# Universality for Lozenge Tiling Local Statistics

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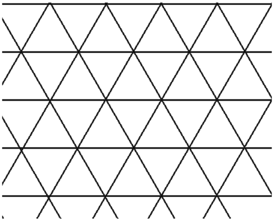
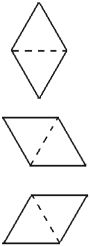
# Lozenge Tilings

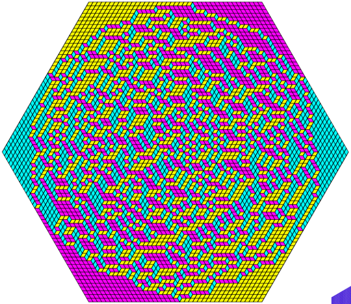
- Consider uniformly random tilings of large, finite subdomains $R$ of the triangular lattice $\mathbb{T}$ using three types of **lozenges**.

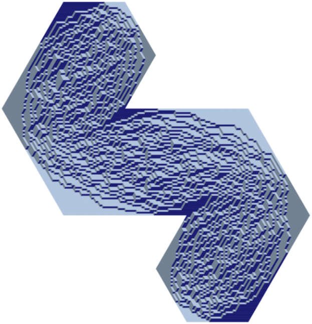


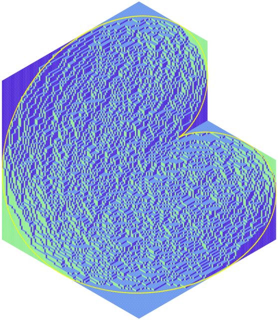
$\mathbb{T}$



# Lozenge Tilings of Different Domains

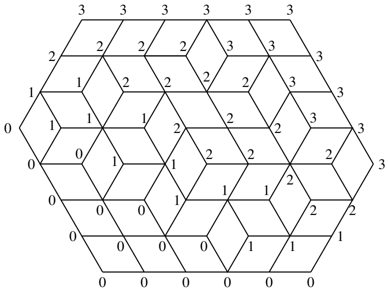


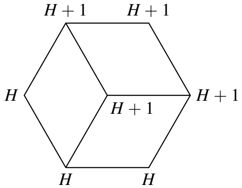




# Height Functions

- Associated with any tiling of $R \subset \mathbb{T}$ is a **height function**.

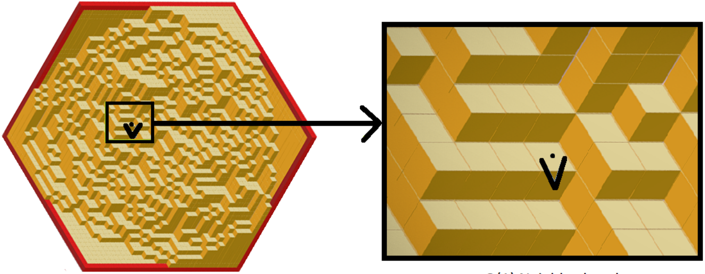



- **Boundary height function**: Restriction of this height function to $\partial R$ (independent of the tiling, up to shifts).

# Local Statistics of Lozenge Tilings

- Consider a uniformly random tiling of a domain $R \subset \mathbb{T}$.
- Fix a vertex $v \in R$ and consider an $O(1)$-neighborhood of $v$.




O(1) Neighborhood

- **Local statistics**: Random tiling on this $O(1)$-neighborhood.

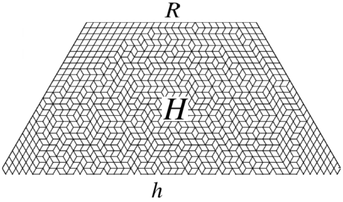
**Question (Kasteleyn, 1961)**

*How do the local statistics around v depend on R?*

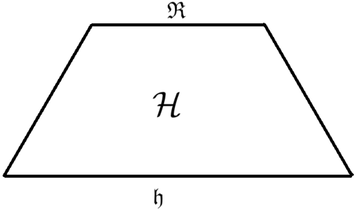
# Global Law

- Fix simply-connected macroscopic domain $\mathfrak{R}$ with piecewise smooth boundary
- Let $N \in \mathbb{Z}_{>0}$ be large and $R = R_N \subset \mathbb{T}$ be simply-connected and tileable
- Boundary height function $h = h_N : \partial R \to \mathbb{Z}$ associated with tiling of $R$
- Assume $N^{-1}R \approx \mathfrak{R}$ and $N^{-1}h(N\cdot) \approx \mathfrak{h}$ for some $\mathfrak{h} : \partial\mathfrak{R} \to \mathbb{R}$
- Height function $H = H_N : R \to \mathbb{Z}$ for uniformly random tiling $\mathcal{M} = \mathcal{M}_N$ of $R$

Discrete




Continuous



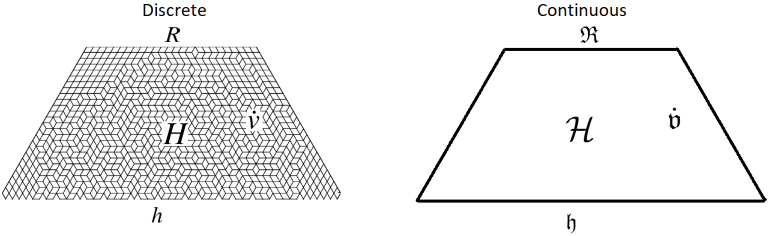

Cohn–Kenyon–Propp (2000): For any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{v\in R} \left|N^{-1}H_N(v) - \mathcal{H}(N^{-1}v)\right| > \delta\right] = 0,$$

where $\mathcal{H} : \mathfrak{R} \to \mathbb{R}$ solves a **variational principle** (unique maximizer of $\int_{\mathfrak{R}} \sigma\big(\nabla\mathcal{H}(z)\big)dz$).

# Local Statistics Results

- Fix $\mathfrak{v} \in \mathfrak{R}$ such that $(s, t) = \nabla \mathcal{H}(\mathfrak{v})$ satisfies $s, t > 0$ and $s + t < 1$.
- Let $v = v_N \in R_N$ be such that $N^{-1} v \approx \mathfrak{v}$.




Sheffield (2003): There exists a **unique infinite-volume, translation-invariant, extremal Gibbs measure** of slope $(s, t)$, called $\mu_{s,t}$.

**Theorem (A., 2019)**

*As $N$ tends to $\infty$, the local statistics of $\mathfrak{M}$ around $v$ are given by $\mu_{s,t}$.*

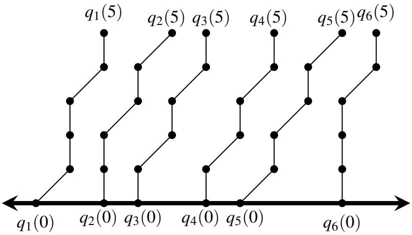
- Predicted by Cohn–Kenyon–Propp (2000)
- **Universality**: Limiting local statistics around $v$ only depend on $\nabla \mathcal{H}(\mathfrak{v})$

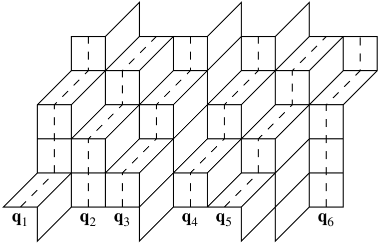
# Previous Results

- Domains
  - Kenyon (1997): Torus
  - Okounkov–Reshetikhin (2001, 2005): $q$-Weighted (skew) plane partitions
  - Baik–Kreicherbauer–McLaughlin–Miller (2007), Gorin (2007): Hexagons
  - Petrov (2012): Trapezoids
  - Gorin (2016): Domains "covered" by trapezoids
  - Laslier (2017): Bounded perturbations of the above
- Many of these results are based on exact determinantal identities for correlation functions
  - Kasteleyn (1961): **(Inverse) Kasteleyn matrix**
  - Okounkov–Reshetikhin (2001): **Schur processes**
- Issues
  - Inverse Kasteleyn matrix entries unstable under perturbations of $R$
  - Schur processes only apply for specific domains

# Non-Intersecting Paths

- A **path** is a integer sequence $\mathbf{q} = \big(q(0), q(1), \ldots, q(t)\big)$ such that $q(i+1) - q(i) \in \{0, 1\}$ for each $i$.
- An ensemble $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$of paths is **non-intersecting** if $q_1(s) < q_2(s) < \cdots < q_n(s)$ for each $s$.




- Bijection between non-intersecting path ensembles and lozenge tilings

# Random Non-Intersecting Path Ensembles

- Fix initial data $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and $\beta \in (0, 1)$.
- Let $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$ be an ensemble of $n$ Bernoulli random walks, with jump probability $\beta$, starting at $a_1, a_2, \ldots, a_n$ and conditioned to never intersect.
- Its probability distribution is given by

$$\mathbb{P}_{\beta;\mathbf{a}}[\mathbf{Q}] = \beta^{|\mathbf{q}(t)|-|\mathbf{a}|}(1-\beta)^{tn-|\mathbf{q}(t)|+|\mathbf{a}|} \prod_{1 \le j < k \le n} \frac{q_k(t) - q_j(t)}{a_k - a_j},$$

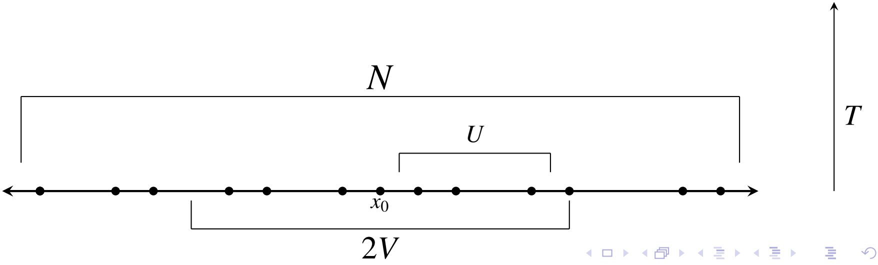
  if $\mathbf{Q}$ is non-intersecting and 0 otherwise, where $\mathbf{q}(t) = \big(q_1(t), q_2(t), \ldots, q_n(t)\big)$ and $|\mathbf{p}| = \sum_{p \in \mathbf{p}} p$.
- Conditional on the ending data $\mathbf{q}(t)$, $\mathbf{Q}$ is uniform on all non-intersecting path ensembles connecting $\mathbf{a}$ to $\mathbf{q}(t)$.

# Universality Results for Non-Intersecting Random Walks

Gorin–Petrov (2016): The model $\mathbb{P} = \mathbb{P}_{\beta;\mathbf{a}}$ is a determinantal point process.

- Explicit kernel amenable to analysis
- If the initial data is sufficiently **regular**, then universal local statistics appear after running $\mathbb{P}_\beta$ for **short time**
  - Scales $1 \ll U \ll T \ll V \ll N$
  - Initial data $\mathbf{a} = (a_1, a_2, \ldots, a_N)$, density $\rho \in (0, 1)$, integer $x_0 \in \mathbb{Z}$
  - Assume $|I \cap \mathbf{a}| \approx \rho U$, for any interval $I \subset [x_0 - V, x_0 + V]$ of length $U$

Then the local statistics of the non-intersecting random walk model $\mathbb{P}_{\beta;\mathbf{a}}$, run for time $T$, converge around site $x_0$ to some measure $\mu_{s,t}$



# Outline

- Tileable $R = R_N \approx N\mathfrak{R} \subset \mathbb{T}$
- Uniformly random tiling $\mathcal{M} = \mathcal{M}_N$ with associated height function $H : R \to \mathbb{Z}$
- Vertex $v = v_N \approx N\mathfrak{v}$ of $R$

We will locally compare $\mathcal{M}$ around $v$ with a $\mathbb{P}_{\beta;\mathbf{a}}$ path ensemble.

1. **Local Law**: Show $H$ is *approximately planar* (with slope $\nabla\mathcal{H}(\mathfrak{v})$) in small disks around $v$. This *verifies the regularity of the initial data* of the path ensemble to be coupled with $\mathcal{M}$.
2. **Comparison**: Couple $\mathcal{M}$ with a random path ensemble $\mathbf{P}$ sampled under some $\mathbb{P}_{\beta;\mathbf{a}}$, such that *the two models likely coincide around $v$*.
3. **Universality**: Use results of Gorin–Petrov to show that the local statistics of $\mathbf{P}$ around $v$ are universal, and *conclude that the same holds for $\mathcal{M}$*.

- Reminiscent of Erdős–Yau "three-step strategy" in random matrix theory, but independent and with very different proofs
- Method potentially also applies to other tiling models (such as domino ones)

# The Local Law

- Assume $\mathfrak{R} = \mathcal{B}_1$ and $\mathcal{B}_N \subset R \subset \mathcal{B}_{N+2}$ (but no assumptions on $h$)
- Global law $\mathcal{H} : \mathcal{B}_1 \to \mathbb{R}$ with $\nabla \mathcal{H}(\mathfrak{v}) = (s, t)$
- Fix $0 < \varepsilon < 1$ and assume $\varepsilon < s, t < s + t < 1 - \varepsilon$

**Proposition (A., 2019)**

*There exists $C = C(\varepsilon) > 1$ such that, for $c = \frac{1}{20000}$ and any $1 \le M \le \frac{N}{\log N}$,*

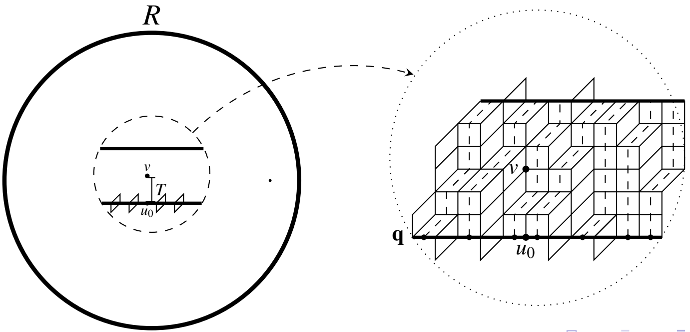
$$\mathbb{P}\left[ \max_{|u-v|<M} \left| M^{-1}\big(H(u) - H(v)\big) - M^{-1}(u - v) \cdot \nabla \mathcal{H}(\mathfrak{v}) \right| > (\log M)^{-c} \right] < CM^{-100}.$$

Here, $M$ can be taken **independently of** $N$.

- If $M$ is close to $N$, then this analyzes global behavior
- If $M$ is close to 1, this analyzes local behavior

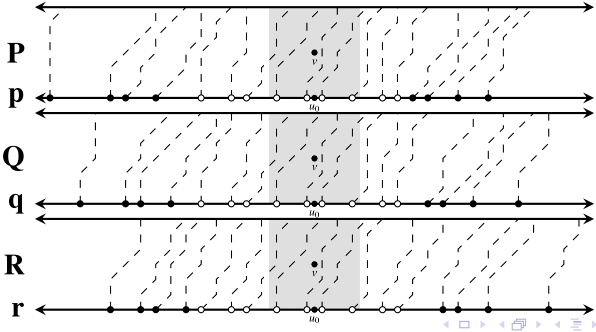
# Outline of the Comparison

- Let $v = (x, y) \in R$.
- Fix an integer $1 \ll T \ll N \sim \text{diam}(R)$.
- Define the vertex $u_0 = v - (0, T) = (x_0, y_0 - T) \in R$.
- Interpret $\mathcal{M}$ as an ensemble $\mathbf{Q}$ of non-intersecting paths, and let $\mathbf{q}$ denote the locations where these paths intersect the horizontal line $\{y = y_0 - T\}$.
- Local law: Approximates density of $\mathbf{q}$ and drifts of paths in $\mathbf{Q}$



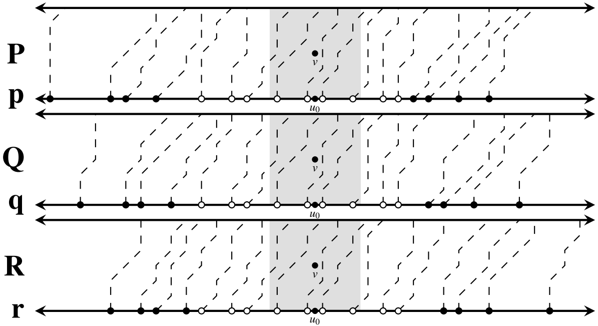
# Outline of the Comparison

- Introduce particle configurations $\mathbf{p}$ and $\mathbf{r}$ that coincide with $\mathbf{q}$ near $u_0$, but are to the left and right of $\mathbf{q}$, respectively, away from $u_0$.
- Define two random path ensembles $\mathbf{P} \sim \mathbb{P}_{\beta_1;\mathbf{p}}$ and $\mathbf{R} \sim \mathbb{P}_{\beta_2;\mathbf{r}}$
    - Select $\beta_1 \approx \beta_2$ such that $\beta_1 < \beta_2$ and the drifts of the $\mathbf{P}$-paths are less than those of the $\mathbf{Q}$-paths, which are less than those of the $\mathbf{R}$-paths
- Show that there exists a coupling between $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ such that $\mathbf{Q}$ is likely bounded between $\mathbf{P}$ and $\mathbf{R}$.



# Outline of the Comparison

- Prove that the expected difference between the height functions for **P** and **R** tends to 0 in a large neighborhood of $u_0$ (containing $v$).
  - Based on explicit identities from Gorin–Petrov and the facts that $\beta_1 \approx \beta_2$, $\mathbf{p}_1 = \mathbf{p}_2$ near $u$, and $\mathbf{p}_1 \approx \mathbf{p}_2$ everywhere
- Using the ordering between $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ and a Markov bound, conclude that they can be coupled to coincide near $v$ with high probability.



# Local Law

- Assume $\mathfrak{R} = \mathcal{B}_1$ and $\mathcal{B}_N \subset R \subset \mathcal{B}_{N+2}$ (but no assumptions on $h$)
- Solution $\mathcal{H} : \mathcal{B}_1 \to \mathbb{R}$ of variational principle, with boundary data approximately (within $O(N^{-1})$ of) $N^{-1}h(N^{-1}\cdot)$
- Set $\nabla\mathcal{H}(\mathfrak{v}) = (s,t)$ and assume $\varepsilon < s, t < s+t < 1-\varepsilon$

**Proposition (A., 2019)**

*There exists $C = C(\varepsilon) > 1$ such that, for $c = \frac{1}{20000}$ and any $1 \le M \le \frac{N}{\log N}$,*

$$\mathbb{P}\left[\max_{|u-v|<M}\left|M^{-1}\big(H(u)-H(v)\big) - M^{-1}(u-v)\cdot\nabla\mathcal{H}(\mathfrak{v})\right| > (\log M)^{-c}\right] < CM^{-100}.$$

Proof uses a **multi-scale analysis** with **effective global laws** for the height function of the tiling.

# Effective Global Laws

An **effective global law** is one of the form

$$\mathbb{P}\left[\max_{v\in R_N}\left|N^{-1}H(v)-\mathcal{H}(N^{-1}v)\right|>\omega_N\right]<CN^{-100},$$

for some explicit $\omega_N$ dependent on $N$.

- Cohn–Kenyon–Propp (2000): Can take $\omega_N=\delta>0$ independent of $N$
- On all known exactly solvable domains, one has $\varpi \ll N^{\delta-1}$, for any $\delta>0$
- Concentration estimates show

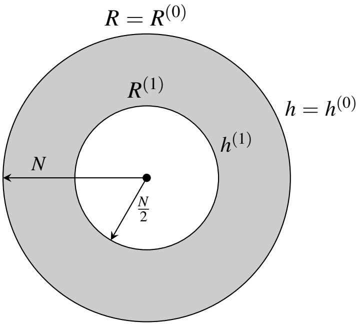
  $$\mathbb{P}\left[\max_{v\in R_N}\left|N^{-1}H(v)-N^{-1}\mathbb{E}\big[H(v)\big]\right|>N^{\delta-1/2}\right]<CN^{-100},$$

  but do not bound $\left|N^{-1}\mathbb{E}\big[H(v)\big]-\mathcal{H}(N^{-1}v)\right|$

Let us outline a proof of the local law assuming one has an effective local law with $\omega_N=(\log N)^{-1-c}$, for some $c>0$.

# Outline of the Local Law

- Original domain $R = R^{(0)} \approx \mathcal{B}_N$
- Random height function $H$ on $R$ with boundary data $h = h^{(0)}$
- Let $R^{(1)} = \mathcal{B}_{N/2} \cap \mathbb{T}$ and $h^{(1)} = H|_{\partial R^{(1)}}$

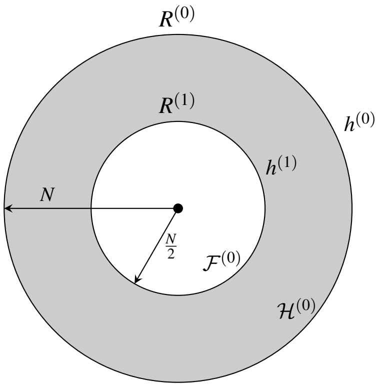



- Condition on the restriction of $H$ to $R^{(0)} \setminus R^{(1)}$
- The restriction of $H$ to $R^{(1)}$ is uniform over height functions with boundary data $h^{(1)}$

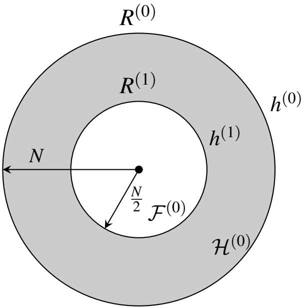
# Outline of the Local Law

Define solutions of variational principle

- Let $\mathcal{H}^{(0)} : \mathcal{B}_1 \to \mathbb{R}$ have boundary data approximately $N^{-1}h^{(0)}(N\cdot)$
- Let $\mathcal{F}^{(0)} : \mathcal{B}_{1/2} \to \mathbb{R}$ have boundary data approximately $N^{-1}h^{(1)}(N\cdot)$



# Outline of the Local Law




- By the assumed effective global law,

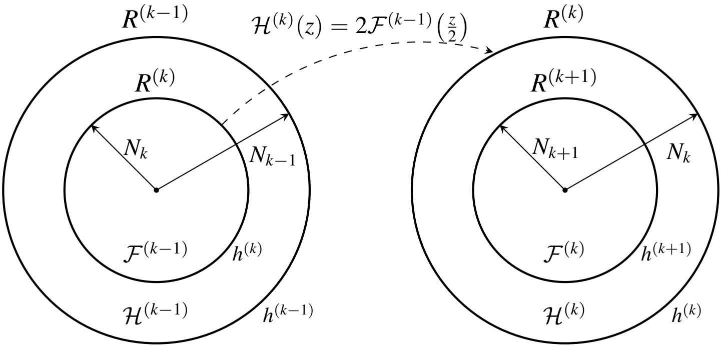
$$\mathbb{P}\left[\max_{v\in\partial R^{(1)}}\left|N^{-1}h^{(1)}(v)-\mathcal{H}^{(0)}(N^{-1}v)\right|>(\log N)^{-1-c}\right]<CN^{-100};$$

$$\mathbb{P}\left[\max_{v\in R^{(1)}}\left|N^{-1}H(v)-\mathcal{F}^{(0)}(N^{-1}v)\right|>2(\log N)^{-1-c}\right]<CN^{-100}.$$

# Outline of the Local Law

Rescale and repeatedly apply this procedure.

- Assume $N = N_0$ is a power of 2, and set $N_k = \frac{N_{k-1}}{2}$ for each $k > 1$
- Define $\mathcal{H}^{(k)}(z) = 2\mathcal{F}^{(k)}\left(\frac{z}{2}\right)$, which solves the variational principle on $\mathcal{B}_1$




We would like to show that $\mathcal{H}^{(k)}$ is approximately linear with slope $(s, t)$, for large $k$.

# Gradient Stability Estimate

To that end, we bound the change in gradient upon passing from $\mathcal{H}^{(0)}$ to $\mathcal{F}^{(0)}$.

- Assume $(s,t) \in [0,1]^2$ satisfies $\varepsilon < s, t < s + t < 1 - \varepsilon$.

**Lemma**

*There exist constants $C = C(\varepsilon) > 1$ and $\delta = \delta(\varepsilon) > 0$ such that the following holds. Suppose $\mathcal{H}_1, \mathcal{H}_2 : \mathcal{B}_1 \to \mathbb{R}$ are solutions of the variational principle with boundary data $\mathfrak{h}_1, \mathfrak{h}_2 : \partial\mathcal{B}_1 \to \mathbb{R}$, respectively. If $\mathfrak{h}_1, \mathfrak{h}_2$ are within $\delta$ of a plane of slope $(s,t)$, then*

$$\sup_{z \in \mathcal{B}_{1/2}} \left| \nabla\mathcal{H}_1(z) - \nabla\mathcal{H}_2(z) \right| < C \sup_{z \in \partial\mathcal{B}_1} \left| \mathfrak{h}_1(z) - \mathfrak{h}_2(z) \right|.$$

Proof uses results of De Silva–Savin (2008) and known estimates on solutions of uniformly elliptic partial differential equations.

# Outline of the Local Law

- By the assumed effective global law,

$$\mathbb{P}\left[\max_{v\in\partial R^{(1)}}\left|N^{-1}h^{(1)}(v)-\mathcal{H}^{(0)}(N^{-1}v)\right|>(\log N)^{-1-c}\right]<CN^{-100};$$

$$\mathbb{P}\left[\max_{v\in R^{(1)}}\left|N^{-1}H(v)-\mathcal{F}^{(0)}(N^{-1}v)\right|>2(\log N)^{-1-c}\right]<CN^{-100}.$$

- Therefore by the maximum principle, with probability $1-2CN^{-100}$,

$$\sup_{z\in\partial\mathcal{B}_{1/2}}\left|\mathcal{H}^{(0)}(z)-\mathcal{F}^{(0)}(z)\right|<3(\log N)^{-1-c}.$$

- Thus, the previous lemma gives, with probability $1-2CN^{-100}$,

$$\sup_{z\in\partial\mathcal{B}_{1/4}}\left|\nabla\mathcal{H}^{(0)}(z)-\nabla\mathcal{F}^{(0)}(z)\right|<C(\log N)^{-1-c}.$$

- Since $\nabla\mathcal{H}^{(1)}\left(\frac{z}{2}\right)=\nabla\mathcal{F}^{(0)}(z)$ this implies, with probability $1-2CN^{-100}$,

$$\sup_{z\in\partial\mathcal{B}_{1/2}}\left|\nabla\mathcal{H}^{(0)}\left(\frac{z}{2}\right)-\nabla\mathcal{H}^{(1)}(z)\right|<C(\log N)^{-1-c}.$$

# Outline of the Local Law

- Recall $N$ is a power of 2 and $N_k = 2^{-k}N$.
- With probability $1 - CN^{-100}$,

$$\sup_{z \in \partial \mathcal{B}_{1/2}} \left| \nabla \mathcal{H}^{(0)}\left(\frac{z}{2}\right) - \nabla \mathcal{H}^{(1)}(z) \right| < C(\log N)^{-1-c}.$$

- More generally, with probability $1 - CN_k^{-100}$,

$$\sup_{z \in \partial \mathcal{B}_{1/2}} \left| \nabla \mathcal{H}^{(k)}\left(\frac{z}{2}\right) - \nabla \mathcal{H}^{(k+1)}(z) \right| < C(\log N_k)^{-1-c}.$$

- Summing over $k \in [0, K-1]$ gives, with probability $1 - CN_K^{-100}$,

$$\sup_{z \in \partial \mathcal{B}_{1/2}} \left| \nabla \mathcal{H}^{(0)}\left(\frac{z}{2^{K+1}}\right) - \nabla \mathcal{H}^{(K+1)}(z) \right| < C(\log N_K)^{-c}.$$

- For $1 \ll M \ll \frac{N}{\log N}$ and $M \in \left[\frac{N}{2^{K+1}}, \frac{N}{2^K}\right)$, we have $\nabla \mathcal{H}^{(0)}\left(\frac{z}{2^{K+1}}\right) \approx (s, t)$.
- Thus, $\mathcal{H}^{(K+1)}$ is approximately a plane of slope $(s, t)$.
- So, on scale $M$, we have $H \approx \mathcal{H}^{(K+1)}$ is nearly planar with slope $(s, t)$.

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# Improved Effective Global Law Without Facets

- **Issue**: We can only prove an effective global law for fully general boundary data when $\omega = (\log N)^{-c}$.
- However, for boundary data giving rise to a limit shape with **no frozen facets**, we have an **improved global law**.

- Integer $N > 0$, real number $\varepsilon > 0$, and domain $R \approx \mathcal{B}_N$
- Solution $\mathcal{G} : \mathcal{B}_1 \to \mathbb{R}$ of the variational principle with boundary data $\mathfrak{g} : \partial\mathcal{B}_1 \to \mathbb{R}$
- Random height function $H : R_N \to \mathbb{Z}$ with boundary data $h : \partial R_N \to \mathbb{Z}$

**Definition**

We say that $(h, g)$ is $\lambda$-confined if the following holds.

- For each $z \in \partial R$, we have $\mathfrak{g}(z) < N^{-1}h(z) < g(z) + \lambda$.
- For each $z \in \mathcal{B}_1$, $\nabla\mathcal{G}(z) = (s_z, t_z)$ satisfies $\varepsilon < s_z, t_z < s_z + t_z < 1 - \varepsilon$.

# Improved Effective Global Law

- Integer $N > 0$, real number $\varepsilon > 0$, and domain $R \approx \mathcal{B}_N$
- Solution $\mathcal{G} : \mathcal{B}_1 \to \mathbb{R}$ of the variational principle with boundary data $\mathfrak{g} : \partial\mathcal{B}_1 \to \mathbb{R}$
- Random height function $H : R_N \to \mathbb{Z}$ with boundary data $h : \partial R_N \to \mathbb{Z}$

**Lemma**

*Assume $(h, g)$ is $\lambda$-confined. Then, for small $c > 0$ and large $C = C(\varepsilon) > 1$,*

$$\mathbb{P}\left[\sup_{v \in R} \left| N^{-1} H(v) - \mathcal{G}(N^{-1} v) \right| > \lambda + N^{-c}\right] < C N^{-100}.$$

- Proof closely follows work of Laslier–Toninelli (2013) and is based on local comparison to hexagons, analyzed by Petrov (2012)
- For $\lambda \sim (\log N)^{-1-c}$, provides an effective global law with $\omega_N \sim (\log N)^{-1-c}$

To establish the local law, show upon reducing scales that the **confinement property is retained** and **apply this improved global law**.