Spins, percolation and height functions

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A new family of models

- Our motivation is to abstract some recurring combinatorial themes present in models of two-dimensional statistical mechanics
- To that end we introduce a four parameter

$$(q, q', a, b) \in \{1, 2, \dots\}^2 \times (0, 1]^2$$

  model of

  1. spins (σ, σ')
  2. height function (h, h')
  3. bond percolation (ω, ω')

  which generalizes the

  - FK-random cluster and Potts models
  - six-vertex model
  - loop $O(n)$ model
  - random current, double random current and XOR-Ising model
- We discuss its basic properties and asymptotic behaviour

The Potts model

Let Q be a finite set with q elements.

For a finite graph $G = (V, E)$ and a *coupling constant* J, the *q-state Potts model* is a probability measure on Q^V given by

$$\mu(\mathrm{s}) = \frac{1}{Z} \exp\Big(-J \sum_{\{v_1, v_2\} \in E} \mathbf{1}\{\mathrm{s}(v_1) \neq \mathrm{s}(v_2)\}\Big), \quad \mathrm{s} \in Q^V,$$

where Z is the partition function.

We say that the model is *ferromagnetic* if $J \geq 0$ and *antiferromagnetic* if $J < 0$.

The Edwards–Sokal coupling

The q-state Potts model is related to the FK(q) *random cluster model* by the classical *Edwards–Sokal coupling*, where for each edge $\{v_1, v_2\}$ satisfying $\mathrm{s}(v_1) = \mathrm{s}(v_2)$, one declares it **open** with probability $1 - e^{-J}$ and independently of other edges.

The resulting configuration of open edges ζ is the random cluster model.

Conditioned on ζ, the spins s can be recovered by choosing a uniform spin from Q independently for each *cluster* of ζ, where a cluster is a connected component of (V, ζ), including isolated vertices.

In particular, if Q is symmetric,

$$\langle \mathrm{s}(v_1)\mathrm{s}(v_2) \rangle = \mu(\mathrm{s}_{v_1}^2)\mu(v_1 \overset{\zeta}{\longleftrightarrow} v_2),$$

where $\{v_1 \overset{\zeta}{\longleftrightarrow} v_2\}$ is the event that v_1 and v_2 are in the same cluster of ω.

The set-up

Let M be a compact, orientable surface with no boundary, or the plane.

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a finite connected graph embedded in M in such a way that each face is a topological disc, and let $\mathsf{G}^* = (\mathsf{U}, \mathsf{E}^*)$ be its dual, where U is identified with the set of faces of G.

For $e \in \mathsf{E} \cup \mathsf{E}^*$, we write $e^* \in \mathsf{E} \cup \mathsf{E}^*$ for its dual edge. Similarly for $\omega \subseteq \mathsf{E} \cup \mathsf{E}^*$, we write $\omega^* = \{e^* : e \in \omega\}$.

For $\omega \subseteq \mathsf{E}$, we write $\omega^\dagger = \mathsf{E}^* \setminus \omega^*$, and for $\omega' \subseteq \mathsf{E}^*$, $(\omega')^\dagger = \mathsf{E} \setminus (\omega')^*$.

1. The spin model

Fix $q, q' \in \{1, 2, \ldots\}$ and let $Q, Q' \subset \mathbb{R}$ satisfy

$$Q = -Q, \quad Q' = -Q', \quad |Q| = q, \quad \text{and} \quad |Q'| = q'.$$

A *spin configuration* on V (resp. U) is any function $\sigma : \mathsf{V} \to Q$ (resp. $\sigma' : \mathsf{U} \to Q'$).

We define *contour configurations*

$$\eta(\sigma) = \{\{v_1, v_2\}^* : \sigma(v_1) \neq \sigma(v_2)\} \subseteq \mathsf{E}^*,$$

and $\eta(\sigma') \subseteq \mathsf{E}$ in a dual fashion. A connected component of η is called a *contour*.

1. The spin model

The configuration space of our (constrained) *spin model* is

$$\Sigma = \{(\sigma, \sigma') \in Q^{\mathsf{V}} \times Q'^{\mathsf{U}} : \eta(\sigma)^* \cap \eta(\sigma') = \emptyset\}.$$

In other words, this is the set of all pairs (σ, σ') whose interfaces do not cross.

Equivalently,

$$(\sigma(v_1) - \sigma(v_2))(\sigma'(u_1) - \sigma'(u_2)) = 0 \qquad (*)$$

for every pair of a primal edge $\{v_1, v_2\}$ and its dual $\{u_1, u_2\}$.

Note that σ is constant on $\eta(\sigma')$ and vice versa.

1. The spin model

We study a probability measure on Σ given by

$$\mathbf{P}(\sigma, \sigma') = \frac{1}{\mathcal{Z}} a^{|\eta(\sigma')|} b^{|\eta(\sigma)|},$$

where $a, b \in (0, 1]$, and $\mathcal{Z}$ is the partition function.

This is equivalent to a pair of independent primal and dual ferromagnetic Potts models with q and q' spins, with coupling constants

$$b = e^{-J}, \quad \text{and} \quad a = e^{-J'},$$

and conditioned on Σ.

2. The height function

$\{v_1, u_1, v_2, u_2\}$ is a *quad*, if $\{v_1, v_2\} \in \mathsf{E}$ and $\{v_1, v_2\}^* = \{u_1, u_2\}$.

Assume that M is of genus zero. For $(\sigma, \sigma') \in \Sigma$, we consider a *height function* $H : \mathsf{V} \cup \mathsf{U} \to \mathbb{R}$ defined up to a constant by the rule: If $u \in \mathsf{U}$ and $v \in \mathsf{V}$ belong to the same quad, then

$$H(u) - H(v) = \sigma(v)\sigma'(u).$$

That these relations are consistent follows from condition (*). Indeed, (*) is equivalent to the fact that the sum of the gradients around each quad is zero.

2. The height function

We will denote by h and h' the restriction of H to V and U respectively. Note that if $\{v_1, u_1, v_2, u_2\}$ is a quad, then

$$h'(u_2) - h'(u_1) = \sigma(v_1)(\sigma'(u_2) - \sigma'(u_1)) = \sigma(v_2)(\sigma'(u_2) - \sigma'(u_1)). \quad (**)$$

Hence, $\eta(\sigma')$ are the *level lines* of h'.

Remark

In higher genera one can define a height function on the universal cover of M. Equivalently, one can talk about the increment of the height function between two points taken along a curve, up to the homotopy of the curve.

3. Bond percolation

Given $(\sigma, \sigma') \in \Sigma$ sampled according to $\mathbf{P}$,

1. Declare each primal edge in $\eta(\sigma')$ and each dual edge in $\eta(\sigma)$ **open**.
2. For each pair of a primal and its dual edge e and e^* such that neither $e \in \eta(\sigma')$ nor $e^* \in \eta(\sigma)$, and independently of other such pairs, declare the state of the edges with the following probabilities:

	$a + b \leq 1$	$a + b \geq 1$
e **open**, e^* **closed**	a	$1 - b$
e **closed**, e^* **open**	b	$1 - a$
both e, e^* **open**	$1 - a - b$	0
both e, e^* **closed**	0	$a + b - 1$

Note that in both cases the probability of opening e and e^* is $1 - b$ and $1 - a$ respectively.

3. Bond percolation

A *cluster* of ω, resp. ω', is a connected component of the graph (V, ω), resp. (U, ω'), including the isolated vertices.

We define

$$\Omega\Sigma = \{(\omega, \omega', \sigma, \sigma') : \sigma \text{ constant on clusters of } \omega \text{ and } \eta(\sigma) \subseteq \omega', \\ \sigma' \text{ constant on clusters of } \omega' \text{ and } \eta(\sigma') \subseteq \omega\},$$

where $(\sigma, \sigma') \in \Sigma$, and we denote by

$$\mathbf{P}(\omega, \omega', \sigma, \sigma')$$

the probability measure on $\Omega\Sigma$ given by the coupling above.

Note that

$$\omega^\dagger \subseteq \omega' \text{ for } a + b \leq 1, \qquad \text{and} \qquad \omega^\dagger \supseteq \omega' \text{ for } a + b \geq 1.$$

3. Bond percolation

Relevant literature:

- C. E. Pfister and Y. Velenik, *Random-cluster representation of the Ashkin-Teller model*, Journal of Statistical Physics 88 (1997Sep), no. 5, 1295–1331.
- A. Glazman and R. Peled, *On the transition between the disordered and antiferroelectric phases of the 6-vertex model*, 2018. arXiv:1909.03436.
- G. Ray and Y. Spinka, *Finitary codings for gradient models and a new graphical representation for the six-vertex model*, 2019. arXiv:1908.09056.

Edwards–Sokal property

Proposition (Conditional laws)

Conditioned on ω,

1. σ is distributed like an independent uniform assignment of a spin from Q to each cluster of ω.

2. σ' is distributed like the q-state Potts model with coupling constant J satisfying $e^{-J} = \frac{a}{1-b}$, and defined on the dual $(\mathsf{V}(\omega), \omega)^*$.

3. in particular, σ and σ' are independent.

Edwards–Sokal property

Proof.

We claim that for fixed (ω, σ') with $\eta(\sigma') \subseteq \omega$, the weight of each consistent configuration $(\omega, \sigma, \sigma')$, i.e., such that σ is constant on the clusters of ω, is equal to

$$a^{|\eta(\sigma')|}(1-b)^{|\omega \setminus \eta(\sigma')|} b^{|\omega^\dagger|},$$

and in particular is independent of σ.

Indeed each edge in

- $\eta(\sigma)$ contributes weight b by the definition of the spin model,
- $\omega^\dagger \setminus \eta(\sigma)$ also contributes weight b since this is the probability that a dual edge $\{u_1, u_2\}$ with $\sigma'(u_1) = \sigma'(u_2)$ ends up in $\omega^\dagger$ in step (2) of the definition of the edge percolation model.

This means that conditioned on (ω, σ'), we have a uniform distribution on all spin configurations σ such that $\eta(\sigma) \subseteq \omega^\dagger$. □

Random cluster model ($a + b = 1$)

Proposition

Assume that M is of genus zero, and $a + b = 1$. Let

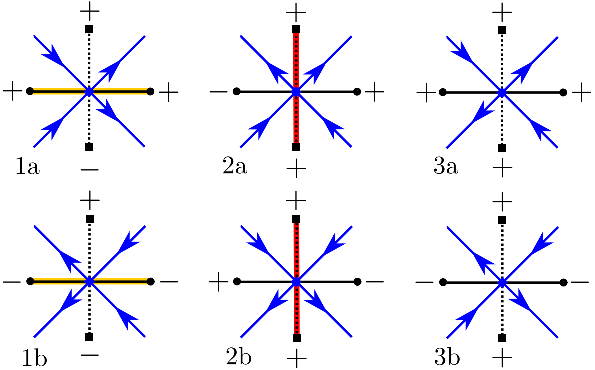
$$p = \frac{q'}{q' + a^{-1} - 1} \in (0, 1],$$

and let $k(\omega)$ be the number of clusters of ω. Then the marginal distribution of $\mathbf{P}$ on ω is given by

$$\mathbf{P}(\omega) \propto (qq')^{k(\omega)} p^{|\omega|} (1 - p)^{|\mathsf{E} \setminus \omega|},$$

which is the FK(qq') *random cluster model* on G with free boundary conditions.

Six-vertex model ($q = q' = 2$)



A primal edge (solid), its dual edge (dashed), and four corresponding medial edges (blue). The sets of yellow primal and red dual edges η and η' are given by a mapping of Rys '63

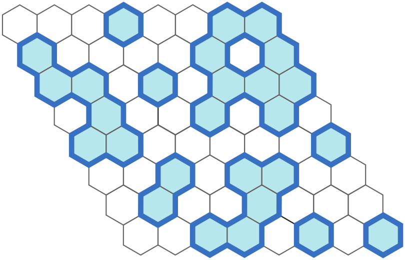
Loop $O(n)$ model ($q' = 2$, $q = n$, $b = 1$)

Proposition

Assume that G is trivalent, and $q' = 2$, $q = n$, $b = 1$. Then

$$\mathbf{P}(\eta) \propto n^{k(\eta)} a^{|\eta|} \propto n^{\#\text{ loops in }\eta} \left(\tfrac{a}{n}\right)^{|\eta|},$$

which is the law of the *loop $O(n)$ model* with $x = a/n$.



Random currents ($q' = 2$, $q = 1$, $a^2 + b^2 = 1$)

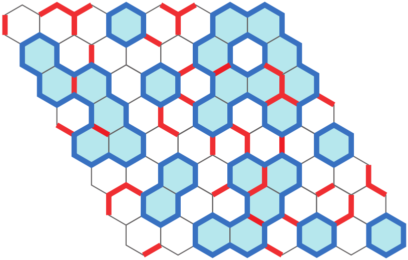
Proposition

Assume that M is of genus zero. Let $a^2 + b^2 = 1$, $q' = 2$ and $q = 1$. Then

$$\mathbf{P}(\eta, \omega) \propto a^{|\eta|}(1-b)^{|\omega \setminus \eta|} b^{|\mathsf{E} \setminus \omega|},$$

which is the law of the sourceless *single random current* with $a = \tanh J$.

Moreover, σ is distributed like the *Ising model*.



Double random currents and XOR-Ising model ($q' = q = 2$, $a^2 + b^2 = 1$)

Proposition

Assume that M is of genus zero. Let $x \in (0, 1]$ be given by $a = 2x/(1 + x^2)$, and let $a^2 + b^2 = 1$ and $q' = q = 2$. Then

$$\mathbf{P}(\eta, \omega) \propto 2^{k(\omega)+|\omega|} x^{|\eta|} (x^2)^{|\omega \setminus \eta|} (1 - x^2)^{|E \setminus \omega|},$$

which is the law of the sourceless *double random current* with $x = \tanh J$, or equivalently $a = \tanh 2J$.

Moreover, σ and σ' are distributed like the *XOR-Ising model*.

The second part of the statement was first discovered during a discussion with Roland Bauerschmidt, Hugo Duminil-Copin, and Aran Raoufi at IHES, Bures-sur-Yvette, in 2017.

An unconstrained spin system

Consider a spin model on $(\mathrm{s}, \mathrm{s}') \in Q^{\mathsf{V}} \times {Q'}^{\mathsf{V}}$ given by the Gibbs-Boltzmann distribution

$$\tilde{\mu}(\mathrm{s}, \mathrm{s}') \propto \exp\Bigg(\sum_{\{v_1,v_2\}\in\mathsf{E}} \delta_{\mathrm{s}(v_1),\mathrm{s}(v_2)}\big(\alpha + \beta\delta_{\mathrm{s}'(v_1),\mathrm{s}'(v_2)}\big)\Bigg),$$

where

$$\alpha = \ln\left(\frac{1-a}{b}\right) \qquad \text{and} \qquad \beta = \ln\left(1 + \frac{q'a}{1-a}\right).$$

This is a special case of the model of Domany & Riedel ’78.

Theorem

Assume that M is of genus zero. Then the distributions of σ under $\mathbf{P}$, and of s under $\tilde{\mu}$ are the same.

Behaviour of height function

Consider the model on $\Lambda_N = \{-N, \ldots, N\}^2$, and let $h' = 0$ on the unbounded face of Λ_N.

Question

What is the behaviour of

$$\mathbf{Var}_{\Lambda_N}[h'(u_0)] \quad \text{as} \quad N \to \infty?$$

- variance bounded $\leftrightarrow$ *localization*
- variance unbounded $\leftrightarrow$ *delocalizatoin*

In the case when $q = q' = 2$ and $a = b$, localization was proved for $a < 1/2$ (Duminil-Copin et al. '16, Glazman & Peled '18), and delocalization for $a = 1/2$ (Duminil-Copin & Sidoravicius & Tassion, Glazman & Peled '18), $a = \sqrt{2}/2$ (Kenyon '99) and its small neighbourhood (Giuliani & Mastropietro & Toninelli '14), and $a = 1$ (Chandgotia et al. 2018).

Height function $\leftrightarrow$ percolation

For $u_1, u_2 \in \mathsf{U}$, let $N(u_1, u_2)$ be the number of clusters of ω *disconnecting* u_1 from u_2.

Theorem

For $a + b \geq 1$, we have

$$\mathbf{Var}[h'(u_1) - h'(u_2)] \asymp \mathbf{E}\big[N(u_1, u_2)\big].$$

Height function $\leftrightarrow$ percolation

Proof.
Fix $u_1, u_2 \in \mathsf{U}$, and let

$$dh' = h(u_2) - h(u_1) \quad \text{and} \quad d\sigma'_{\mathcal{C}} = \sigma'_{\mathcal{C}}(u_2) - \sigma'_{\mathcal{C}}(u_1).$$

Note that if $\mathcal{C}$ does not disconnect u_1 from u_2, then $d\sigma'_{\mathcal{C}} = 0$.

We claim that

$$dh' = \sum_{\mathcal{C}} \sigma(\mathcal{C}) d\sigma'_{\mathcal{C}}.$$

Indeed, let $\gamma = \{\tilde{u}_1, \ldots, \tilde{u}_l\}$ be a path of faces with $\tilde{u}_1 = u_1$ and $\tilde{u}_l = u_2$. Let v_j be one of the two vertices of the edge dual to $\{\tilde{u}_j, \tilde{u}_{j+1}\}$.
By (**), we have

$$dh' = \sum_{j=1}^{l} \sigma(v_j)(\sigma'(\tilde{u}_j) - \sigma'(\tilde{u}_{j+1})).$$

Height function $\leftrightarrow$ percolation

Therefore we have

$$
\begin{aligned}
\mathbf{Var}[dh'] =& \mathbf{E}\Big[\Big(\sum_{\mathcal{C}} \sigma(\mathcal{C}) d\sigma'_{\mathcal{C}}\Big)^2\Big] \\
=& \sum_{\omega \subseteq \mathsf{E}} \sum_{\mathcal{C}_1, \mathcal{C}_2 \subseteq \omega} \mathbf{E}[\sigma(\mathcal{C}_1) d\sigma'_{\mathcal{C}_1} \sigma(\mathcal{C}_2) d\sigma'_{\mathcal{C}_2} \mid \omega] \mathbf{P}(\omega) \\
=& \sum_{\omega \subseteq \mathsf{E}} \sum_{\mathcal{C}_1, \mathcal{C}_2 \subseteq \omega} \mathbf{E}[\sigma(\mathcal{C}_1)\sigma(\mathcal{C}_2) \mid \omega] \mathbf{E}[d\sigma'_{\mathcal{C}_1} d\sigma'_{\mathcal{C}_2} \mid \omega] \mathbf{P}(\omega) \\
=& \mathbf{E}[\sigma_0^2] \sum_{\omega \subseteq \mathsf{E}} \sum_{\mathcal{C} \subseteq \omega} \mathbf{E}[(d\sigma'_{\mathcal{C}})^2 \mid \omega] \mathbf{P}(\omega) \\
=& \mathbf{E}[\sigma_0^2] \mathbf{E}\Big[\sum_{\mathcal{C}} (d\sigma'_{\mathcal{C}})^2\Big] \\
=& \mathbf{E}[\sigma_0^2] \sum_{d \neq 0} d^2 \mathbf{E}[N_d] \\
\asymp& \mathbf{E}[N_{\neq 0}],
\end{aligned}
$$

where $N_d = N_d(u_1, u_2)$ is the number of clusters $\mathcal{C}$ of ω such that $d\sigma'_{\mathcal{C}} = d$.

Height function $\leftrightarrow$ percolation

Proposition

Assume that $a+b \geq 1$. Then

$$\mathbf{E}[N_{\neq 0}] \geq (1 - \tfrac{1}{q'})(\mathbf{E}[N'] - 1).$$

Proof.

- Let $\mathcal{C}'_1, \ldots, \mathcal{C}'_{N'}$ be the clusters of ω' that disconnect u_1 from u_2.
- If two consecutive clusters $\mathcal{C}'_l, \mathcal{C}'_{l+1}$ are assigned different spins, then there exists a circuit in $\eta(\sigma')$ disconnecting $\mathcal{C}'_i$ from $\mathcal{C}'_{l+1}$, and hence also disconnecting u_1 from u_2.

- Conditioned on σ' and ω', we recover ω by choosing randomly edges from $(\omega')^\dagger$ and adding them to $\eta(\sigma')$.
- This means that for every pair $\mathcal{C}'_l, \mathcal{C}'_{l+1}$ with different spin σ', there exists at least one cluster $\mathcal{C}$ of ω, disconnecting u_1 from u_2.
- Moreover, at least one of these clusters must satisfy $d\sigma'_{\mathcal{C}}(u_1, u_2) \neq 0$ (since the sum of $d\sigma'_{\mathcal{C}}$ over all such clusters is nonzero).
- This means that $N_{\neq 0}$ is at least equal to the number of pairs $\mathcal{C}'_l, \mathcal{C}'_{l+1}$ with different spin σ'.
- The latter is equal in distribution to the number of nearest neighbour disagreements in an i.i.d. sequence of length N'.

This finishes the proof of Proposition and Theorem. $\square$

Height function $\leftrightarrow$ percolation

Theorem

Consider a subsequential limit $\mathbf{P}_{\mathbb{Z}^2} = \lim_{k\to\infty} \mathbf{P}_{\mathbb{T}_{N_k}}$ of the *self-dual model* with $q = q'$ and $a = b > 1/2$, and assume that

$$\mathbf{P}_{\mathbb{Z}^2}(\omega \text{ percolates}) = 0.$$

Then

$$\mathbf{P}_{\mathbb{Z}^2}(\text{infinitely many clusters of } \omega \text{ surround the origin}) = 1.$$

and

$$\lim_{|u_1-u_2|\to\infty} \lim_{k\to\infty} \mathbf{Var}_{\mathbb{T}_{N_k}}[h'(u_1) - h'(u_2)] = \infty,$$

where the height increment $h'(u_1) - h'(u_2)$ is computed along one of the shortest paths from u_1 to u_2 in the dual torus $\mathbb{T}^*_{N_k}$.

Thank you!