A geometric flow of Balanced metrics

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Bridging the Gap between Kähler and non-Kähler Complex Geometry
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Some classes of Hermitian metrics

An n-dimensional Hermitian manifold (M, ω) is

- *Kähler* if $d\omega = 0$;
- *Balanced* if $d^*\omega = 0$ ($\iff d\omega^{n-1} = 0$);
- *Pluriclosed* if $\partial\bar{\partial}\omega = 0$;
- *Gauduchon* if $\partial\bar{\partial}\omega^{n-1} = 0$;
- *Strongly Gauduchon* if $[\partial\omega^{n-1}]_{\bar{\partial}} = 0$;
- *Asteno-Kähler* if $\partial\bar{\partial}\omega^{n-2} = 0$.

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- *Asteno-Kähler* if $\partial\bar{\partial}\omega^{n-2} = 0$.

Main problems faced in the talk: improve a balanced metric on a fixed cohomology class/ study the existence of different kind of special metrics.

Approaches: use a geometric flow/ work on homogeneous spaces.

Balanced Metrics ($d^*\omega = 0$)

Some good reasons for studying balanced metrics:

• A metric is balanced if and only if $\Delta_\partial f = \Delta_{\bar\partial} f = 2\Delta_d f$ for every smooth map f (Gauduchon '77).

• The twistor space of an anti-self-dual, oriented 4-dimensional Riemannian manifold always has a balanced metric (Gauduchon '81).

• Every compact complex manifold bimeromorphic to a compact Kähler manifold is balanced (Alessandrini-Bassanelli '93). Hence Moishezon manifolds and complex manifolds in the Fujiki class $\mathcal{C}$ are balanced.

• Any left-invariant Hermitian metric on a complex Lie group is balanced.

• The balanced condition can be characterized in terms of currents, in particular Calabi-Eckmann manifolds have no balanced metrics (Michelson '82).

• On a balanced manifold ω^{n-1} is calibration.

Some generalizations of the Kähler-Ricci flow

Some geometric flows of Hermitian non-Kähler metrics in the literature are generalizations of the Kähler-Ricci flow:

Hermitian curvature flows (Streets,Tian, Ustinovskiy...),

$$\partial_t \omega_t = -S(\omega_t) + Q(T_t, \bar{T}_t) \quad \begin{cases} \text{Hermitian curvature flow} \\ \text{Pluriclosed flow} \\ \text{Ustinovskiy flow} \end{cases}$$

Chern-Ricci-flow (Gill,Tosatti,Weinkove...)

$$\partial_t \omega_t = -\rho(\omega_t)$$

Notation. Given a Hermitian manifold M, $\omega = \frac{i}{2} g_{r\bar{s}} dz^r \wedge dz^{\bar{s}}$, R and T are the curvature and the torsion of the Chern connection and

$$S_{i\bar{j}} = g^{r\bar{s}} R_{r\bar{s}i\bar{j}}\,, \quad \rho_{i\bar{j}} = g^{r\bar{s}} R_{i\bar{j}r\bar{s}}$$

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Calabi functional: $\mathrm{Ca}\colon \{\text{Kähler forms} \in [\omega_0]\} \to \mathbb{R}^+$

$$\mathrm{Ca}\colon \quad \omega \mapsto \int_M s_\omega^2 \, \omega^n$$

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Constant scalar curvature $\Longrightarrow$ Extremal

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$$\textit{Calabi flow (CF)} \quad \partial_t \omega_t = i\partial\bar{\partial} s_t\,, \quad \omega_{|t=0} = \omega_0$$

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Theorem [Chen-He '08]. *CF is well-posed. The flow is stable near CSC Kähler metrics and it exists as far as the Ricci curvature is bounded.*

CALABI FLOW AS A FLOW OF $(n-1, n-1)$-FORMS

The Calabi flow can be alternatively written in terms of $(n-1, n-1)$-forms as

$$\partial_t \omega_t^{n-1} = i\partial\bar{\partial} *_t \left(\rho(\omega_t) \wedge \omega_t\right), \quad \omega_{|t=0} = \omega_0.$$

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This new flow moves the form ω_t^{n-1} in the Bott-Chern cohomology class

$$[\omega_0^{n-1}]_{BC} = \left\{\omega_0^{n-1} + i\partial\bar{\partial}\vartheta \ : \ \vartheta \in \Lambda^{n-2,n-2}\right\} \in H_{BC} = \frac{\ker d}{\operatorname{im} \partial\bar{\partial}}.$$

The following decomposition holds

$$\Omega = \ker \Delta^{BC} \oplus \operatorname{im} \partial\bar{\partial} \oplus (\operatorname{im} \partial^* + \operatorname{im} \bar{\partial}^*)$$

(Schweitzer '07).

Generalizations of the Calabi-flow

It is quite natural to consider the flow of balanced metrics

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Another flow which is natural to consider in the balanced setting is the Laplacian-type flow

$$\partial_t \omega_t^{n-1} = \Delta_t^{BC} \omega_t^{n-1} \quad \omega_{|t=0} = \omega_0$$

which is inspired by the *Laplacian flow* in G_2-geometry.

A G_2-structure on a 7-dimensional manifold is a section φ of an open subbundle $\Lambda^3_+ \subseteq \Lambda^3$.

φ determines a metric g_φ and an orientation.

φ is *torsion-free* if $d\varphi = d^*\varphi = 0$.

The *Laplacian flow (LF)* is the geometric flow

$$\partial_t \varphi_t = \Delta_{\varphi_t} \varphi_t \,, \quad d\varphi_t = 0 \,.$$

(Byant '05)

Theorem [Bryant-Xu '11]. *LF is well-posed.*

Laplacian flow in G_2-geometry

$$
\begin{array}{ccc}
\Lambda^2 & \xrightarrow{\ d\ } & \Lambda^3 \\
\big\downarrow \Delta & & \\
\Lambda^2 & \xleftarrow{\ d^*\ } & \Lambda^3
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Lambda^3_+ \cap [\varphi_0] & \xrightarrow{P(\varphi)=\Delta_\varphi \varphi} & \Lambda^3 \\
\big\uparrow \varphi_0 + d & & \big\uparrow d \\
\Lambda^2_+ \subseteq \Lambda^2 & \xrightarrow{\ p(\sigma)\ } & \Lambda^2
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$$p(\sigma) = d^*\varphi_0 + \Delta_{\varphi_0 + d\sigma}\sigma.$$

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$p(\sigma) = d^* \varphi_0 + \Delta_{\varphi_0 + d\sigma} \sigma$. If

$$P_{*|\varphi} = L_\varphi \, , \quad p_{*|\sigma} = l_\sigma$$

$-L_\varphi, -l_\varphi$ are not elliptic,

Laplacian flow in G_2-geometry

$$\begin{array}{ccc} \Lambda^2 & \xrightarrow{d} & \Lambda^3 \\ \big\downarrow \Delta & & \\ \Lambda^2 & \xleftarrow{d^*} & \Lambda^3 \end{array} \qquad \begin{array}{ccc} \Lambda^3_+ \cap [\varphi_0] & \xrightarrow{P(\varphi)=\Delta_\varphi \varphi} & \Lambda^3 \\ \big\uparrow \varphi_0 + d & & \big\uparrow d \\ \Lambda^2_+ \subseteq \Lambda^2 & \xrightarrow{p(\sigma)} & \Lambda^2 \end{array}$$

$p(\sigma) = d^* \varphi_0 + \Delta_{\varphi_0 + d\sigma} \sigma$. If

$$P_{*|\varphi} = L_\varphi \,, \quad p_{*|\sigma} = l_\sigma$$

$-L_\varphi, -l_\varphi$ are not elliptic, but there exists $V\colon \Lambda^3_+ \to \Gamma(M)$ such that

$$\text{if} \quad \tilde{P}(\varphi) = \Delta_\varphi \varphi + \mathcal{L}_{V(\varphi)} \varphi \,, \quad \varphi \in \Lambda^3_+ \cap [\varphi_0]$$

$$\text{then} \quad \tilde{P}_{*|\varphi} \circ d = -\Delta_\varphi \circ d + \text{l.o.t.} \,, \quad \tilde{p}_{*|\varphi} = -\Delta_\varphi + \text{l.o.t.}$$

The well-posedness of the LF follows via a DeTurck trick.

The balanced flow

$$\begin{array}{ccc} \Lambda^{n-2,n-2} & \xrightarrow{i\partial\bar\partial} & \Lambda^{n-1,n-1} \\ \downarrow \Delta^{A} & & \downarrow \Delta^{BC} \\ \Lambda^{n-2,n-2} & \xrightarrow{i\partial\bar\partial} & \Lambda^{n-1,n-1} \end{array} \qquad \begin{array}{ccc} \Lambda_{+}^{n-1,n-1} \cap [\omega^{n-1}]_{BC} & \xrightarrow{P} & \Lambda^{n-1,n-1} \\ \uparrow \omega_0^{n-1}+i\partial\bar\partial & & \uparrow \\ \Lambda_{+}^{n-2,n-2} & \xrightarrow{p} & \Lambda^{n-2,n-2} \end{array}$$

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$-L_\varphi, -l_\varphi$ are not elliptic for both the balanced flows introduced, but if

$$P(\omega^{n-1}) = \, i\partial\bar\partial *_\omega (\rho(\omega) \wedge \omega) + (n-1)\Delta^{BC}_\omega \, \omega^{n-1}$$

then

$$P_{*|\varphi} \circ i\partial\bar\partial = -(n-1)\Delta^{BC} \circ i\partial\bar\partial + \text{l.o.t.} \,, \quad p_{*|\varphi} = -(n-1)\Delta^{A} + \text{l.o.t.}$$

Well-posedness of the Balanced flow

Theorem [Bedulli-V. '18]. *Let (M, ω_0) be a compact balanced manifold. The geometric flow (BF)*

$$\partial_t \omega_t^{n-1} = i\partial\bar{\partial} *_t (\rho(\omega_t) \wedge \omega_t) + (n-1)\Delta_t^{BC} \omega_t^{n-1}\,,$$
$$d\omega_t^{n-1} = 0\,, \quad \omega_{|t=0} = \omega_0$$

is well-posed. The solution ω_t satisfies, $\omega_t^{n-1} \in [\omega_0^{n-1}]_{BC}$ and if ω_0 is Kälher it reduces to the Calabi flow.

Remark. The short-time existence is not free since flow is parabolic "only along $\partial\bar{\partial}$-exact forms".

In general (BF) cannot be reduced to a scalar flow and $T_{\max} < \infty$.

Open problem. Study the short-time existence of

$$\partial_t \omega_t = i\partial\bar{\partial} *_t (\rho(\omega_t) \wedge \omega_t)\,, \qquad \partial_t \omega_t^{n-1} = \Delta_t^{BC} \omega_t^{n-1}$$

Theorem. [Bedulli-V.]. *Let* $(M, \bar{\omega})$ *be a compact Ricci-flat Kähler manifold. Then there exists* $\delta > 0$ *such that if* ω_0 *is a balanced metric on* M *satisfying* $\|\omega_0 - \bar{\omega}\|_{C^\infty} < \delta$*, then (BC) starting from* ω_0 *exists for all* $t \in [0, \infty)$ *and as* $t \to \infty$ *it converges in* C^∞ *topology to a balanced form* ω *satisfying*

$$i\partial\bar{\partial} * (\rho \wedge \omega) + (n-1)\Delta^{BC} * \omega = 0 \qquad (1)$$

Stability of the Balanced flow

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$$i\partial\bar{\partial} * (\rho \wedge \omega) + (n-1)\Delta^{BC} * \omega = 0 \tag{1}$$

Related Problems. The result is similar to the stability of the Laplacian flow in G_2-geometry (Lotay-Wei) and suggest that ω_t should converge to $\bar{\omega}$.

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Improve the result.

A remark on Extremal Balanced metrics

Balanced metrics satisfying

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can be seen as a generalization of extremal Kähler metrics to the balanced frameworks.

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$$\mathrm{Ca}\colon [\omega_0^{n-1}]_{BC} \cap \Lambda_+^{n-1,n-1} \to \mathbb{R}_+ \,, \quad \mathrm{Ca}(\omega^{n-1}) = \int_M s_\omega^2 \, \omega^n$$

In this way

$$\omega \text{ extremal} \iff 2(n-1) i\partial\bar{\partial} s \wedge \rho = i\partial\bar{\partial}((2\Delta s + s^2)\omega)$$

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Problem. *Study the interplay between the two notions of extremal balanced metrics*

Balanced flow: a different approach

Let (M, ω) be a Hermitian manifold and

$$C^\infty_\omega(M) = \left\{ v \in C^\infty(M) \ : \ \omega_v^{n-1} = \omega^{n-1} + i\partial\bar{\partial}(v\omega^{n-2}) > 0 \right\}$$

which is open in $C^\infty(M)$.

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$$\begin{array}{lll} \omega \ \text{balanced} & \Longrightarrow & \omega_v \ \text{balanced} \\ \omega \ \text{Gauduchon} & \Longrightarrow & \omega_v \ \text{Gauduchon} \\ \omega \ \text{Strongly Gauduchon} & \Longrightarrow & \omega_v \ \text{Strongly Gauduchon} \end{array}$$

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ω	balanced	$\Longrightarrow$	ω_v	balanced
ω	Gauduchon	$\Longrightarrow$	ω_v	Gauduchon
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$\left\{\omega_v^{n-1}\right\} \subset \Lambda_+^{n-1,n-1} \cap [\omega^{n-1}]_{BC}$. We introduce

$$\partial_t \omega_t^{n-1} = i\partial\bar\partial(s_{\omega_t}\omega^{n-2})\,, \quad \omega_{|t=0} = \omega_0 \in \{\omega_v\}$$

which is inspired by the $(n-1)$-plurisubharmonic flow

$$\partial_t \omega_t^{n-1} = -(n-1)\rho(\omega_t) \wedge \omega^{n-2}\,, \quad \omega_{|t=0} = \omega_0\,.$$

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always has a unique short-time solution $\{\omega_t\}_{t\in[0,T_{max})}$. $\{\omega_t\}$ *is balanced for every* t*. If further* $c_1(M) \leq 0$, ω *is Kähler-Einstein and* ω_0 *is close enough to* ω *in* C^∞*-topology, then* $\{\omega_t\}$ *is defined for any positive* t *and converges in* C^∞*-topology to* ω.

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$$\partial_t \omega_t^{n-1} = i\partial\bar{\partial}(s_{\omega_t}\omega^{n-2}) \quad \text{is equivalent to} \quad \partial_t u_t = s_{u_t}$$

which is elliptic in very strong sense (Whisken-Polden, Mantegazza-Martinazzi) $\Longrightarrow$ short-time existence.

An open conjecture

Conjecture [Fino-V.]. *Every compact complex manifold admitting a pluriclosed and a balanced metric is Kähler.*

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- Chiose proved that a compact complex manifold of in the Fujiki class $\mathcal{C}$ has a pluriclosed metric if and only if it is Kähler.
- Li, Fu and Yau found a new class of non-Kähler balanced manifolds by using conifold transactions. Such examples include the connected sums M_k of k-copies of $S^3 \times S^3$, $k \geq 1$. M_k has no pluriclosed metrics.

An open conjecture

- Chiose, Rasdeaconu, Suvaina proved that the conjecture is true on compact 3-folds such that for every Gauduchon metric ω

$$H_A^{2,2} \ni [\omega^2]_A \text{ contains a balanced metric}$$

Theorem [Fino-V.]. *The conjecture is true in 2-step nilmanifolds with invariant complex structures and on 3-dimensional solvmanifolds with invariant complex structures and holomorphically trivial canonical bundle.*

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It is quite natural to consider the same problem for other classes of Hermitian metrics

Theorem [Grantcharov-Fino-V.]. *The homogeneous space* $\mathrm{SU}(5)/T^2$ *simply connected and has an invariant complex structure which admits both balanced and astheno-Kähler metrics, but does not admit any pluriclosed metric.*

Thank you!