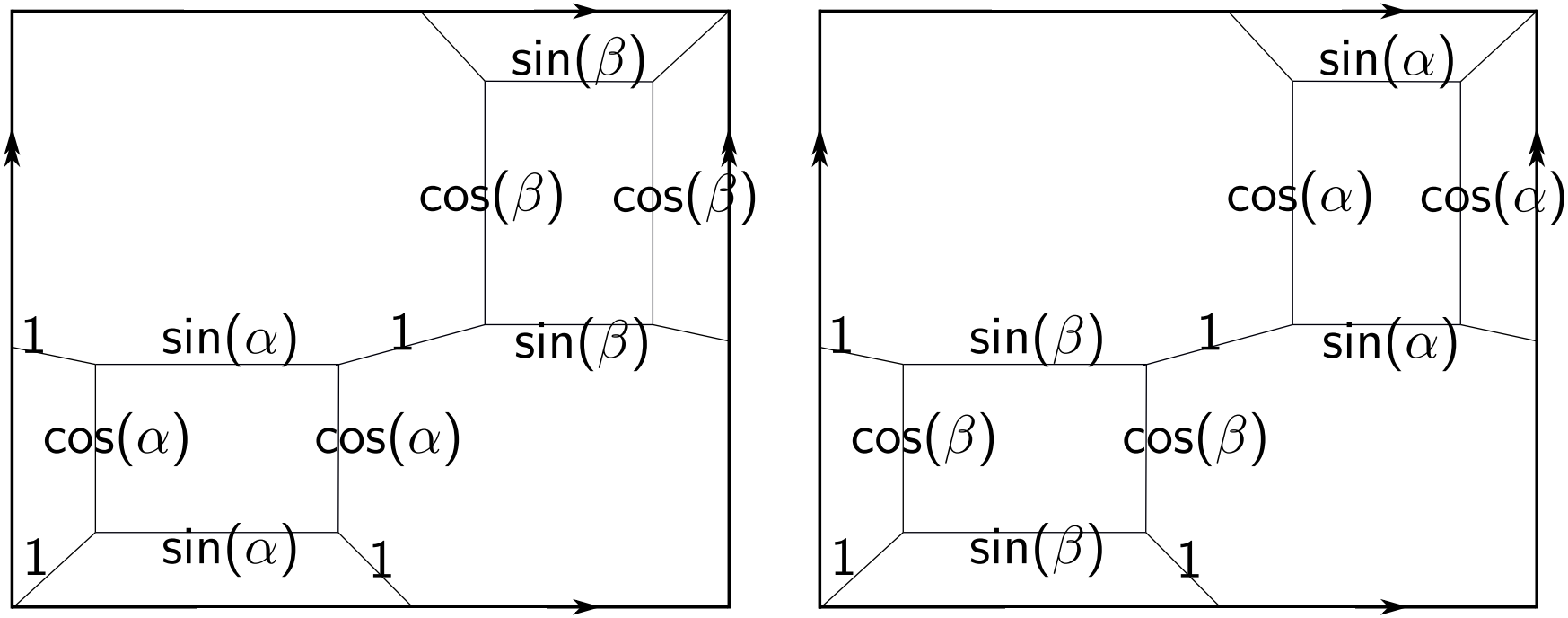

"Staggered" 6V models, in a gas phase for $\alpha \neq \beta$.

Idea of proof

Find a relation on Kasteleyn matrices.

Proposition *[M. 2018]*

Consider a graph on the sphere, torus, or the whole plane, equipped with a free-fermion 8V-model. Let

- K be the Kasteleyn matrix of the (non-bipartite) dimers from (a, b, c, d),
- K_1, K_2 those of (bipartite) dimers from (a_1, b_1, c_1) and (a_2, b_2, c_2),

then

$$K^{-1} = \frac{1}{2}\left((I + T)K_1^{-1} + (I - T)K_2^{-1}\right).$$

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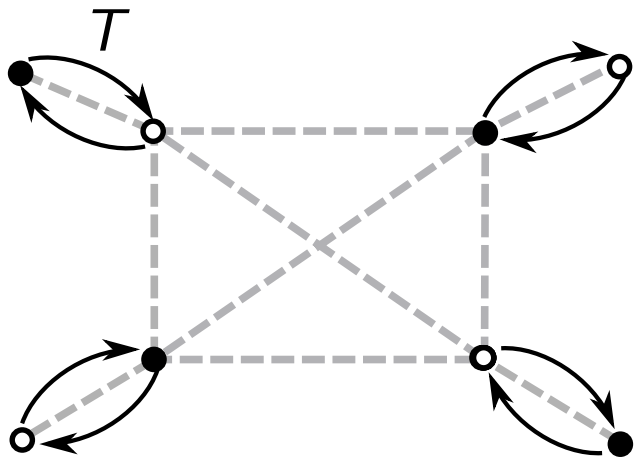

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Consequence: Correlations of the 8V-model can actually be expressed in terms of those of 6V-models.

Remark: The relation can be generalized into

$$K^{-1}_{\alpha,\beta} = \frac{1}{2}\left((I+T)K^{-1}_{\alpha,\beta'} + (I-T)K^{-1}_{\alpha',\beta})\right).$$

Non-bipartite to bipartite (2)

Theorem *[M. 2018]*

For a finite planar graph, equipped with a free-fermion 8V-model a, b, c, d,

$$Z_{8V}(a, b, c, d)^2 = Z_{6V}(a_1, b_1, c_1)\ Z_{6V}(a_2, b_2, c_2).$$

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Moreover, let τ, τ' be random 8V-configuration sampled from the Boltzmann measure of (a, b, c, d), and τ_1, τ_2 from those of $(a_1, b_1, c_1), (a_2, b_2, c_2)$, all independent. Then

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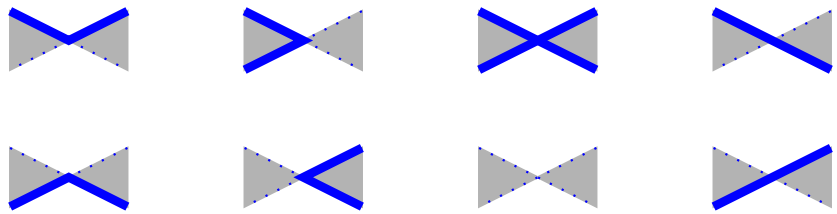
Remark: $\tau_{\alpha,\beta} \triangle \tau_{\alpha',\beta'} \stackrel{d}{=} \tau_{\alpha,\beta'} \triangle \tau_{\alpha',\beta}$

Sketch of proof

1. Duality

[Kramers-Wannier 1941], [Wu 1969], [Kadanoff-Ceva 1971], [Dubédat 2011]...

Other encoding of configurations:



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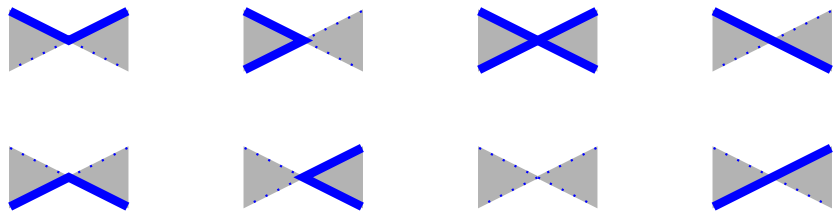

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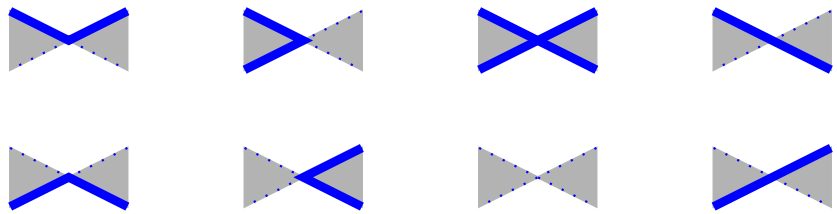

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Compatibility: $H = \ker(\Psi)$ where

$$\Psi : \left(\mathbb{Z}_2^2\right)^E \to \mathbb{Z}_2^{V \cup F}$$

$$(x_e, y_e)_{e \in E} \mapsto \left(\sum_{e \sim v} y_e, \sum_{e \sim f} x_e \right)_{v \in V, f \in F}$$

Sketch of proof

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$$\left(\mathbb{Z}_2^2\right)^E \xrightarrow{\Psi} (\mathbb{Z}_2)^{V \cup F}$$

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Sketch of proof

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$$(\mathbb{Z}_2)^{V\cup F} \xrightarrow{\Phi} \left(\mathbb{Z}_2^2\right)^E \xrightarrow{\Psi} (\mathbb{Z}_2)^{V\cup F}$$

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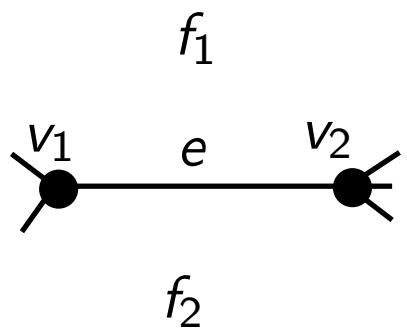
$$\Phi\left(\sigma_V, \sigma_F\right)_e = \left(\sigma_{v_1} + \sigma_{v_2}, \sigma_{f_1} + \sigma_{f_2}\right)$$



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By equipping $(\mathbb{Z}_2)^{V\cup F}$ with the standard scalar product $(\cdot,\cdot)$ and $\left(\mathbb{Z}_2^2\right)^E$ with the symplectic one $\langle\cdot|\cdot\rangle$:

$$\langle (x_e, y_e)|(x'_e, y'_e)\rangle = \sum_{e\in E} x_e y'_e + x'_e y_e,$$

one has $\Phi = \Psi^*$. Hence

$$H = H^{\perp}.$$

Sketch of proof

1. Duality

We have a space $(\mathbb{Z}_2^2)^E$ equipped with a (symplectic) form $\langle \cdot | \cdot \rangle$, so there is a Fourier transform:

$$\hat{g}(\tau) = \sum_{\tau'} (-1)^{\langle \tau | \tau' \rangle} g(\tau'),$$

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Applying this to the set of 8V-configurations $H = H^\perp$ and with g being the weight function, we get Wu's abelian duality.

Sketch of proof

1. Duality

$$Z_{8V}(a, b, c, d) = Z_{8V}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$$

where

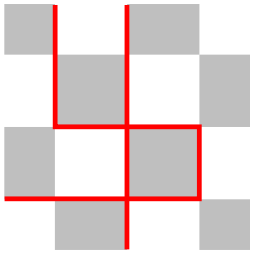
$$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

More generally, allows for tracking *order-disorder* variables.

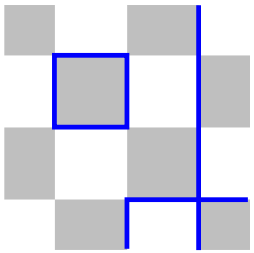
Sketch of proof

2. Spin switching

τ



τ'



Sketch of proof

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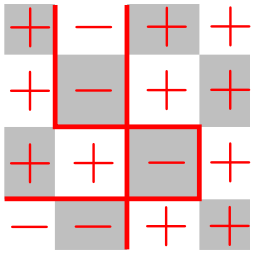

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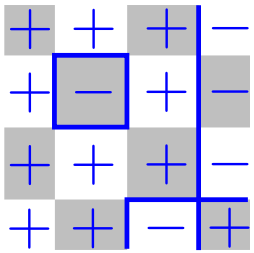

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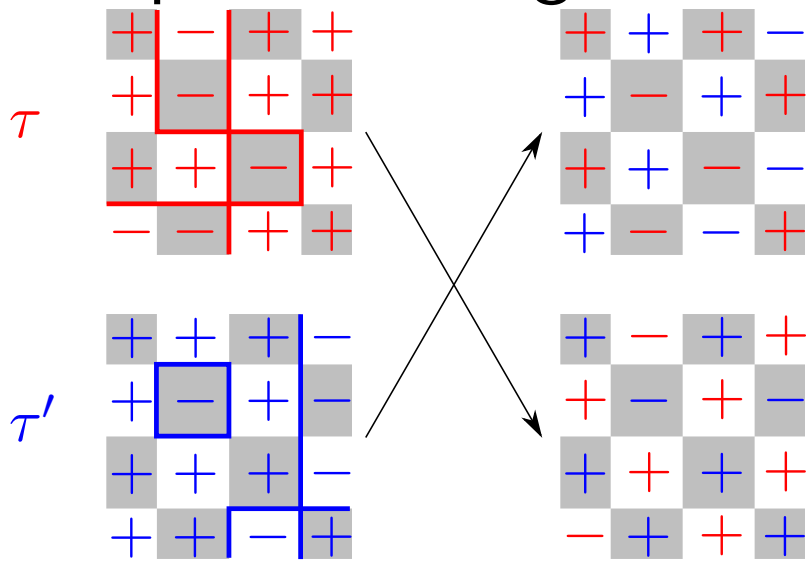

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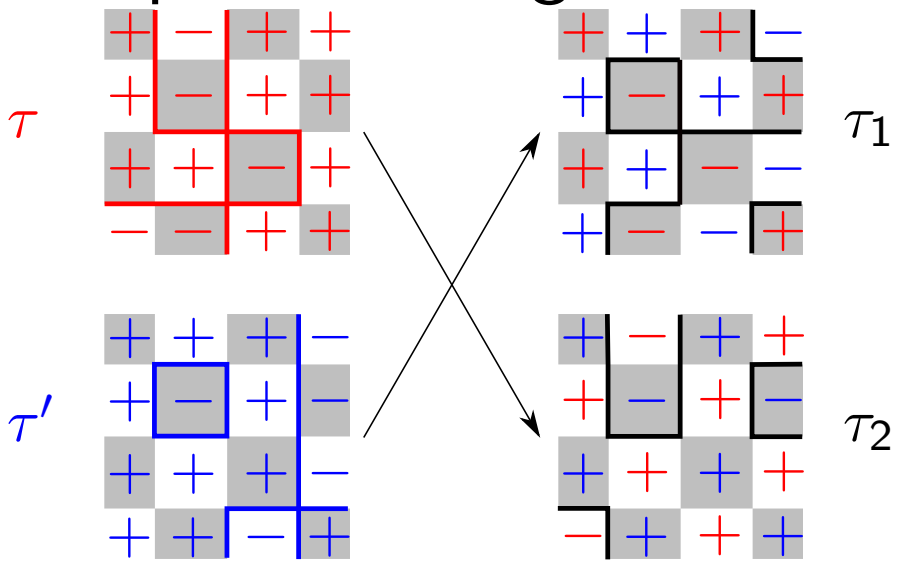

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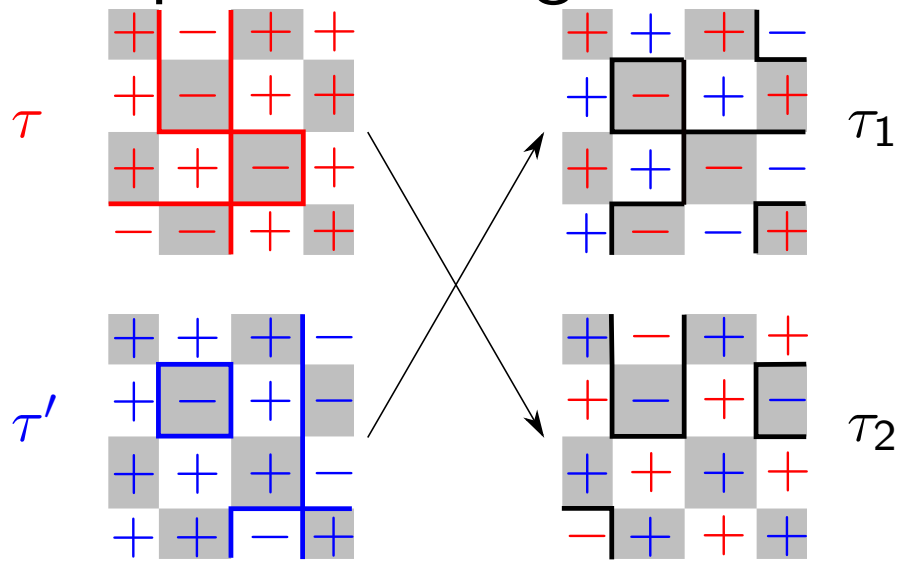
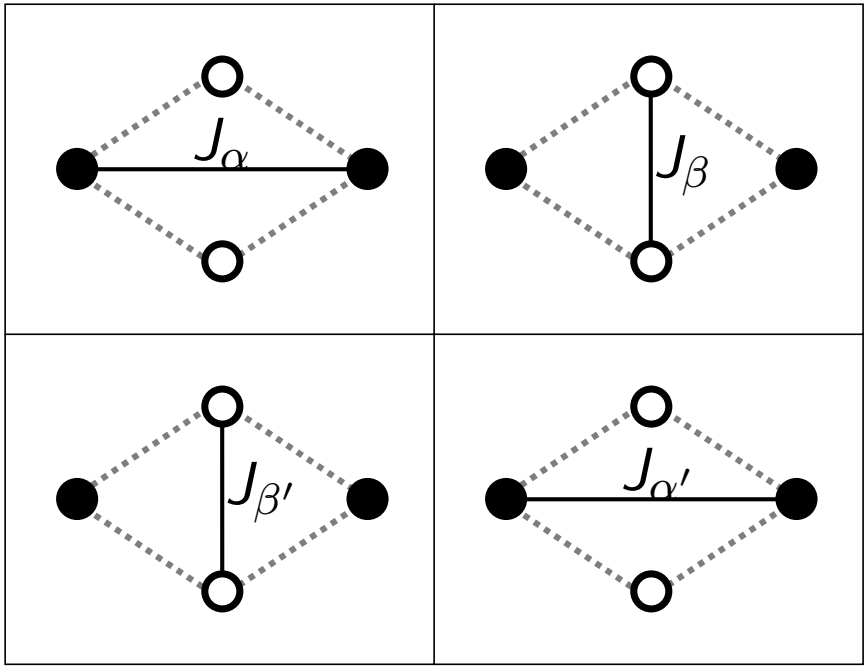


If $ab = cd$ and $a'b' = c'd'$,

$$
\begin{aligned}
&Z_{8V}(a, b, c, d) \\
&\times Z_{8V}(a', b', c', d') \\
&= Z_{8V}(a_1, b_1, c_1, d_1) \\
&\times Z_{8V}(a_2, b_2, c_2, d_2)
\end{aligned}
$$

Sketch of proof

3. Wrapping up



$\longrightarrow 8V^{\Gamma}_{\alpha,\beta} \dashleftarrow\dashrightarrow 8V_{\alpha,\beta}$ *(f-f)*

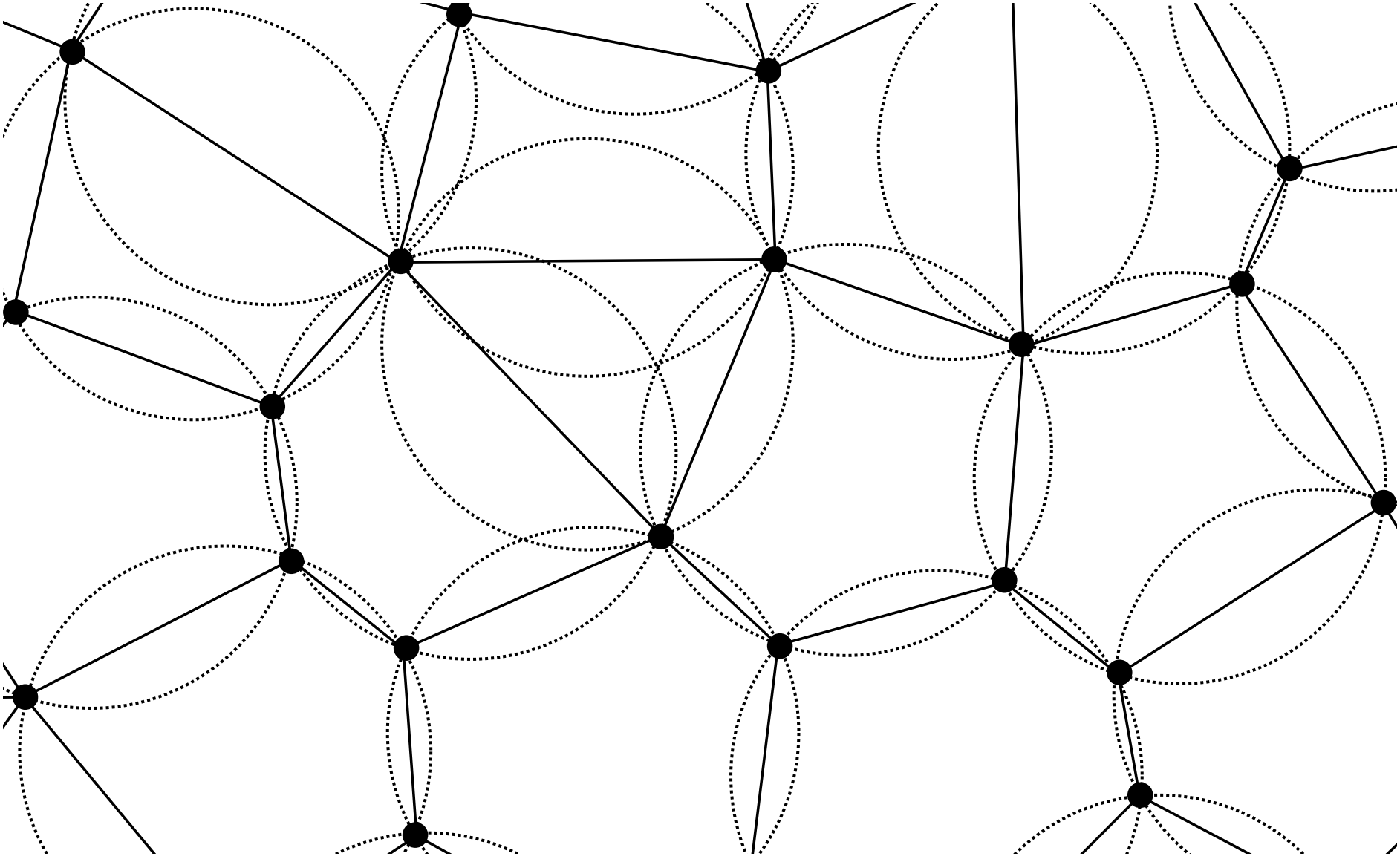
$\longrightarrow 8V^{\Gamma}_{\alpha',\beta'} \dashleftarrow\dashrightarrow 8V_{\alpha',\beta'}$

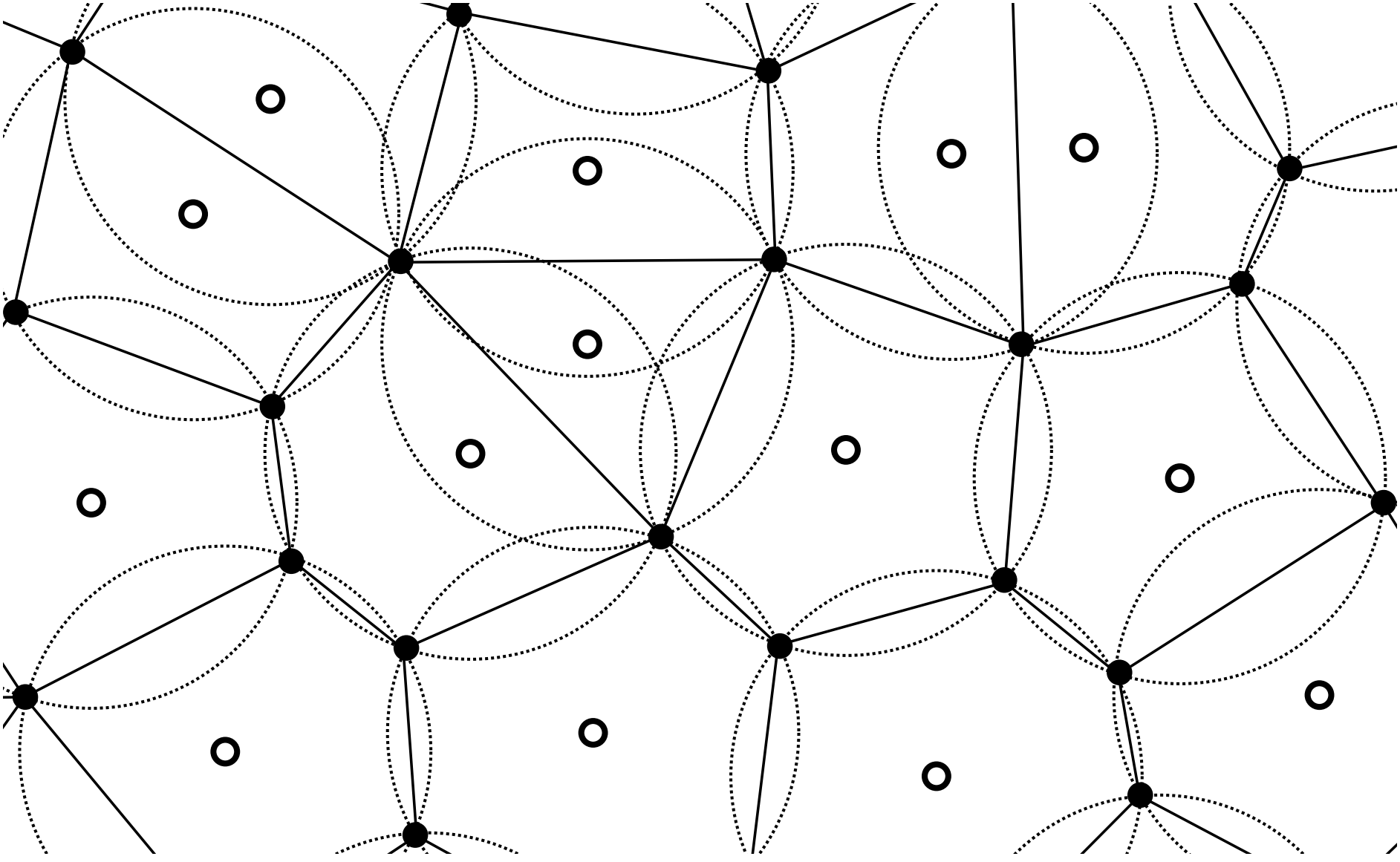
$8V^{\Gamma}_{\alpha,\beta'}$ $8V^{\Gamma}_{\alpha',\beta}$

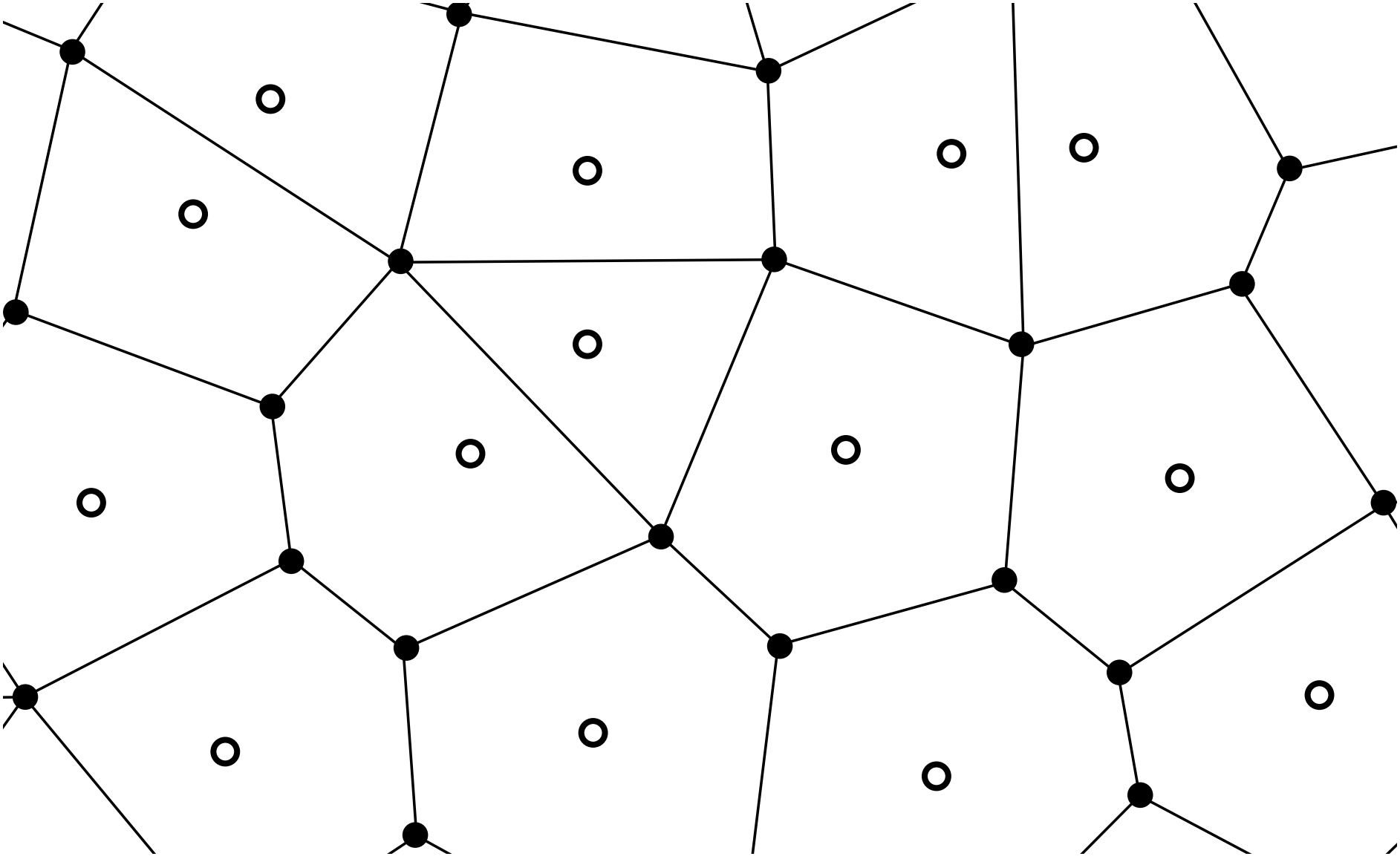
$\updownarrow$ $\updownarrow$

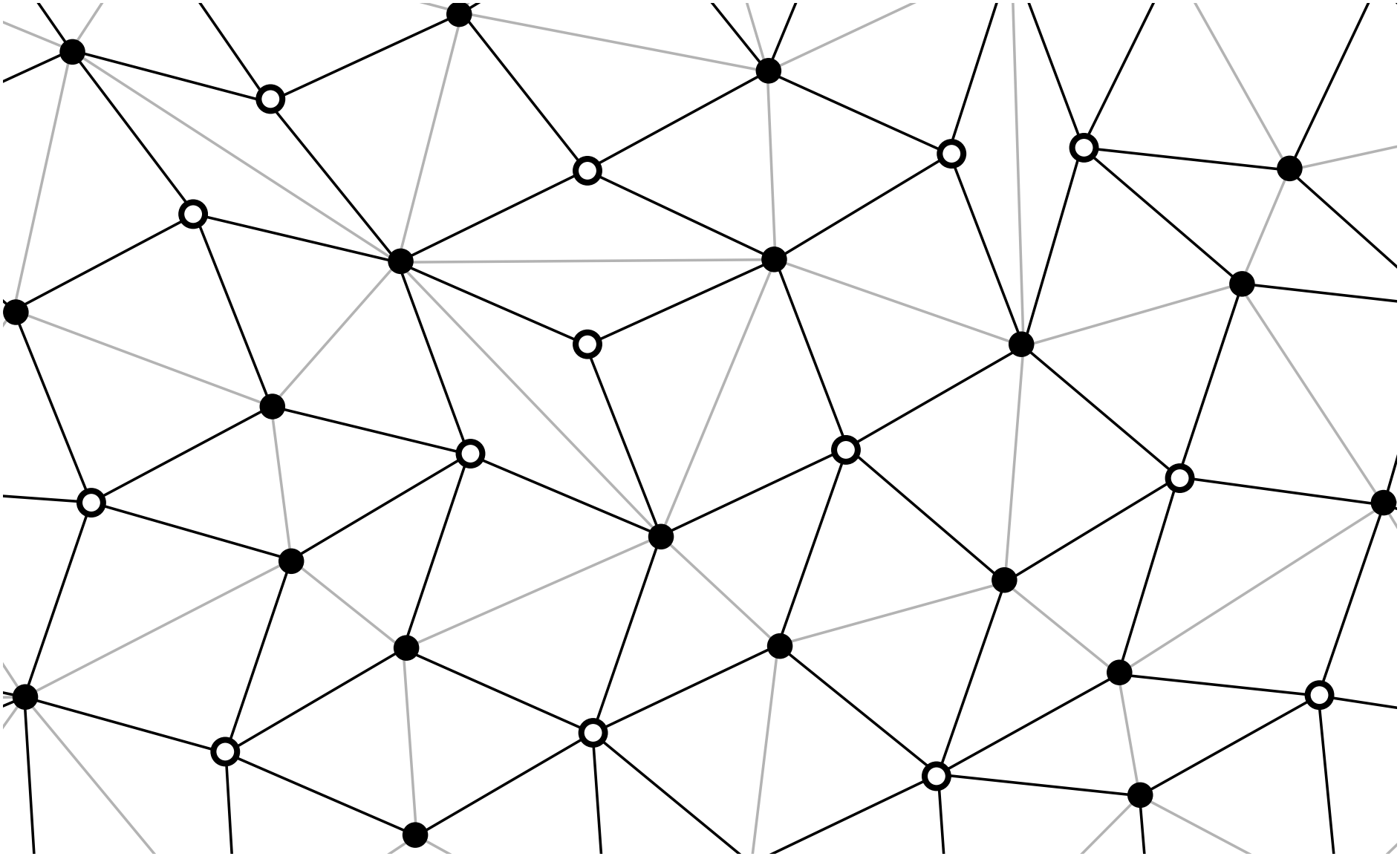
$8V_{\alpha,\beta'}$ $8V_{\alpha',\beta}$

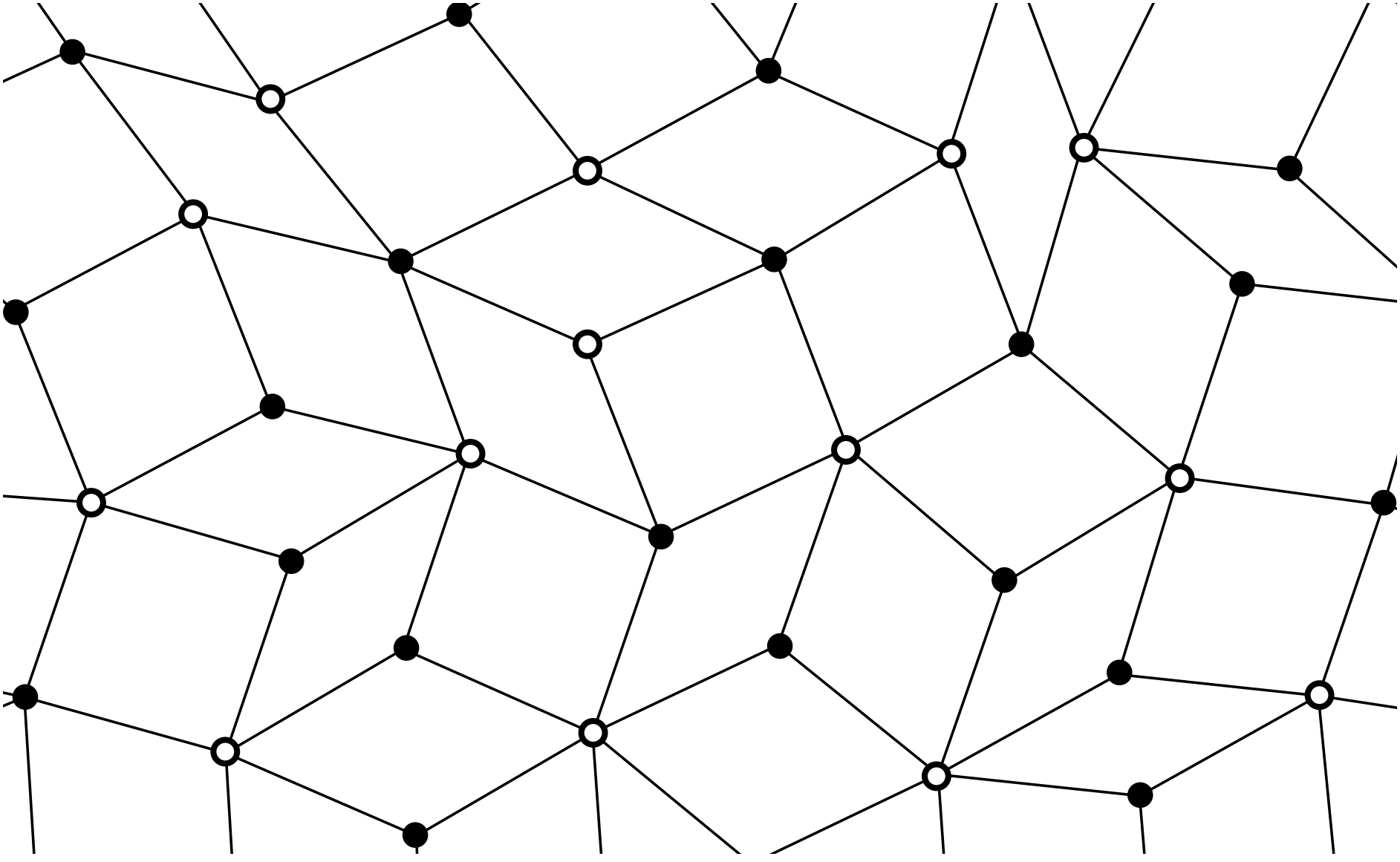
Application to the isoradial Z-invariant setting

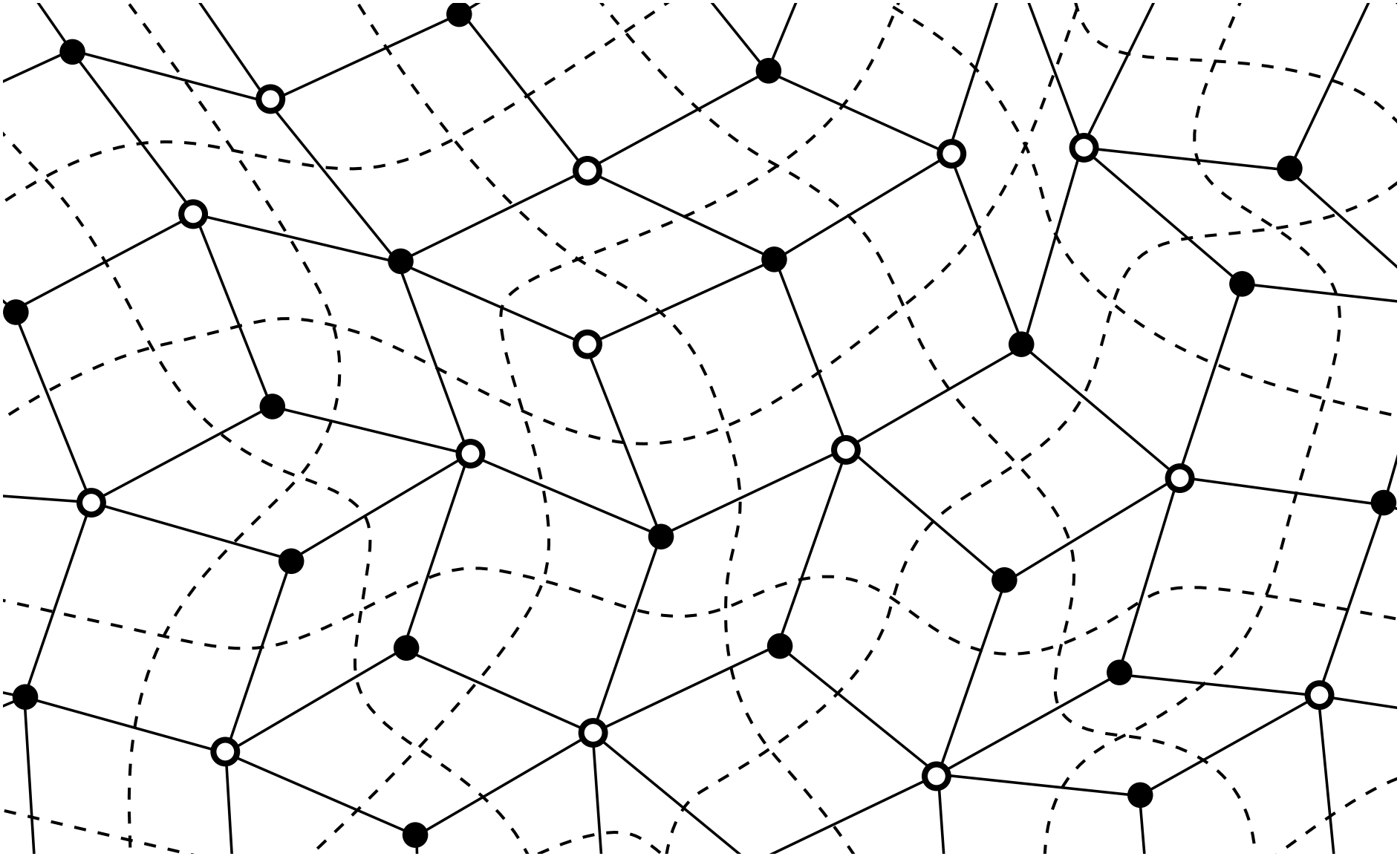


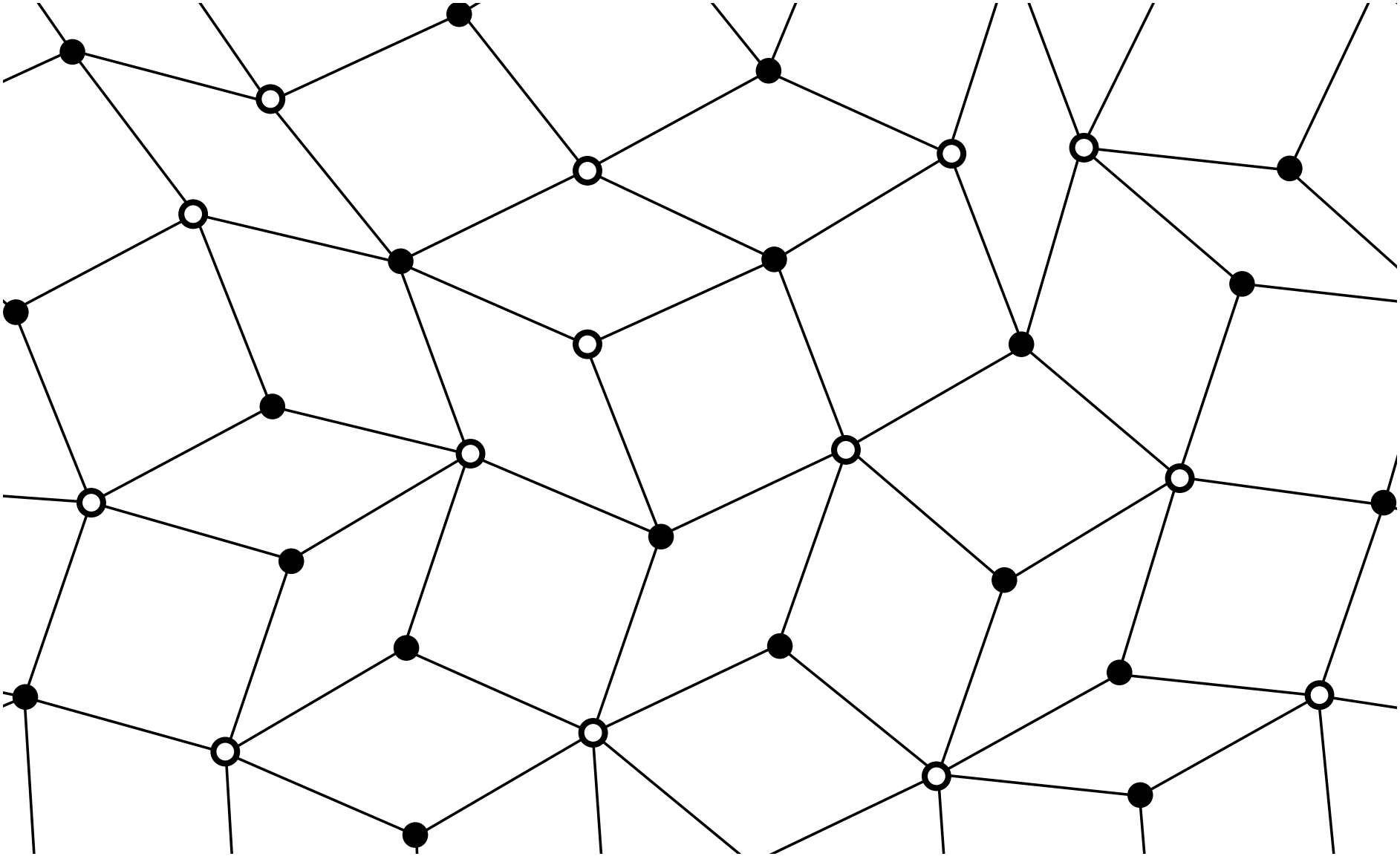


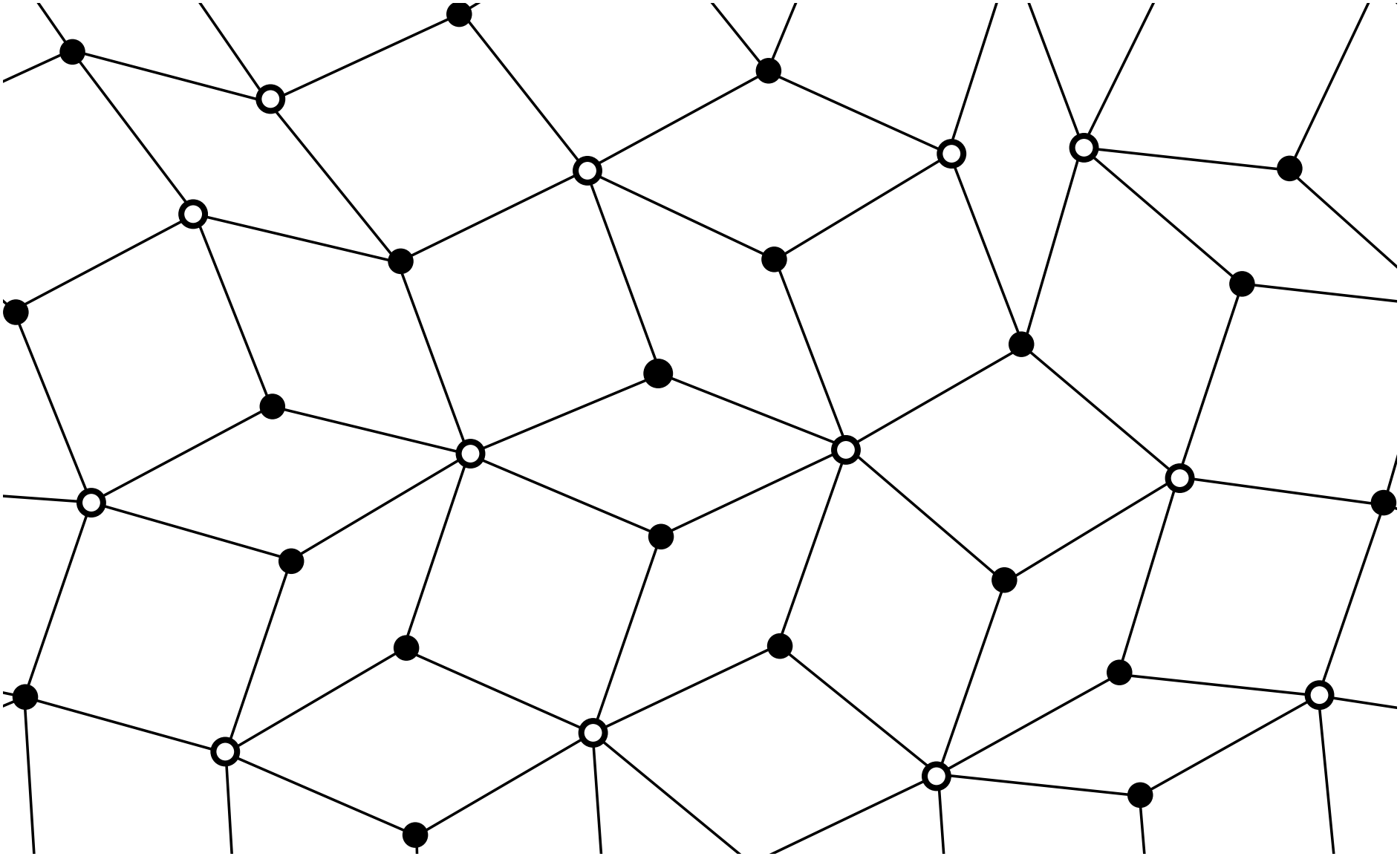




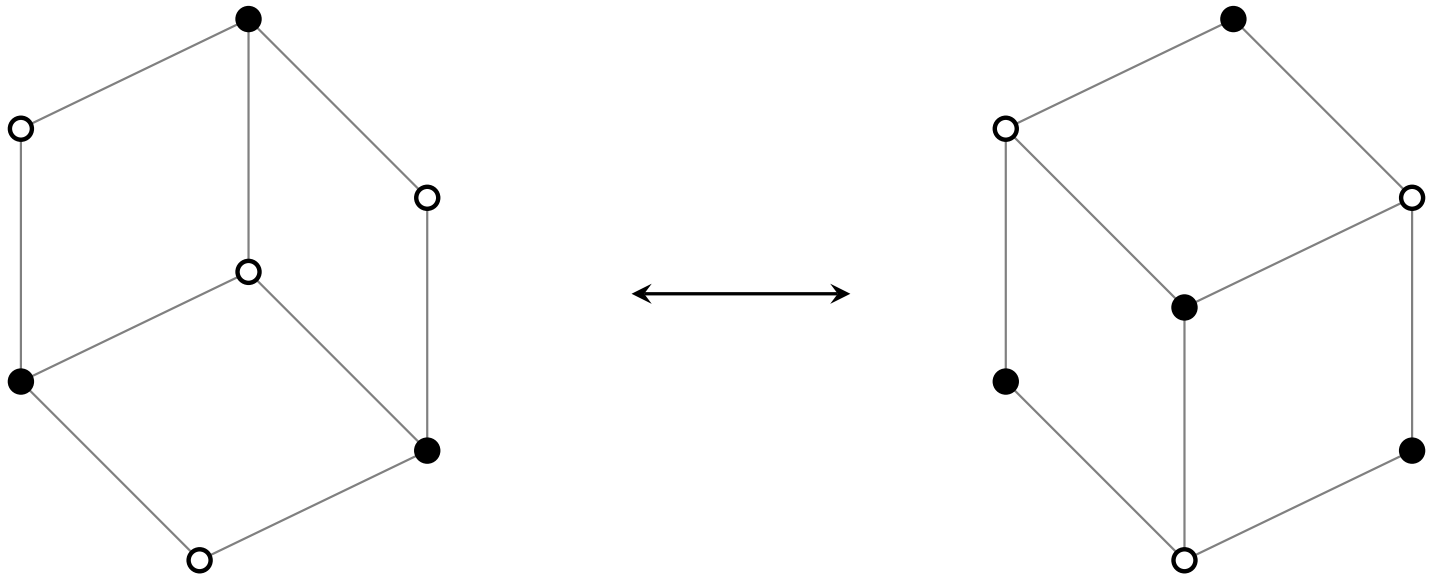








Star-triangle move for lozenges



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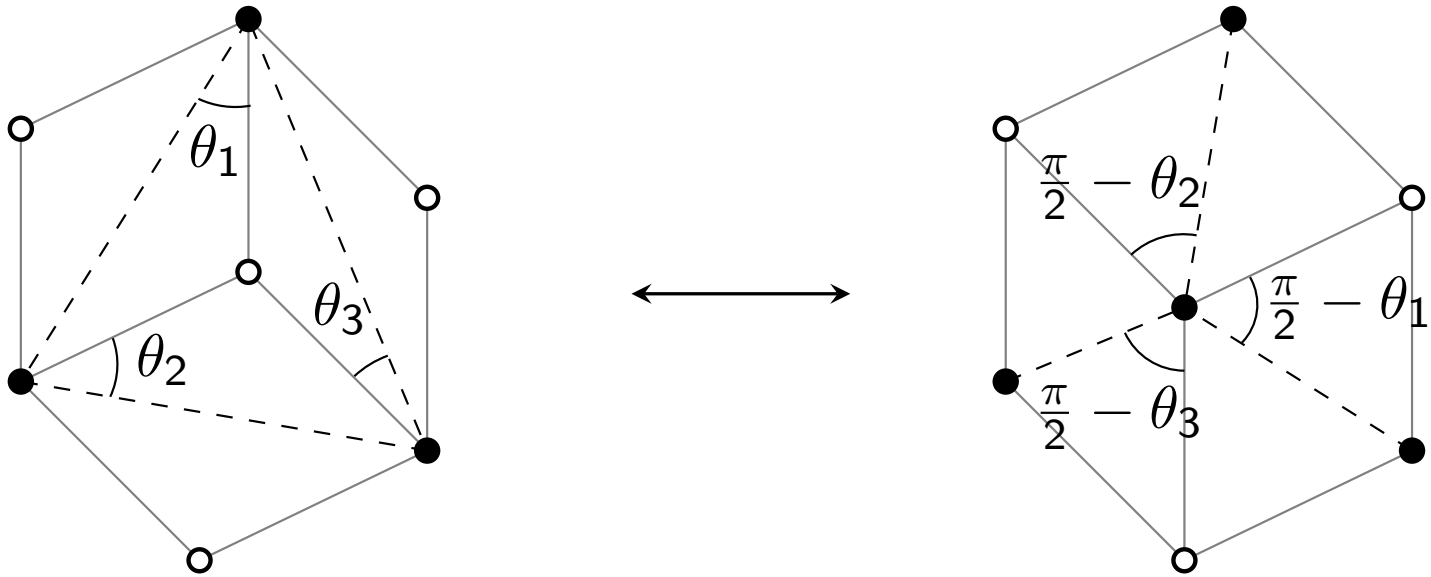
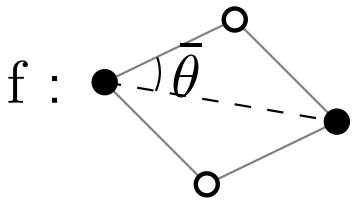


Star-triangle move for lozenges

Free-fermion 8V weights on the face of lozenges given by the geometry ($k, \ell \in [0, 1)$):



$$a(\mathrm{f}) = \mathrm{sn}\,(\theta|k) + \mathrm{sn}\,(\theta|\ell)$$
$$b(\mathrm{f}) = \mathrm{cn}\,(\theta|k) + \mathrm{cn}\,(\theta|\ell)$$
$$c(\mathrm{f}) = 1 + \mathrm{sn}\,(\theta|k)\,\mathrm{sn}\,(\theta|\ell) + \mathrm{cn}\,(\theta|k)\,\mathrm{cn}\,(\theta|\ell)$$
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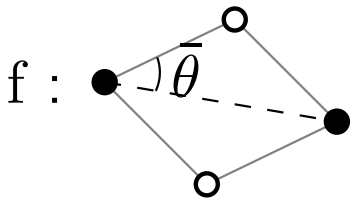

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Proposition *[M. 2018]*

This 8V model is invariant in distribution under star-triangle transformation of lozenges.

Star-triangle move for lozenges

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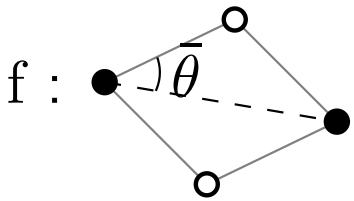

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Proposition *[M. 2018]*

This 8V model is invariant in distribution under star-triangle transformation of lozenges.

Z-**invariant** regime (Baxter).
Corresponding 6V models in *[Boutillier-de Tilière-Raschel 2016]*

Gibbs measure

Fix $k, \ell \in [0, 1)$ and the Z-invariant weights of the 8V model.

Theorem *[M. 2018]*

For any isoradial graph, and any $k, \ell \in [0, 1)$, there exists an ergodic Gibbs measure $\mathcal{P}_{k,\ell}$ on 8V configurations.

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$$\mathcal{P}_{k,\ell}(e_1, \ldots, e_n \in \tau) = \sqrt{\det \left(K_{k,\ell}^{-1} \right)_V}.$$

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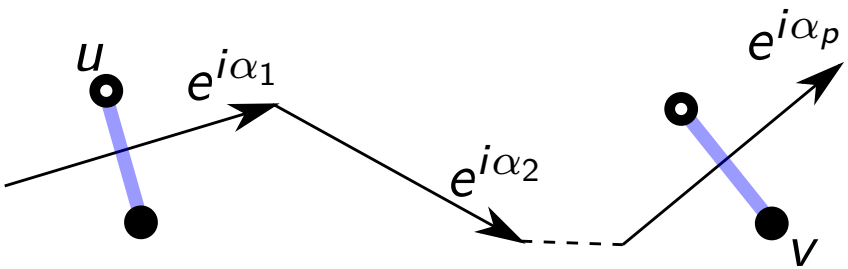
The operator $K_{k,\ell}^{-1}$ is explicit and **local**:



$$K_{k,\ell}^{-1}[u,v] = g(\alpha_1, \dots, \alpha_p).$$

Correlations

Theorem

If $0 < k < \ell < 1$, as $|x - y| \to \infty$,

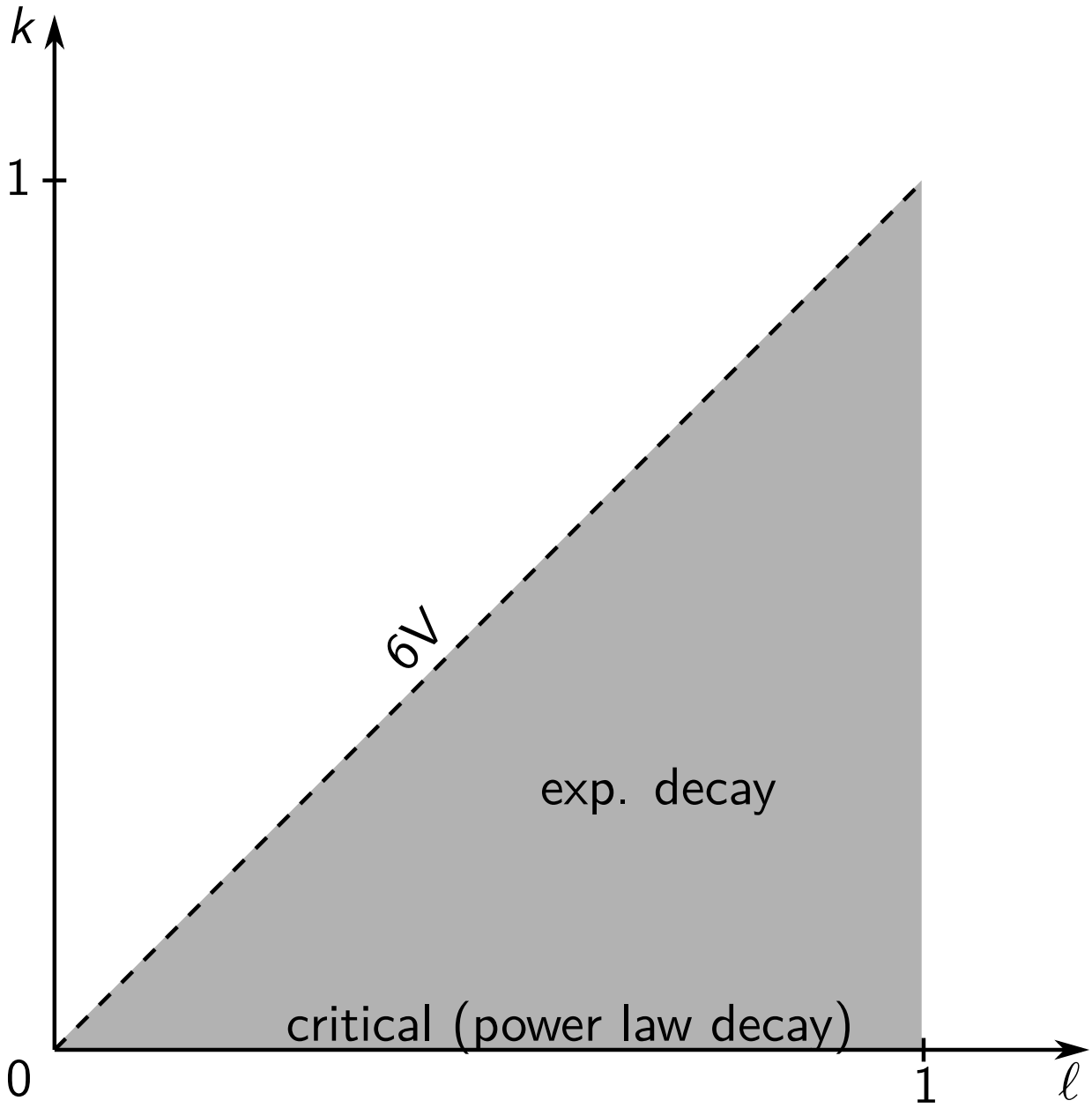
$$K_{k,\ell}^{-1}[x, y] \sim |x - y|^{-\frac{1}{2}} \exp\left(-\frac{|x - y|}{\xi_k}\right).$$

When $k \to 0$,

$$\xi_k = \Theta\left(k^{-2}\right) = \Theta\left((\beta - \beta_c)^{-1}\right).$$

Critical exponent $\nu = 1$ (universality class of the Ising model).

Correlations



Summary

In free-fermion vertex models, there exists a generic relation

$$8V^2 = 6V_1 \times 6V_2.$$

It can be made global, local, algebraic, probabilistic...

Perspectives

- Computation of density of states a, b, c, d.
- Geometric interpretation/embedding? (Regge symmetry)
- Analytic extension?