Towards Spacetime Entanglement Entropy for Interacting Theories

Yasaman K. Yazdi (University of Alberta, Canada),
Ravi Kunjwal (Perimeter Institute for Theoretical Physics, Canada),
Heidar Moradi (University of Cambridge, UK)

March 31, 2019 – April 7, 2019

1 Overview

Entanglement entropy was originally conceived in pursuit of an understanding of black hole entropy [1]. The Bekenstein-Hawking thermodynamic entropy of a black hole is proportional to the area of its event horizon. However, the microscopic origin of this entropy has been a mystery. Entanglement entropy, similarly, in many different situations obeys a spatial "area law". This means that in units of the UV cutoff of the theory, it scales proportional to the spatial area of the boundary of the region whose entropy of entanglement is being considered. The area law nature of entanglement entropy, as well as the fact that it has both quantum (entanglement) and gravitational (geometric) features, make it a strong candidate for being the microscopic origin of black hole entropy. Since its conception in 1983, entanglement entropy has also found many important applications in other areas of modern physics such as condensed matter physics [2, 3] and information theory [4]. This has strengthened its role as a foundational concept in fundamental theoretical physics.

Our main motivation in the work we carried out relates back to the original goal of understanding black hole entropy as well as other questions in quantum gravity. Numerous studies have already been made on the connection between entanglement entropy and black hole entropy [5, 6, 7, 8]. While these studies have shed some light on the issue, they have not been conclusive. One of the challenges in studying the entanglement entropy of a black hole is that black holes are truly global and spacetime objects. One would have to know the entire causal history of a spacetime to characterize a black hole. The information in a subregion, much less a moment in time, would not suffice. This is in stark contrast to conventional quantum systems which are typically characterized by states at a "moment in time". There are arguments related to the well-posedness of the initial value problem that in some classical and semiclassical cases justify the characterization of a physical system using data at a moment in time or on a Cauchy surface. In going beyond classical theory and entering the regime of nontrivial semiclassical and full quantum gravity one must insist on finding spacetime tools to probe quantum properties such as entanglement entropy. Freeing quantum characteristics from spatial surfaces would also pave the way for studying dynamical causal structures and spacetimes. These dynamical scenarios are inevitable in full quantum gravity, so it is a worthwhile investment to build a framework that could study them.

Thus we wish to cast entanglement entropy in an explicitly spacetime form. An additional motivation for doing this for entanglement entropy in particular is the crucial role played by the UV cutoff. Entanglement entropy and its properties are quantified with respect to a UV cutoff, without which one would get infinite values. In order for entanglement entropy to serve as an objective measure for a theory in a spacetime without any special frames, we need the UV cutoff to not belong to any special frame. This is only possible if the cutoff is a spacetime rather than spatial cutoff.

2 Recent Developments and Open Problems

Major strides were made towards an intrinsically spacetime definition of entanglement entropy in work by Sorkin [9]. In this work entanglement entropy was defined in a spacetime framework for a gaussian scalar field in an arbitrary spacetime. This was achieved by expressing the entropy in terms of the spacetime two-point function (or Wightman function) of the theory. Explicitly, in terms of the Wightman function $W(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle$ and the spacetime commutator $\Delta(x,y) = [\phi(x), \phi(y)]$, the entropy is

$$S = \sum_{\lambda} \lambda \ln |\lambda|, \tag{1}$$

where λ are solutions of the generalized eigenvalue problem

$$Wv = i\lambda \Delta v, \qquad \Delta v \neq 0. \tag{2}$$

Such a definition was possible because the two-point function contains all the information in a gaussian theory. This definition has been explored in some follow-up work in both continuum spacetimes [10] and discrete causal sets [11]. In both cases the characteristic area law scaling of the entropy was obtained with respect to a covariant UV cutoff set by a smallest eigenvalue of Δ.

To understand the nature of entanglement in general physical theories, which tend to be interacting, it is important to go beyond gaussian theories. The focus of our workshop was to make progress in this direction. An open question is how the entropy can be described by spacetime correlators in such theories. For non-gaussian theories Wick's theorem fails to hold and generally all higher n-point correlation functions are needed to specify the theory. One would therefore expect extensions to Sorkin's formula to depend on these higher order correlators.

3 Scientific Progress Made

In order to gain deeper intuition about the problem and as a first step towards generalizing the entropy definition (1)-(2), we tested the formula under perturbations away from a gaussian theory. The density matrix we considered (expressed in the block-diagonal q-basis of [9]) was

$$\rho_{qq'} = \langle q|\rho|q'\rangle = Ne^{-A/2(q^2+q'^2)-C/2(q-q')^2-\left(\lambda_1 \frac{q^4+q'^4}{2}+\lambda_2(q^3q'+qq'^3)+\lambda_3 q^2q'^2\right)}, \tag{3}$$

where N is a normalization constant, A and C are constant coefficients, and $\lambda_i \ll 1$ set the strength of the perturbations. Equation (3) is the most general quartic perturbation of a gaussian density matrix that is symmetric in $q \leftrightarrow q'$. Expectation values with respect to such non-gaussian states do not factorize in the sense of Wick's theorem and one would expect higher order correlator contributions to the formulas (1)-(2).

For a given density matrix, the entanglement entropy is typically directly computed through

$$S = -\mathrm{tr}\rho \log \rho, \tag{4}$$

which can be straightforwardly calculated using the replica trick [12, 13],

$$S = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{tr}\left(\rho^n\right). \tag{5}$$

Using the replica trick, the entropy can be computed to first order in λ_i using a formalism reminiscent of path integrals. We find

$$\begin{aligned} S = &-\frac{\mu \log \mu + (1-\mu)\log(1-\mu)}{1-\mu} - \frac{3\mu \log \mu}{(\mu+1)(\mu-1)^5\beta^2}\lambda_1 \\ &- \frac{3(\mu+1)\log \mu}{2(\mu-1)^5\beta^2}\lambda_2 - \frac{3(1+\mu+\mu^2)\log \mu}{(\mu+1)(\mu-1)^5\beta^2}\lambda_3 + \mathcal{O}(\lambda^2). \end{aligned} \tag{6}$$

where

$$\mu = \frac{\sqrt{1+2C/A}-1}{\sqrt{1+2C/A}+1}, \quad \text{and} \quad \beta = \frac{1}{2}\left(A\sqrt{1+2C/A}+A+C\right). \tag{7}$$

It is useful to express the result in terms of the gaussian form with a perturbed μ which we will call $\mu_{replica}$

$$S = -\frac{\mu_{replica}\log\mu_{replica} + (1-\mu_{replica})\log(1-\mu_{replica})}{1-\mu_{replica}}, \tag{8}$$

where

$$\mu_{replica} = \mu + \frac{3\mu}{\beta^2(\mu+1)(\mu-1)^3}\lambda_1 + \frac{3(\mu+1)}{2\beta^2(\mu-1)^3}\lambda_2 + \frac{1+\mu+\mu^2}{\beta^2(\mu+1)(\mu-1)^3}\lambda_3 + \mathcal{O}(\lambda^2). \tag{9}$$

In general we expect that the formulas (1)-(2) need to be generalized to include contributions of higher order correlators. However, as a first approximation we tried to compute the formula in the non-gaussian case, by replacing the two-point correlator W with its non-gaussian counterpart. We can express the result as

$$S = -\frac{\mu_{correlation}\log\mu_{correlation} + (1-\mu_{correlation})\log(1-\mu_{correlation})}{1-\mu_{correlation}}. \tag{10}$$

To our surprise we found that

$$\mu_{replica} = \mu_{correlation}, \tag{11}$$

despite being computed using very different methods. This implies that Sorkin's proposal still holds for non-gaussian theories (at least to first order in perturbation theory), and nothing besides the (non-gaussian) two-point correlator contributes. This is a nontrivial result and, to our knowledge, the first such example.

4 Outcome of the Meeting

The main outcome of our meeting was to show that the entropy definition (1)-(2) continues to hold to first order in perturbation theory for the quartic perturbations that we considered. More precisely, for our non-gaussian theory, the same gaussian entropy formula holds but with W and Δ replaced by their perturbation-corrected versions. We have written a paper [14] containing the details of our findings. Our results indicate that S may universally depend on W, or at least primarily on W. If this turns out to be true, then it is an example of a physical insight we have gained as a result of working in this spacetime correlation framework for entanglement entropy.

Our work at this stage serves as an important proof of principle that it is possible to formulate entanglement entropy in terms of correlations functions for theories beyond gaussian theories. This opens the door to extending such studies to general interacting theories and conformal field theories.

References

[1] R. D. Sorkin, On the Entropy of the Vacuum Outside a Horizon, in Tenth International Conference on General Relativity and Gravitation (held Padova, 4-9 July, 1983), Contributed Papers, vol. 2, pp. 734736. (1983) arXiv:1402.3589 [gr-qc].

[2] A. Kitaev, and J. Preskill, Topological Entanglement Entropy, Phys. Rev. Lett. 96 110404, 2006, arXiv:hep-th/0510092.

[3] M. Levin, and X-G Wen, Detecting Topological Order in a Ground State Wave Function, Phys. Rev. Lett. 96, 110405, 2006.

[4] J. Pye, W. Donnelly and A. Kempf, Locality and Entanglement in Bandlimited Quantum Field Theory, Phys. Rev. D 92 (2015) 105022, arXiv:1508.05953.

[5] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, A Quantum Source of Entropy for Black Holes, Phys. Rev. , vol. D34, pp. 373383, 1986.

[6] T. Jacobson, and A. Satz, Black Hole Entanglement Entropy and the Renormalization Group Physical Review D - Particles, Fields, Gravitation and Cosmology. 87.

[7] S. N. Solodukhin, Entanglement entropy of Black Holes, Living Rev.Rel. 14 (2011) 8.

[8] R. Emparan, Black hole Entropy as Entanglement Entropy: A Holographic Derivation, JHEP 0606 (2006) 012.

[9] R. D. Sorkin, Expressing Entropy Globally in Terms of (4D) Field-Correlations, arXiv:1205.2953 (2012); J. Phys. Conf. Ser. 484 012004 2014; http://www.pitp.ca/personal/rsorkin/some.papers/143.s.from.w.pdf.

[10] M. Saravani, R. D. Sorkin, and Y. K. Yazdi, Spacetime Entanglement Entropy in 1+1 Dimensions, Class. Quantum Grav. 31 (2014) 214006, arXiv:1311.7146.

[11] R. D. Sorkin, and Y. K. Yazdi, Entanglement Entropy in Causal Set Theory, (2018) Class. Quantum Grav. 35 074004, arXiv:1611.10281.

[12] J. F. Koksma, T. Prokopec and M. G. Schmidt, Entropy and Correlators in Quantum Field Theory, Annals Phys. **325**, 1277 (2010) arXiv:1002.0749.

[13] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. , vol. B424, pp. 443467, 1994.

[14] Y. Chen, R. Kunjwal, H. Moradi, Y. K. Yazdi and M. Zilhão, Towards Spacetime Entanglement Entropy for Interacting Theories, arXiv:2002.00966 [hep-th].