Limit shape of perfect matchings on square-hexagon lattice

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Perfect matching: definition

- $G = (V, E)$: $|V| < \infty$, $|E| < \infty$
- Dimer configuration (perfect matching): subset of edges such that each vertex is incident to exactly one edge.
- Edge weight: $w : E(G) \to \mathbb{R}^+ \cup \{0\}$;
- $\mathrm{P}(M) = \frac{1}{Z} \prod_{e \in M} w(e)$;
- Partition function $Z = \sum_M \prod_{e \in M} w(e)$.
- When $w(e) = 1$, $\forall e \in E(G)$, Z is the total number of perfect matchings.

Perfect matching, dimer, and tiling

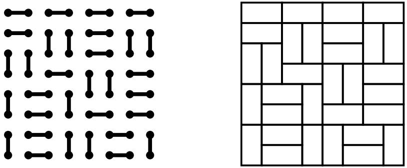

Figure: Perfect matching on square grid and domino tiling (by R. Kenyon)

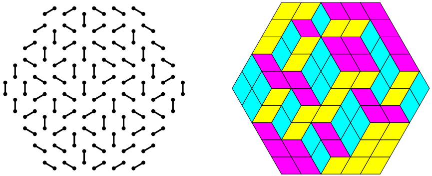

Figure: Perfect matching on hexagonal lattice and lozenge tiling (by R. Kenyon)

Asymptotic behavior

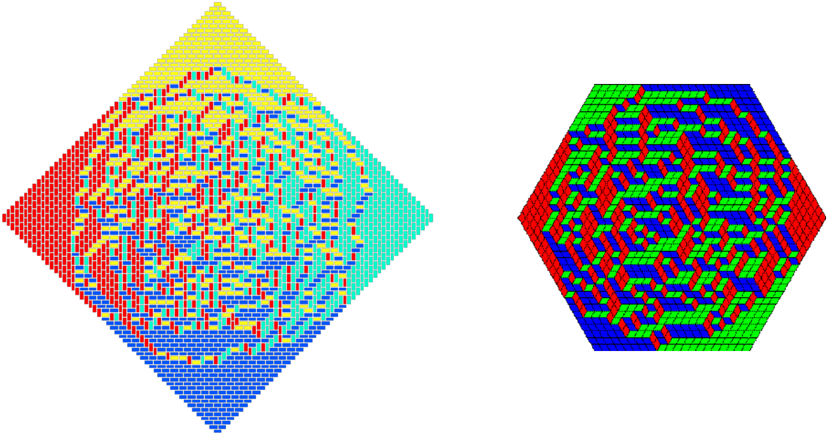

Figure: Limit shapes of uniform random tilings on square grid and hexagonal lattice (by James Propp)

Previous work

- (Cohn, Kenyon and Propp 2001) A variational principle for domino tilings
- (Okounkov, Reshetikhin 2001) Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram
- (Kenyon, Okounkov 2007) Explicitly solved the variational problem, obtain limit shape and frozen boundary for uniform lozenge tilings with certain boundary condition
- (Petrov, Gorin, Panova, Bufetov, Knizel, 2012-2018) Uniform perfect matchings on hexagon lattice and square grid with certain boundary conditions: limit shape and height fluctuation
- (Boutillier, Bouttier, Chauy, Corteel, Ramassamy, 2015) Dimers on rail yard graph as a Schur process, correlation function

Whole-plane lattice: local structures

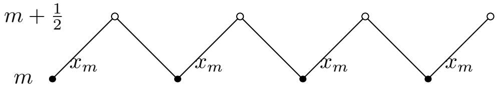

Figure: between levels m and $m+\frac{1}{2}$

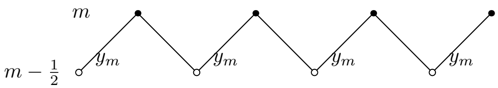

Figure: between levels $m-\frac{1}{2}$ and m when $a_m = 0$

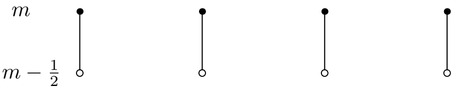

Figure: between levels $m-\frac{1}{2}$ and m when $a_m = 1$

Contracting lattice example: square grid

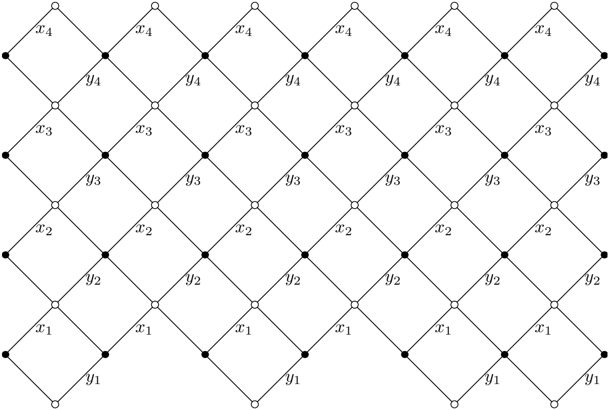

Figure: Rectangular Aztec diamond with $N = 4$, $m = 2$, $\Omega = (1, 3, 5, 6)$, and $a_i = 0$.

Contracting lattice examples: hexagon lattice

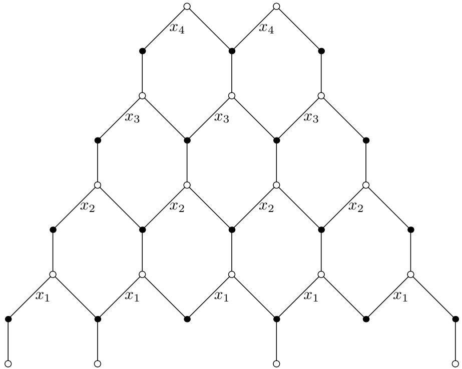

Figure: Contracting hexagon lattice with $N = 4$, $m = 2$, $\Omega = (1, 2, 4, 6)$, and $a_i = 1$.

Contracting lattice examples: square-hexagon lattice

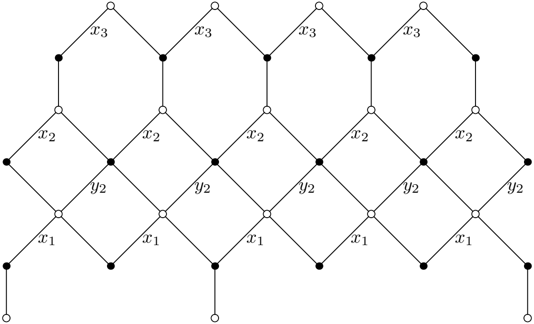

Figure: Contracting square-hexagon lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6), (a_1, a_2, a_3) = (1, 0, 1)$.

Overview

- Uniform boundary condition: $\Omega = (1, \ell, 2\ell, \ldots, N\ell)$; $\ell \in \mathbb{N}^+$ fixed.
- Piecewise boundary condition:

  - $\frac{x_{i+1,N}}{x_{i,N}} \leq N^{-\alpha}$, where $1 \leq i \leq n-1$, $\alpha > 0$.
  - $x_1 = x_2 = \ldots = x_n = 1$, $\{y_i\}_{i \in I_2}$ periodic.

Partition, Schur function and counting measure

- Partition of length N: an N-tuple of non-increasing, nonnegative integers: $\mu = (\mu_1 \geq \mu_2 \geq \ldots, \geq \mu_N \geq 0)$.
- $\mathbb{GT}_N^+$: all the partitions of length N.
- $\lambda \in \mathbb{GT}_N^+$.
- rational Schur function

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det_{i,j\in 1,\ldots,N}(u_i^{\lambda_j+N-j})}{\prod_{1\leq i<j\leq N}(u_i - u_j)}$$

- Counting measure:

$$m(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right). \tag{1}$$

Correspondence between partition and dimer configurations

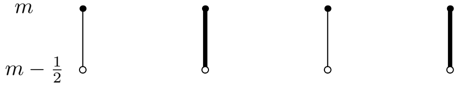

Figure: partition on the level-$\left(m-\frac{1}{2}\right)$ row: (1 0)

- $m \in \{0, 1, 2, \ldots\}$;
- P-vertices (resp. Q-vertices): vertices incident to at least one present edge (resp. vertices incident to only absent edges) between Levels $m-\frac{1}{2}$ and m .
- Assume at level $m-\frac{1}{2}$ of $\mathcal{R}(\Omega, \check{a})$ has p_j P-vertices and q_j Λ-vertices.
- The dimer configuration on level $m-\frac{1}{2}$ corresponds to a partition $\lambda^{(p_j)} \in \mathbb{GT}_{p_j}$, where for $1 \leq i \leq p_j$, $\lambda_i^{(p_j)}$ is the number of Q-vertices on the right of the ith P-vertex (counting from the left)

Partition function and Schur polynomial

Theorem

(Boutillier and Li, 2017) The dimer partition function on $\mathcal{R}(\Omega, \check{a})$ with these weights is given by

$$Z = \left[\prod_{i \in I_2} \Gamma_i \right] s_\omega(x_1, \ldots, x_N)$$

where

$$\begin{aligned} \Gamma_i &= \prod_{t=i+1}^{N} (1 + y_i x_t) \\ I_2 &= \{k \in \{1, \ldots, N\} | a_k = 0\}. \end{aligned}$$

Uniform bottom boundary condition and periodic weights

- ℓ: fixed positive integer
- Uniform boundary condition:
  $\omega = ((N-1)(\ell-1), (N-2)(\ell-1), \ldots, \ell-1, 0)$.
- $s_\omega(x_1, \ldots, x_N) = \prod_{1 \leq i < j \leq N} \frac{x_i^\ell - x_j^\ell}{x_i - x_j}$.
- $x_i = x_{[i \mod n]}$; $y_i = y_{[i \mod n]}$.

Limit shape for uniform bottom boundary condition and periodic weights

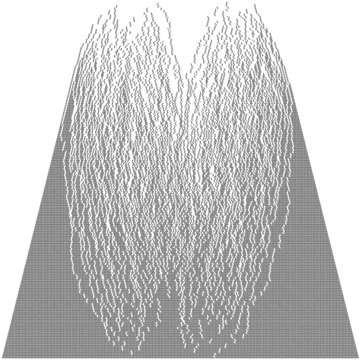

Figure: Limit shape of perfect matchings on the square-hexagon lattice with weights $y_1 = 3, x_1 = 0.3, x_2 = 0.8, y_3 = 0.5, x_3 = 1.4, x_4 = 1.8$ and $\ell = 1$

Limit shape for uniform bottom boundary condition and periodic weights

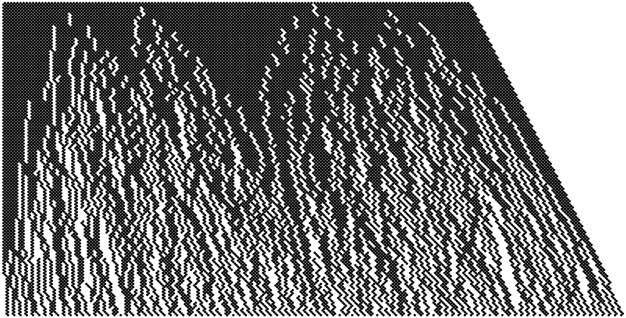

Figure: Limit shape of perfect matchings on the square-hexagon lattice with weights $y_1 = 3, x_1 = 10, x_2 = 0.1, y_3 = 0.5, x_3 = 3.0, x_4 = 0.3$ and $m = 2$

Limit shape for uniform bottom boundary condition and periodic weights

- $\kappa \in (0,1)$.
- $$Q_\kappa(u) = \frac{1}{1-\kappa} \tag{2}$$
  $$\left[\frac{1}{n}\sum_{1\leq j\leq n}\log\left(\frac{u^m - x_j^m}{u-x_j}\right) + \frac{\kappa}{n}\sum_{i\in\{1,2,\ldots,n\}\cap I_2}\log(1+y_i u)\right] \tag{3}$$
- $\mathbf{m}^\kappa$: the limit of the counting measure for the partition corresponding to the dimer configuration on the $\lfloor 2\kappa N\rfloor$th row of the square-hexagon lattice, counting from the bottom.
- (Boutillier, Li 2017)
  $$\int_{\mathbb{R}} x^p \mathbf{m}^\kappa(dx)$$
  $$= \frac{1}{2(p+1)\pi\mathbf{i}}\oint_{C_{x_1,\ldots,x_n}} \frac{dz}{z}\left(zQ_\kappa'(z) + \sum_{j=1}^{n}\frac{z}{n(z-x_j)}\right)^{p+1} := I_p,$$

Proof of the limit shape with uniform bottom boundary condition

- $1 \leq k \leq 2N+1$, $t = \lfloor \frac{k}{2} \rfloor$.
- ρ^k: probability measure for random partitions corresponding to the dimer configurations on the kth row of $\mathcal{R}(\Omega, \check{a})$, counting from the bottom.
- $X^{(N-t)} = (x_{\overline{t+1}}, \ldots, x_{\overline{N}})$ where $\overline{i} = [i \mod n]$.
- $Y^{(t)} = (x_{\overline{1}}, \ldots, x_{\overline{t}})$.
- Schur generating function (definition):

$$\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1, \ldots, u_{N-t}) = \sum_{\lambda \in \mathbb{GT}^+_{N-t}} \rho^k(\lambda) \frac{s_\lambda(u_1, \ldots, u_{N-t})}{s_\lambda(X^{(N-t)})}.$$

Proof of the limit shape with uniform bottom boundary condition

- By Schur branching formula,

$$\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1, \ldots, u_{N-t}) = \frac{s_\omega(u_1, \ldots, u_{N-t}, Y^{(t)})}{s_\omega(X^{(N)})} \prod_{i \in \{1,2,\ldots,t\} \cap I_2} \prod_{j=1}^{N-t} \left(\frac{1 + y_{\bar{i}} u_j}{1 + y_{\bar{i}} x_{\overline{t+j}}} \right)$$

- $V_N = \prod_{1 \leq i < j \leq N} (u_i - u_j)$;
- $\mathcal{D}_p = \frac{1}{V_N} \sum_{i=1}^{N} \left(u_i \frac{\partial}{\partial u_i} \right)^p V_N$;
- $\lambda^{(N-t)}$: partition corresponding to dimer configurations on the $(2t)$th or $(2t+1)$th row of $\mathcal{R}(\Omega, \check{a})$, counting from the bottom.

Proof of the limit shape with uniform bottom boundary condition

- $$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda^{(N-t)}\right] dx\right)^m = \frac{1}{[(1-\kappa)N]^{m(l+1)}}(\mathcal{D}_p)^m \mathcal{S}_{\rho^k, X^{(N-t)}}(u_1, \ldots, u_{N-t})\Big|_{(u_1,\ldots,u_N)=(x_1,\ldots,x_N)}$$
- Analyzing the leading terms,
$$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda^{(N-t)}\right] dx\right) \approx I_p$$
$$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda^{(N-t)}\right] dx\right)^2 \approx I_p^2$$

Frozen Boundary

- $\mathcal{R} := \frac{1}{N}\mathcal{R}(\Omega, \check{a})$;
- Liquid region $\mathcal{L}$: the set of (χ, κ) inside $\mathcal{R}$ such that the density of $\mathbf{m}^{\kappa}$ there is neither 0 nor 1.
- Fact: density $f(x)$ of a measure η and Stieltjes transform: $f(x) = -\lim_{\epsilon \to 0^+} \frac{1}{\pi}\mathrm{Im}[\mathrm{St}_{\eta}(x + i\epsilon)]$.

Frozen boundary when $m = 1$

$$U(z) = \frac{z}{n} \sum_{i \in \{1,2,\ldots,n\} \cap I_2} \frac{y_i}{1 + y_i z}, \qquad V(z) = \sum_{j=1}^{n} \frac{z}{n(z - x_j)}.$$

Frozen boundary

$$(\chi, \kappa) = \left(\frac{U(z)V'(z) - U'(z)V(z)}{V'(z) - U'(z)}, \frac{V'(z)}{V'(z) - U'(z)} \right);$$

A cloud curve of rank $m + n$.

Frozen boundary when $m = 2$

$$W(z) = \frac{z}{n} \sum_{1 \leq j \leq n} \frac{1}{z + x_j};$$

then

$$\begin{aligned} \chi &= \frac{W'(z)U(z) + V'(z)U(z) - U'(z)V(z) - W'(z)V(z)}{V'(z) - U'(z)} + W(z), \\ \kappa &= \frac{V'(z) + W'(z)}{V'(z) - U'(z)}; \end{aligned}$$

for (χ, κ) on the frozen boundary. If we have m' distinct values of x_i's in the fundamental domain, then for $z = x_i$, we get that the the points $(U(x_j) + W(x_j), 1)$ are m' tangent points of the frozen boundary to the line $\kappa = 1$.

A formula for Schur function

- $\lambda(N) \in \mathbb{GT}_N^+$
- Σ_N: symmetric group of N-elements.
- $\sigma \in \Sigma_N$.
- $\Sigma_N^X = \{\sigma \in \Sigma_N : x_i = x_{\sigma(i)}\}$.
- $[\Sigma_N/\Sigma_N^X]^r$: all the right cosets of Σ_N^X in Σ_N
- $\eta_j^\sigma(N) = |\{k : k > j, x_{\sigma(k)} \neq x_{\sigma(j)}\}|$.
- $\Phi^{(i,\sigma)}(N) = \{\lambda_j(N) + \eta_j^\sigma(N), x_{\sigma(j)} = x_i\}$.
- $\phi^{(i,\sigma)}(N)$: the partition obtained by decreasingly ordering elements in $\Phi^{(i,\sigma)}(N)$.

A formula for Schur function

Theorem

(Li, 2018) $\lambda \in \mathbb{GT}_N^+$

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\overline{\sigma} \in [\Sigma_N / \Sigma_N^X]^r} \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma)}(N)|} \right) \left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1) \right) \times \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{x_{\sigma(i)} - x_{\sigma(j)}} \right)$$

Corollary

- $1 \leq k \leq N$ and $k = qn + r$, where $r < n$ and q, r are positive integers.
- $$w_i = \begin{cases} u_i & \text{if } 1 \leq i \leq k \\ x_i & \text{if } k+1 \leq i \leq N \end{cases}$$
- $$s_\lambda(w_1, \ldots, w_N) = \sum_{\overline{\sigma} \in [\Sigma_N / \Sigma_N^X]^r} \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma)}|} \right) \left(\prod_{i=1}^{r} s_{\phi^{(i,\sigma)}} \left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{qn+i}}{x_i}, 1, \ldots, \right.\right.$$
$$\times \left(\prod_{i=r+1}^{n} s_{\phi^{(i,\sigma)}} \left(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i} \ldots, \frac{u_{(q-1)n+i}}{x_i}, 1, \ldots, 1 \right) \right)$$
$$\times \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}} \right)$$

Assumptions on edge weights

- $x_{1,N} > x_{2,N} > \ldots > x_{n,N} > 0$;
- $\frac{N}{n}$ is a positive integer;
- $x_{i,N} = x_{j,N}$ if $[i \mod n] = [j \mod n]$;
- $$\liminf_{N\to\infty} \frac{\log\left(\min_{1\le i<j\le n} \frac{x_{i,N}}{x_{j,N}}\right)}{\log N} \ge \alpha > 0,$$
- α: a sufficiently large positive constant independent of N.

Piecewise boundary condition

- ω: partition on the bottom boundary.
- Let $1 \leq i \leq n-1$, for any $p \geq \frac{iN}{n} > q$, $\omega_p > \omega_q$.
- $\mu_1 > \ldots > \mu_t$ are all the distinct elements in $\omega_1, \ldots, \omega_N$, with t a finite integer independent of N.
- $1 \leq p < q \leq s$, $C_1 N \leq \mu_p - \mu_q \leq C_2 N$

Limit shape with piecewise boundary condition

$$\int_{\mathbb{R}} x^p \mathbf{m}^\kappa(dx) = \frac{1}{2n(p+1)\pi \mathbf{i}} \sum_{i=1}^{n} \oint_{C_1} \frac{dz}{z} \left(zQ'_{i,\kappa}(z) + \frac{n-i}{n} + \frac{z}{n(z-1)} \right)^{p}$$

where for $i = 1$

$$Q_{i,\kappa}(z) = \left[H_{\mathbf{m}_i}(z) - (n-i)\log z + \kappa \sum_{r \in \{1,2,\ldots,n\} \cap I_2} \log \frac{1 + y_r z x_1}{1 + y_r x_1} \right]$$

for $2 \leq i \leq n$,

$$Q_{i,\kappa}(z) = \frac{1}{(1-\kappa)n} \left[H_{\mathbf{m}_i}(z) - (n-i)\log z \right]$$

Limit shape with piecewise boundary condition

- $\overline{\sigma}_0 \in [\Sigma_N/\Sigma_N^X]^r$, such that $x_{\sigma_0(1)} \geq x_{\sigma_0(2)} \geq \ldots \geq x_{\sigma_0(N)}$.
- $\mathbf{m}_i$: the limit of the counting measure for $\phi^{(i,\sigma_0)}$.
- $H'_{\mathbf{m}_i}(u) = \frac{\mathrm{St}^{(-1)}_{\mathbf{m}_i}(u)}{u} - \frac{1}{u-1}$.
-

$$s_\lambda(x_1,\ldots,x_N) \approx \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma_0)}(N)|}\right)\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)\right) \times \left(\prod_{i<j, x_{\sigma_0(i)} \neq x_{\sigma_0(j)}} \frac{1}{x_{\sigma_0(i)} - x_{\sigma_0(j)}}\right)$$

Proof of Limit Shape

$$
\begin{aligned}
&s_\lambda(w_1,\ldots,w_N) \approx \\
&\left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma_0)}|}\right)\left(\prod_{i=1}^{r} s_{\phi^{(i,\sigma_0)}}\left(\frac{u_i}{x_i},\frac{u_{n+i}}{x_i}\ldots,\frac{u_{qn+i}}{x_i},1,\ldots,1\right)\right) \\
&\times\left(\prod_{i=r+1}^{n} s_{\phi^{(i,\sigma_0)}}\left(\frac{u_i}{x_i},\frac{u_{n+i}}{x_i}\ldots,\frac{u_{(q-1)n+i}}{x_i},1,\ldots,1\right)\right) \\
&\times\left(\prod_{i<j,x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{1}{w_{\sigma_0(i)}-w_{\sigma_0(j)}}\right)
\end{aligned}
$$

Asymptotic Analysis

- $|\Sigma_N| = N!$, $\Sigma_N^X = \left[\left(\frac{N}{n}\right)!\right]^n$, hence $\lim_{N\to\infty} \frac{1}{N} \log |(\Sigma_N/\Sigma_N^X)^r| = n$.
- for any $\sigma \in \Sigma_N$, $\left| \frac{\prod_{i<j, x_{\sigma_0(i)} \neq x_{\sigma_0(j)}} (x_{\sigma_0(i)} - x_{\sigma_0(j)})}{\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} (x_{\sigma(i)} - x_{\sigma(j)})} \right| = 1$.
- $\left| \frac{x_{\sigma_0(1)}^{\lambda_1} \cdots x_{\sigma_0(N)}^{\lambda_N}}{x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(N)}^{\lambda_N}} \right| \geq \left(\min_{1\leq i<j\leq n} \frac{x_i}{x_j} \right)^{D(\sigma_0,\sigma)}$; where $D_{\sigma_0,\sigma}$ is a certain function measuring the difference of σ_0 and σ
- $\left| \frac{\prod_{i<j, x_{\sigma_0(i)} \neq x_{\sigma_0(j)}} x_{\sigma_0(i)}}{\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} x_{\sigma(i)}} \right| = \left| \prod_{x_i \neq x_j, \sigma_0^{-1}(i) > \sigma_0^{-1}(j), \sigma^{-1}(i) < \sigma^{-1}(j)} \frac{x_j}{x_i} \right| \geq 1$
- $s_{\phi^{(i,\sigma)}}(1, \ldots, 1) = \prod_{1\leq j<k\leq \frac{N}{n}} \frac{\phi_j^{(i,\sigma)} - \phi_k^{(i,\sigma)} + k - j}{k-j}$.
- $s_{\phi^{(i,\sigma_0)}}(1, \ldots, 1) \geq 1$.
- Under the assumption of edge weights,

$$\left| \frac{s_{\phi^{(i,\sigma_0)}}(1, \ldots, 1)}{s_{\phi^{(i,\sigma)}}(1, \ldots, 1)} \right| \geq N^{-CD(\sigma_0,\sigma)}$$

for some constant C.

Proof of Limit Shape

$$\lim_{N\to\infty} \frac{1}{N} \log \left[\frac{s_\lambda(w_{1,N}, \ldots, w_{N,N})}{s_\lambda(x_{1,N}, \ldots, x_{N,N})} \right] = \sum_{i=1}^{k} [P_i(u_i)]$$

where

1. if $[i \mod n] \neq 0$, $P_i(u) = \frac{H_{\mathbf{m}_{[i \mod n]}}(u)}{n} - \frac{(n-[i \mod n])\log(u)}{n}$.
2. if $[i \mod n] = 0$, $P_i(u) = \frac{H_{\mathbf{m}_n}(u)}{n}$.

Frozen Boundary

- $\mathrm{St}_{\mathbf{m}^\kappa}(x) = \sum_{i=1}^{n} \log(z_i(x))$; $z_i(x)$ is a root of $F_{i,\kappa}(z) = x$, and

$$F_{i,\kappa}(z) = zQ'_{i,\kappa}(z) + \frac{n-i}{n} + \frac{z}{n(z-1)}.$$

- $F_{i,\kappa}(z) = x$ has at most one pair of complex conjugate roots.
- For $1 \leq i \leq n$, the condition on (χ, κ) such that $F_{i,\kappa}(z) = \frac{\chi}{1-\kappa}$ has double roots are disjoint cloud curves.

Limit shape with piecewise boundary condition

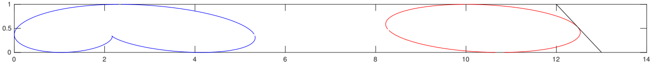

Figure: Frozen boundary for a contracting hexagonal lattice when $n = 2$, $(r_1, r_2, r_3, r_4) = (12, 8, 5, 2)$, represented by the union of the red curve and the blue curve.

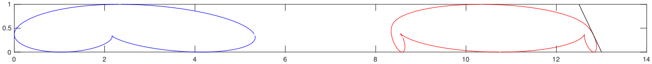

Figure: Frozen boundary for a contracting square hexagon lattice with $n = 2$, $|I_2 \cap \{1, 2\}| = 1$ when $(r_1, r_2, r_3, r_4) = (12, 8, 5, 2)$, $c_r = \frac{1}{2}$, represented by the union of the red curve and the blue curve.

Gaussian Unitary Ensemble (GUE)

- GUE: a random Hermitian matrix whose eigenvalues $\epsilon_1 \geq \epsilon_2 \geq \dots \epsilon_k$ have a distribution $\mathbb{P}_{\mathbb{GUE}_k}$ on $\mathbb{R}^k$ with a density with respect to the Lebesgue measure on $\mathbb{R}^k$ proportional to:

$$\prod_{1 \leq i < j \leq k} (\epsilon_i - \epsilon_j)^2 \exp\left(-\sum_{i=1}^{k} \epsilon_i^2\right),$$

Dimers near the top and GUE

- $x_1 = \ldots = x_N = 1.$
- $y_i = y_{[i \mod n]}.$
- $\lambda^k(N)$ be the signature corresponding to the dimer configuration incident to the $(N-k+1)$th row of white vertices in $\mathcal{R}(\Omega(N), \check{a})$, and for $1 \leq l \leq k$,
- $b_{kl}^N = \lambda_l^k(N) + N - l.$
- $\psi_1 = \int_{\mathbb{R}} x d\mathbf{m}^1$; $\psi_2 = \int_{\mathbb{R}} x^2 d\mathbf{m}^1$ where $\mathbf{m}^1$ is the limit counting measure of signatures on the top of $\mathcal{R}(\Omega(N), \check{a})$.
- $$\tilde{b}_{kl}^{(N)} = \frac{\frac{b_{kl}^{(N)}}{\sqrt{N}} - \sqrt{N}\left(\psi_1 - \frac{1}{2} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,\ldots,n\}} \frac{y_i}{1+y_i}\right)}{\psi_2 - \psi_1^2 - \frac{1}{12} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,2,\ldots,n\}} \frac{y_i}{(1+y_i)^2}}, \quad 1 \leq l \leq k.$$

Theorem

(Boutillier and Li 2017) For any fixed k, the distribution of $\left(\tilde{b}_{kl}^{(N)}\right)_{l=1}^{k}$ converges weakly to $\mathbb{P}_{\mathbb{GUE}_k}$ as $N \to \infty$.

- $(q_1, \ldots, q_k) \in \mathbb{R}^k$ be a random vector with distribution $\mathbb{P}$
- $Q = \mathrm{diag}[q_1, \ldots, q_k]$.
- $\mathbb{P}$ is $\mathbb{P}_{\mathbb{GUE}_k}$ if and only if for any diagonal matrix P,

$$\mathbb{E} \int_{U(k)} \exp[\mathrm{Tr}(PUQU^*)]dU = \exp\left(\frac{1}{2}\mathrm{Tr}P^2\right).$$

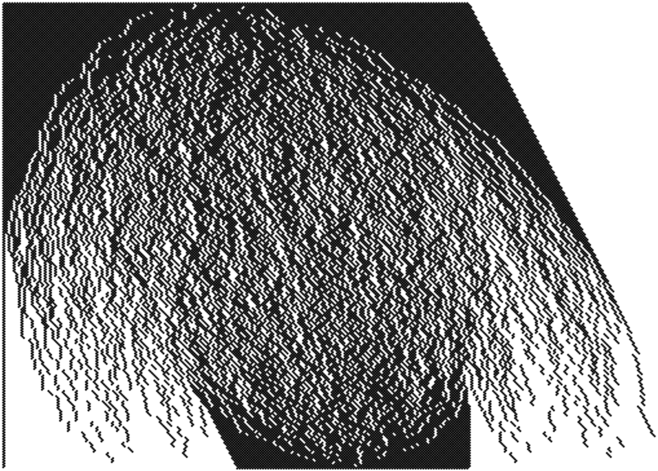

Figure: Limit shape of perfect matchings on the square-hexagon lattice with periodic weights $x_1 = x_2 = x_3 = x_4 = 1, y_1 = 3, y_3 = 0.5$.

Thank you!