Adler-Moser polynomials, Gross-Pitaeskii, and KP-I

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The Traveling Wave Gross-Pitaeskii equation

- This talk concerns

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- Travelling Waves of Gross-Pitaeskii equation:

$$i\partial_t \Phi = \Delta\Phi + \Phi\left(1-|\Phi|^2\right) \quad \text{in} \quad \mathbb{R}^2.$$

Travelling waves $U(x-ct, y)$

Another motivation

- Superfluids passing an obstacle:

$$\varepsilon^2 \Delta u + u - |u|^2 u = 0 \text{ in } \mathbb{R}^2 \backslash \Omega, \; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

- Let $u_\varepsilon = \rho_\epsilon e^{i\frac{\Phi_\varepsilon}{\varepsilon}}$ be a vortex free solution. Then $\rho_\varepsilon \to \rho, \Phi_\epsilon \to \Phi$

$$\begin{cases} \nabla(\rho^2 \nabla\Phi) = 0 \text{ in } \mathbb{R}^2 \backslash \Omega, \\ \rho^2 = 1 - |\nabla\Phi|^2, \\ \frac{\partial\Phi}{\partial\nu} = 0 \text{ on } \partial\Omega, \\ \nabla\Phi(x) \to (0, \delta) \text{ as } |x| \to +\infty. \end{cases}$$

(Irrotational Flow)

- $$u = u_\varepsilon U = \rho_\varepsilon e^{i\frac{\phi_\varepsilon}{\varepsilon}} U$$

  Then U satisfies

  $$\epsilon^2 \Delta U + 2\epsilon^2 \nabla \rho_\epsilon \nabla U + 2i\epsilon \nabla \Phi_\epsilon \nabla U + U\rho_\epsilon^2 (1 - |U|^2) = 0.$$

  $$x = x_0 + \varepsilon y,$$

  $$2i\epsilon \nabla \Phi_\epsilon \nabla U \to 2\nabla \Phi(x_0) \nabla U$$

  The limit equation is the travelling wave GP (rescaled).

  $$\Delta U + 2i\nabla \Phi(x_0) \nabla U + (\rho(x_0))^2 U(1 - |U|^2) = 0.$$

Ref: FH Lin-Wei 2018

Two limits

$$-ic\partial_x U = \Delta U + U\left(1 - |U|^2\right) \quad \text{in} \;\; \mathbb{R}^2.$$

$$0 < c < \sqrt{2} \text{ (sound speed)}$$

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- $c \to 0$: Ginzburg-Landau equation and Adler-Moser polynomials.

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- $c \to 0$: Ginzburg-Landau equation and Adler-Moser polynomials.
- $c \to \sqrt{2}$: KP-I equation (*Kadomtsev – Petviashvili*)

$$\partial_t u + \partial_x^3 u + 3\partial_x\left(u^2\right) - \partial_x^{-1}\partial_y^2 u = 0.$$

Jone-Roberts Program

- Jones-Roberts program(1970'): Existence of travelling waves $U(x-ct,y)$ with $c \in \left(0,\sqrt{2}\right)$, from physical point of view.

  This is called Jones-Roberts Program.

- Rigorous mathematical proof by Bethuel-Gravejat-Saut-2009, using variational method.
- No finite energy travelling wave with $c \geq \sqrt{2}$ (Gravejat-2003).

Variational Method

Energy functional:

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |u|^2)^2$$

Momentum

$$P[u] = \frac{1}{2}\int_{\mathbb{R}^2} < i\nabla u, u - 1 >$$

(variational method)

$$\inf\{E[u] \ \ |P[u] = C\}$$

Bethuel-Gravejat-Saut (2008,2009) proved existence of least energy traveling waves when $0 < c < \sqrt{2}$.

We are interested in the full solution structure of

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Recent numerical simulation by Chiron-Scheid: Multiple branches of travelling waves for the Gross Pitaevskii equation, 2017 provides evidence of abundance of higher energy solutions. Our first aim to construct these higher energy solutions.

Part I: small speed case

$$c = \varepsilon << 1$$

$$-i\varepsilon\partial_x U = \Delta U + U\left(1 - |U|^2\right) \quad \text{in } \mathbb{R}^2.$$

Small speed case: $0 < c = \varepsilon << 1$

$\varepsilon = 0$, Ginzburg-Landau

$$\Delta u + u(1 - |u|^2) = 0 \text{ in } \mathbb{R}^2$$

Degree ± 1 Vortex solution

$$v_+ = S(r)e^{i\theta}, \; v_- = S(r)e^{-i\theta}$$

Theorem Lin-Wei 2010: Traveling wave solution with two opposite vortices

$$u_\varepsilon(z) \sim v_+(z - \varepsilon^{-1}\vec{e_2})v_-(z + \varepsilon^{-1}\vec{e_2})$$

This is also the least energy travelling wave solutions.

Force of attractions between ± 1 vortices $\approx \frac{1}{d}$

Lorentz forces between these "charged" vortices is $\simeq$ speed of motion ε

Balancing $\varepsilon \simeq \frac{1}{d}$ (repelling due to opposite signs of charges).

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Question: are there travelling multi-vortex solutions? If there are, where are they located?

Multi-vortex travelling waves

Theorem (Liu-Wei 2018)

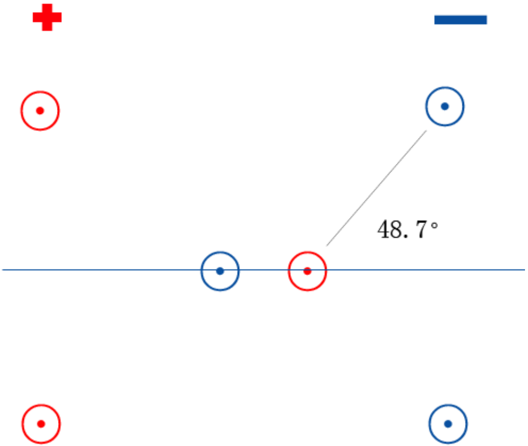
Let $N \leq 34$. *For* ε *small, there is a solution* $U = u + o(1)$, *where*

$$u = \prod_{k=1}^{N(N+1)/2} \left[v_+ \left(z - \varepsilon^{-1} p_k \right) v_- \left(z - \varepsilon^{-1} q_k \right) \right],$$

where $p_1, \ldots, p_{N(N+1)/2}$ *are roots of an Adler-Moser polynomial* A_N *and*

$$q_k = -p_k.$$

Travelling 6-Vortex Solutions: $N = 2$



Remarks:

- For any N, theorem will be true, if A_N has no repeated root.
- For $N \leq 34$, computer software verifies that A_N has no repeated root.
- If A_{N-1} and A_N have no common root, then A_N has no repeated root.
- Conjecture: A_N has no repeated root for any N.

Vortex location and Adler-Moser polynomials

- The error:

$$E(u) := \varepsilon i \partial_x u + \Delta u + u\left(1 - |u|^2\right).$$

- $u \sim \Pi_k u_k$, $u_k = v_+(z - \varepsilon^{-1} p_k)$ or $u_k = v_-(z - \varepsilon^{-1} q_k)$
- Let $|u_k|^2 - 1 = \rho_k$.

$$|u|^2 - 1 = \prod_k (1 + \rho_k) - 1 = \sum_k \rho_k + \sum_{k \geq 2} Q_k,$$

where $Q_k = \sum_{i_1 < i_2 < \cdots < i_k} \left(\rho_{i_1} \cdots \rho_{i_k}\right)$ (small terms).

- At the main order,

$$E(u) \sim \varepsilon i \sum_k \left(\partial_x u_k \prod_{j \neq k} u_j\right) + \sum_{k,j,k \neq j} \left(\left(\nabla u_k \nabla u_j\right) \prod_{l \neq i,j} u_l\right).$$

Projection of error on the kernel: translating modes

Around the vortex point $\varepsilon^{-1}p_k$, for some constant α_0:

- $\nabla u_k \nabla u_j$ term:

$$\int_{|z-\varepsilon^{-1}p_k|\le C\varepsilon^{-1}} \left(\nabla u_k \nabla u_j e^{-i\theta_j}\right) \overline{\partial_x u_k} \sim i\alpha_0 \varepsilon \operatorname{Re} \frac{1}{p_k - p_j},$$

$$\int_{|z-\varepsilon^{-1}p_k|\le C\varepsilon^{-1}} \left(\nabla u_k \nabla u_j e^{-i\theta_j}\right) \overline{\partial_y u_k} \sim i\alpha_0 \varepsilon \operatorname{Im} \frac{1}{p_k - p_j},$$

- $i\varepsilon\partial_x u_k$ term:

$$\int_{|z-\varepsilon^{-1}p_k|\le C\varepsilon^{-1}} i\varepsilon\partial_x u_k \overline{(\partial_x u_k)} \sim i\varepsilon\alpha_0,$$

$$\int_{|z-\varepsilon^{-1}p_k|\le C\varepsilon^{-1}} i\varepsilon\partial_x u_k \overline{(\partial_y u_k)} \sim 0.$$

Projected equations for translating vortices

- Let $\mu \in \mathbb{R}$ be fixed. Let $p_1, \ldots, p_m$ $(q_1, \ldots, q_n)$denote the (scaled) position of degree $1(-1)$ vortices.

$$\begin{cases} \sum_{j \neq \alpha} \frac{1}{p_\alpha - p_j} - \sum_j \frac{1}{p_\alpha - q_j} = \mu, \text{for } \alpha = 1, \ldots, m, \\ \sum_{j \neq \alpha} \frac{1}{q_\alpha - q_j} - \sum_j \frac{1}{q_\alpha - p_j} = -\mu, \text{for } \alpha = 1, \ldots, n. \end{cases}$$

- If $\mu \neq 0$, then necessarily $m = n$.
- The case of $\mu = 0$ is corresponding to stationary vortex configuration.
- The question is how to find these points $(p_1, \ldots, p_n, q_1, \ldots, q_n)$.

Kirchhoff-Routh Hamiltonian

$$\begin{cases} \sum_{j \neq \alpha} \frac{1}{p_\alpha - p_j} - \sum_j \frac{1}{p_\alpha - q_j} = \mu, \text{for } \alpha = 1, \dots, m, \\ \sum_{j \neq \alpha} \frac{1}{q_\alpha - q_j} - \sum_j \frac{1}{q_\alpha - p_j} = -\mu, \text{for } \alpha = 1, \dots, n. \end{cases}$$

is the translating vortices for the Kirchhoff-Routh Halmitonian

$$\frac{dz_i^*}{dt} = \sum_{k \neq i} \frac{\Gamma_k}{z_k - z_i}$$

where

$$\Gamma_k = \pm 1$$

Kirchhoff-Routh Halmitonian

$$\frac{dz_i^*}{dt} = \sum_{k\neq i} \frac{\Gamma_k}{z_k - z_i}$$

Dynamics of Vortices in Euler Flows:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \nabla p && \text{in } \mathbb{R}^2 \times (0, T) \\ \mathbf{u}\cdot\nu &= 0 && \text{on } \partial\Omega \times (0, T) \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega \times (0, T) \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 && \text{in } \Omega \end{aligned}$$

$\mathbf{u}(x, t) : \bar{\Omega} \times [0, T) \to \mathbb{R}^2$, $p(x, t) : \bar{\Omega} \to \mathbb{R}$.
Ω smooth, bounded domain in $\mathbb{R}^2$ or entire space.

Γ_k–circulation of vortices.

Rigorous verification: Davila-del Pino-Wei, arXiv:1803.00066, Gluing methods for vortex dynamics in Euler flows

The generalized Tkachenko equation

$$\begin{cases} \sum\limits_{j\neq\alpha} \frac{1}{p_\alpha - p_j} - \sum\limits_{j} \frac{1}{p_\alpha - q_j} = \mu, \text{ for } \alpha = 1, ..., m, \\ \sum\limits_{j\neq\alpha} \frac{1}{q_\alpha - q_j} - \sum\limits_{j} \frac{1}{q_\alpha - p_j} = -\mu, \text{ for } \alpha = 1, ..., n. \end{cases}$$

- Let $P(z) = \prod\limits_j (z - p_j)$, $Q(z) = \prod\limits_j (z - q_j)$ be the generating polynomials. Then (Tkachenko 1964) Tkachenko equation:

$$P''Q - 2P'Q' + PQ'' = 2\mu\left(P'Q - PQ'\right).$$

- The Adler-Moser polynomials provide a sequence of polynomial solutions to the Tkachenko equation (Bartman 1983).
- There many other polynomials which are solutions to the Tkchenko equation (Demina-Kudryashov 2011) but they don't satisfy the nondegeneracy conditions below.

Adler-Moser Polynomials

- Let $K = (k_2, ...,)$ be parameters. Define $\theta_n(z; K)$ by

$$\exp\left(z\lambda - \sum_{j=2}^{+\infty} \frac{k_j \lambda^{2i-1}}{2j-1}\right) = 1 + \sum_{n=1}^{+\infty} \theta_n(z; K)\, \lambda^n$$

- $\theta_1(z; K) = z$, $\theta_3(z; K) = -\frac{k_2}{3} + \frac{z^3}{6}$,

$$\theta_5(z; K) = -\frac{k_3}{5} - \frac{k_2}{6} z^2 + \frac{1}{120} z^5.$$

- $\theta'_{n+1} = \theta_n$.

The Adler-Moser and modified Adler-Moser polynomials

- The Adler-Moser polynomial:

$$\Theta_n\left(z;K\right) := c_n W\left(\theta_1, ..., \theta_{2n-1}\right).$$

Constant c_n is chosen such that leading coefficient is 1.

- Θ_n is of degree $n\left(n+1\right)/2$.
- $\Theta_1\left(z;K\right) = z$, $\Theta_2\left(z;K\right) = z^3 + k_2$, and

$$\Theta_3\left(z;K\right) = z^6 + 5k_2 z^3 - 9k_3 z - 5k_2^2.$$

- The modified Adler-Moser polynomial:

$$\tilde{\Theta}_n\left(z;K\right) := c_n e^{-\mu z} W\left(\theta_1, ..., \theta_{2n-1}, e^{\mu z}\right).$$

- $Q = \Theta_n(z, K)$, $P = \tilde{\Theta}_n(z, \mu, K)$ satisfies the Tkachenko equation (Bartman 1983).

Symmetric vortex configuration

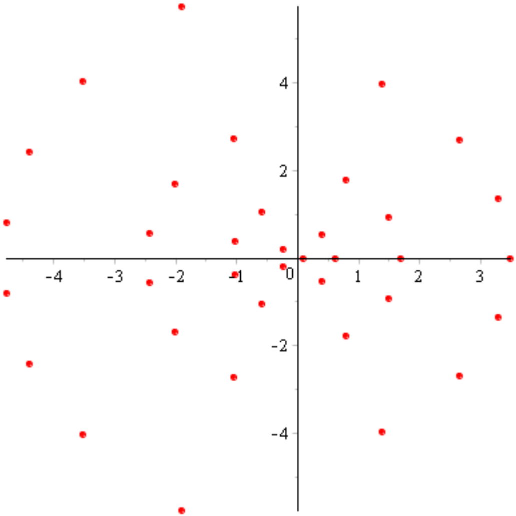
Take $\mu = 1$ and $K_0 := -\frac{1}{2}(1, 1, ...,)$.
Define

$$A_n := \Theta_n\left(z + \frac{1}{2}; K_0\right), B_n = \tilde{\Theta}_n\left(z + \frac{1}{2}; K_0\right).$$

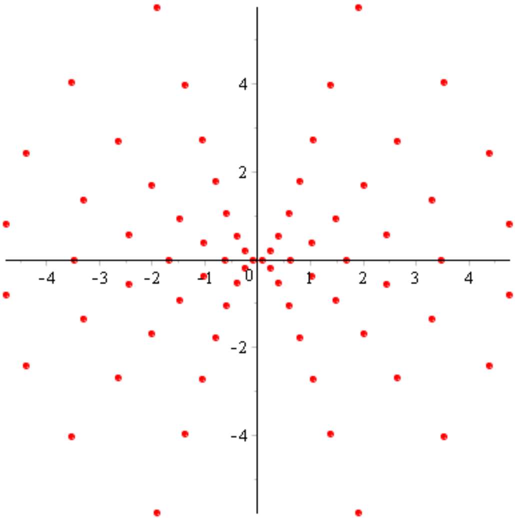
- A_n, B_n have real coefficients and $B_n(z) = A_n(-z)$.
- The roots of these two polynomials give us a “symmetric” translating-vortex configuration.

$$\begin{cases} \sum_{j \neq \alpha} \frac{1}{p_\alpha - p_j} - \sum_j \frac{1}{p_\alpha - q_j} = \mu, \text{for } \alpha = 1, ..., m, \\ \sum_{j \neq \alpha} \frac{1}{q_\alpha - q_j} - \sum_j \frac{1}{q_\alpha - p_j} = -\mu, \text{for } \alpha = 1, ..., n. \end{cases}$$

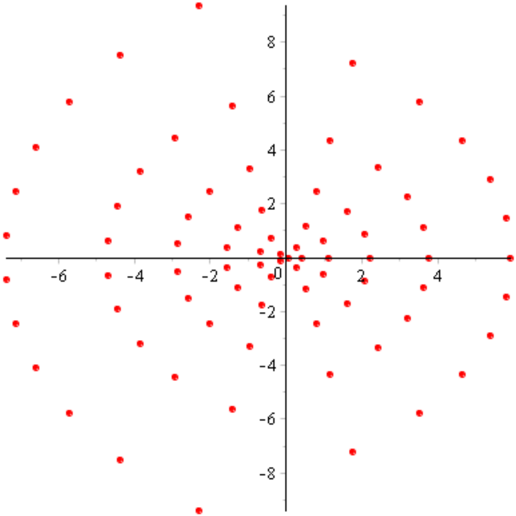
Roots of A_8



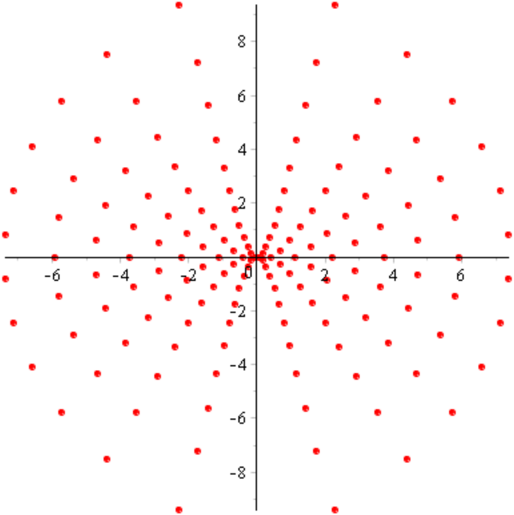
Roots of A_8 and B_8



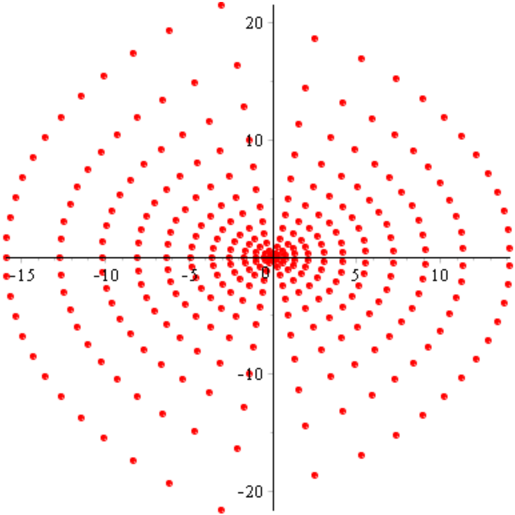
Roots of A_{12}



Roots of A_{12} and B_{12}

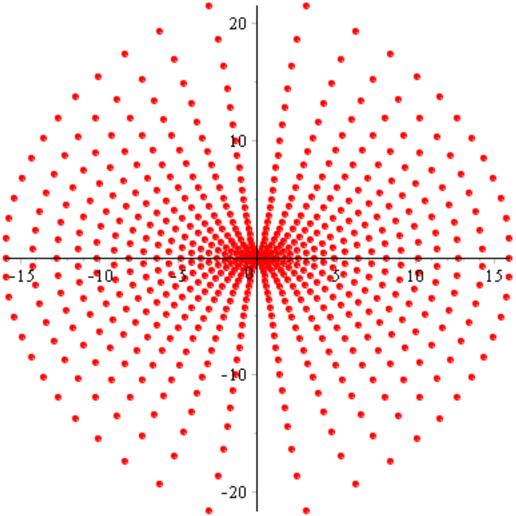


Roots of A_{25}



Roots of A_{25} and B_{25}

Approximately(but not exactly) on (25)circles and lines.



The force map

Let $\mathbf{p} = \left(p_1, \ldots, p_{n(n+1)/2}\right), \mathbf{q} = \left(q_1, \ldots, q_{n(n+1)/2}\right)$. Define the force map F :

$$(\mathbf{p}, \mathbf{q}) \to \left(F_1, \ldots, F_{n(n+1)/2}, G_1, \ldots, G_{n(n+1)/2}\right),$$

where

$$F_k = \sum_{j \neq k} \frac{1}{p_k - p_j} - \sum_j \frac{1}{p_k - q_j},$$
$$G_k = \sum_{j \neq k} \frac{1}{q_k - q_j} - \sum_j \frac{1}{q_k - p_j}.$$

Nondegeneracy of the symmetric vortex-configuration

- Let $a = \left(a_1, ..., a_{n(n+1)/2}\right)$, $b = \left(b_1, ..., b_{n(n+1)/2}\right)$ represent the roots of A_n and B_n.
- To carry out the construction, we need Nondegeneracy: The linearization of the map F at (a, b) has no nontrivial "symmetric" kernel.
- $DF|_{(a,b)}$ always has non-symmetric kernels, arising from the variation of the parameters k_j.
- How to prove nondegeneracy?

Nondegeneracy of the symmetric vortex-configuration

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- To carry out the construction, we need Nondegeneracy: The linearization of the map F at (a, b) has no nontrivial "symmetric" kernel.
- $DF|_{(a,b)}$ always has non-symmetric kernels, arising from the variation of the parameters k_j.
- How to prove nondegeneracy?
- Claim: If A_n has no repeated roots, then nondegeneracy holds.

Proof of Nondegeneracy

- Recursive relation of A_n:

$$A''_{n+1}A_n - 2A'_{n+1}A'_n + A_{n+1}A''_n = 0.$$

- Let $\phi_n = \frac{A_{n+1}}{A_n}$ and $\psi_n(z) = \frac{B_n}{A_n}e^{\mu z}$. Darboux transformation between ψ_n and ψ_{n+1}

$$\psi_{n+1} = \frac{W(\psi_n, \phi_n)}{\phi_n}.$$

- Tkachenko equation

$$A''_n B_n - 2A'_n B'_n + A_n B''_n = 2\mu \left(A'_n B_n - A_n B'_n\right).$$

Linearize the recursive relation

$$\xi_n'' A_{n+1} - 2\xi_n' A_{n+1}' + \xi_n A_{n+1}'' + A_n'' \xi_{n+1} - 2A_n' \xi_{n+1}' + A_n \xi_{n+1}'' = 0.$$

- Let $f_n := \left(\frac{\xi_n}{A_n}\right)'$:

$$f_n' + 2\left(\ln \frac{A_n}{A_{n+1}}\right)' f_n + f_{n+1}' + 2\left(\ln \frac{A_{n+1}}{A_n}\right)' f_{n+1} = 0.$$

- Given f_{n+1}, solve for f_n :

$$f_n = -f_{n+1} + 2\frac{A_{n+1}^2}{A_n^2} \int_0^z \frac{A_n^2}{A_{n+1}^2} f_{n+1}' ds.$$

Linearize the Darboux transformation

Linearizing the Darboux transform

$$\psi_{n+1} = \frac{W(\psi_n, \phi_n)}{\phi_n}.$$

at (ψ_n, ϕ_n), we get

$$\sigma_n' - \sigma_n (\ln \phi_n)' = \psi_n (f_{n+1} - f_n) - \sigma_{n+1}.$$

Hence from σ_{n+1}, we get

$$\sigma_n = \phi_n \int_0^z \phi_n^{-1} (\psi_n (f_{n+1} - f_n) - \sigma_{n+1}) \, ds.$$

Transform the kernel to $n = 0$

- Linearize the Tkachenko equation

$$P''Q - 2P'Q' + PQ'' = 2\mu\left(P'Q - PQ'\right)$$

at (A_0, B_0) yields

$$(\sigma_0 e^{\mu z})' + 2e^{2\mu z} f_0 = 0.$$

- Analyzing the singularities of f_n and σ_n (corresponding to roots of A_j), we obtain $\sigma_0 = f_0 = 0$. (Simplicity of roots needed.)
- All kernels of $DF|_{(a,b)}$ are corresponding to the variation of the parameters k_j.
- As a result, symmetric kernel is trivial.

Part II: Transonic limit: $c \to \sqrt{2}$

$$-i\varepsilon\partial_x U = \Delta U + U\left(1 - |U|^2\right) \quad \text{in } \mathbb{R}^2.$$

Bethuel-Gravejat-Saut-2008 proved: Let

$$\varepsilon = \sqrt{2 - c^2}$$

$$\eta_c = 1 - |u_c|^2 .$$

Then (under certain energy bound of the travelling wave u_c) as $c \to \sqrt{2}$(transonic limit):

$$\frac{1}{\varepsilon^2}\eta_c\left(\frac{x}{\varepsilon}, \frac{\sqrt{2}y}{\varepsilon^2}\right) \to \text{traveling wave solution of KP-I.}$$

$$-c\partial_x u + \partial_x^3 u + 3\partial_x\left(u^2\right) - \partial_x^{-1}\partial_y^2 u = 0.$$

KP-I: an integrable system

The KP-I equation(Kadomtsev-Petviashvili 1970):

$$\partial_t u + \partial_x^3 u + 3\partial_x \left(u^2\right) - \partial_x^{-1}\partial_y^2 u = 0.$$

- KP equation is integrable
  ——Lax pair, Inverse scattering, Backlund transformation, Hirota's direct method, Darboux Transformation...
  ——Explicit soliton solutions, exponentially localized in certain directions.
- Analysis of the inverse scattering transform of KP-I (Manakov et al., Ablowitz-Fokas, X. Zhou, Ablowitz-Villarroel...).

Lump solution

Consider travelling wave solution $u\left(x-ct, y\right)$:

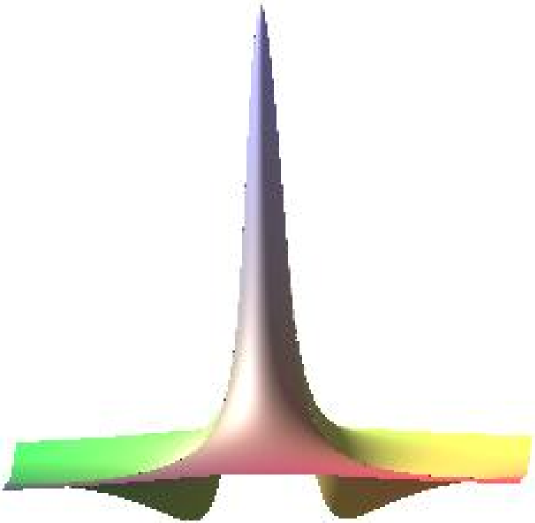
$$\partial_x^2\left(\partial_x^2 u - cu + 3u^2\right) - \partial_y^2 u = 0.$$

It has the following family of lump solutions (Manakov et al.-1977; Ablowitz-Satsuma-1979)

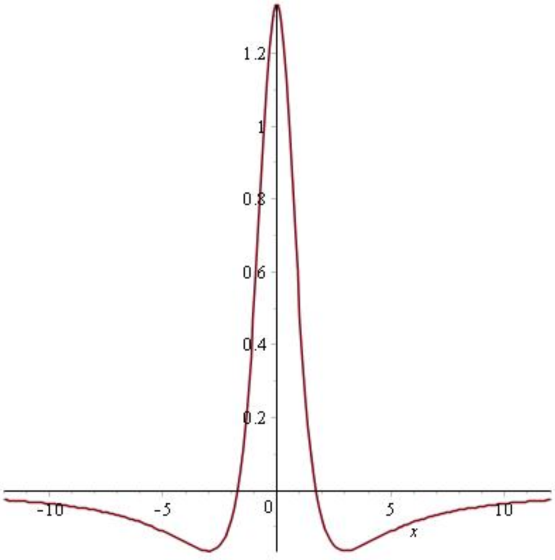
$$u = Q(x-ct, y) = \frac{4\left(-\left(x-ct\right)^2 + cy^2 + \frac{3}{c}\right)}{\left(\left(x-ct\right)^2 + cy^2 + \frac{3}{c}\right)^2}.$$

Nonradial, decays in all directions at the order $O\left(r^{-2}\right)$.

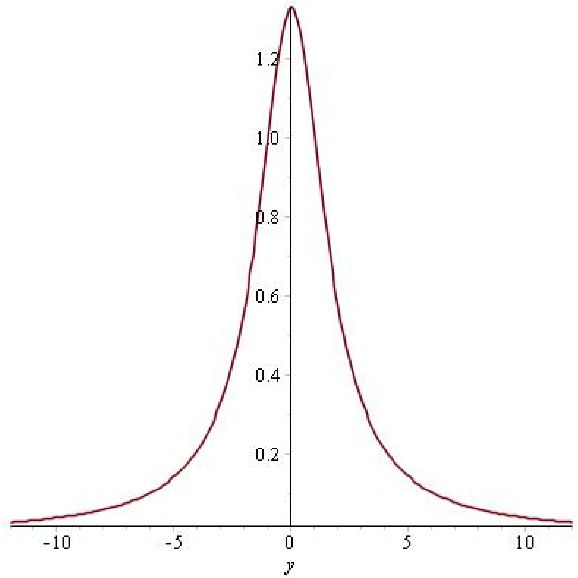
Lump



x-slice when $y = 0$:



y-slice when $x = 0$:



Open questions about lump solution

$$\partial_t u + \partial_x^3 u + 3\partial_x \left(u^2\right) - \partial_x^{-1}\partial_y^2 u = 0.$$

$$u = Q(x - ct, y) = \frac{4\left(-\left(x-ct\right)^2 + cy^2 + \frac{3}{c}\right)}{\left(\left(x-ct\right)^2 + cy^2 + \frac{3}{c}\right)^2}.$$

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Question 1: Is Q nondegenerate? (If so, we can use this solution to construct traveling wave solutions to Gross-Pitaeskii)

Open questions about lump solution

$$\partial_t u + \partial_x^3 u + 3\partial_x \left(u^2\right) - \partial_x^{-1}\partial_y^2 u = 0.$$

$$u = Q(x-ct, y) = \frac{4\left(-\left(x-ct\right)^2 + cy^2 + \frac{3}{c}\right)}{\left(\left(x-ct\right)^2 + cy^2 + \frac{3}{c}\right)^2}.$$

Question 1: Is Q nondegenerate? (If so, we can use this solution to construct traveling wave solutions to Gross-Pitaeskii)

Question 2: Morse index of Q, spectral property of Q? (Chiron-Scheid-2017: numerically Morse index 1.)

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Question 1: Is Q nondegenerate? (If so, we can use this solution to construct traveling wave solutions to Gross-Pitaeskii)

Question 2: Morse index of Q, spectral property of Q? (Chiron-Scheid-2017: numerically Morse index 1.)

Question 3: Is Q orbtitally stable?

Ground state lump solution of generalized KP-I equation

- For $1 < p < 5$, generalized KP-I equation:

$$\partial_x^2 \left(\partial_x^2 u - u + u^p\right) - \partial_y^2 u = 0$$

has a lump type solution (ground state), by variational arguments. No explicit formula is available. $p = 5$ is the critical exponent. (de Bouard-Saut 1997)

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- They are orbital stable when $p \in \left(1, \frac{7}{3}\right)$, unstable for $p \in \left(\frac{7}{3}, 5\right)$. (Yue Liu-Xiaoping Wang-1997; de Bouard–Saut-1997). Numerical study by Klein-Saut-2012.

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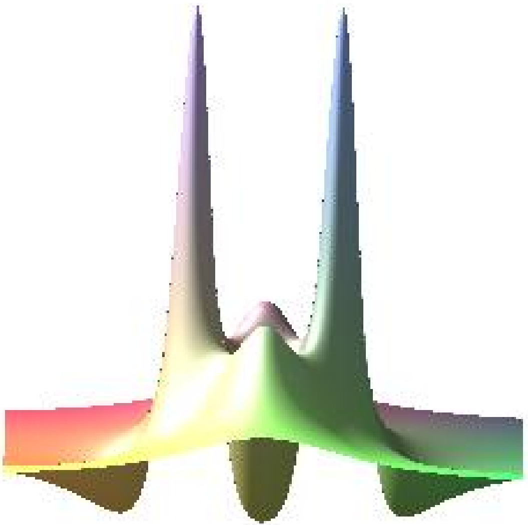
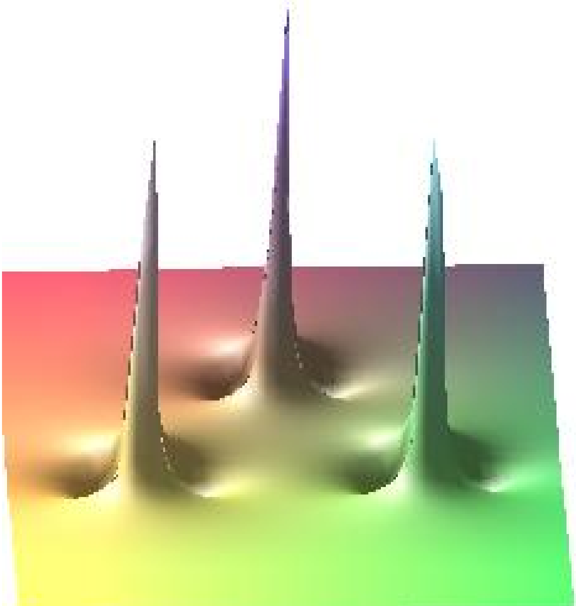
- For $p = 2$, it is not known whether the standard lump could be obtained this way. (Uniqueness of the ground state is still open).
- For $p = 2$, higher energy solitary wave solutions also exist(Pelinovsky-Stepanyants-1994). Related to the Calogero-Moser system. Stability issue more complicated.

Multilump



Multilump



Nondegeneracy of the standard lump

Let Q be the standard lump solution($p = 2$, speed $c = 1$) of the standard KP-I equation.

Theorem (Liu-Wei-2017)

Let ϕ be a solution of the linearized KP-I equation:

$$\partial_x^2 \left(\partial_x^2 \phi - \phi + 6Q\phi \right) - \partial_y^2 \phi = 0.$$

Suppose ϕ is smooth and decaying at infinity:

$$\phi(x, y) \to 0, \ \text{as } x^2 + y^2 \to +\infty.$$

Then $\phi = c_1 \partial_x Q + c_2 \partial_y Q$, for some constants c_1, c_2.

A family of y-periodic solutions bifurcating from 1D solution

$$\partial_x^2 \left(\partial_x^2 u - u + 3u^2\right) - \partial_y^2 u = 0.$$

One dimensional soliton solution

$$w(x) = \frac{1}{2} \cosh^{-2} \left(\frac{x}{2}\right)$$

Two-dimensional lump solution

$$Q(x, y) = \frac{4\left(-x^2 + 3\right)}{\left(x^2 + y^2 + 3\right)^2}$$

A family of y-periodic solutions bifurcating from 1D solution

Let $k, b \in R$, with $k^2 + b^2 = 1$. Define

$$\Gamma_k = \cosh\left(k\left(x\right)\right) + \sqrt{\frac{1-4k^2}{1-k^2}}\cosh\left(kbiy\right).$$

$$Q_k(x,y) = 2\partial_x^2 \ln \Gamma_k$$

Then $Q_k(x,y)$ are solutions to KP-I. They are periodic in y, with period $t_k := \frac{2\pi}{k\sqrt{1-k^2}}$.

- As $k \to 0$, $t_k \to +\infty$, the solutions $2\partial_x^2 \ln \Gamma_k$ converge to the lump Q.
- As $k \to \frac{1}{2}$, $\Gamma_k \to \cosh \frac{x}{2}$, the solutions $2\partial_x^2 \ln \Gamma_k$ converge to the one dimensional solution $\frac{1}{2}\cosh^{-2}\left(\frac{x}{2}\right)$.

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Remark: Similar as the equation $-\Delta u = u^p - u$.

Nondegeneracy of periodic solutions

Let Q_k be the periodic solution corresponding to Γ_k.

Theorem (Liu-Wei-2017)

Let ϕ be a solution of the linearized KP-I equation:

$$\partial_x^2 \left(\partial_x^2 \phi - \phi + 6 Q_k \phi\right) - \partial_y^2 \phi = 0.$$

Suppose ϕ is smooth, $\phi(x, y + t_k) = \phi(x, y)$, and

$$\phi(x, y) \to 0, \text{ as } |x| \to +\infty.$$

Then $\phi = c_1 \partial_x Q_k + c_2 \partial_y Q_k$, for some constants c_1, c_2.

Morse index and orbital stability of the lump solution

As an application of the previous theorems, we get

Theorem (Liu-Wei-2017)

The operator

$$L\eta := -\partial_x^2 \eta + \eta - 6Q\eta + \partial_x^{-2} \partial_y^2 \eta$$

has exactly one negative eigenvalue. As a consequence, the lump Q is orbitally stable: For any $\varepsilon > 0$, there exists $\delta > 0$, such that, if $u(x, y, t)$ is solution of KP-I with $\|u(\cdot, \cdot, 0) - Q\| < \delta$, then for all $t \in (0, +\infty)$,

$$\inf_{\gamma_1, \gamma_2 \in R} \|u(\cdot, \cdot, t) - Q(\cdot + \gamma_1, \cdot + \gamma_2)\| < \varepsilon.$$

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$$\inf_{\gamma_1, \gamma_2 \in R} \|u(\cdot, \cdot, t) - Q(\cdot + \gamma_1, \cdot + \gamma_2)\| < \varepsilon.$$

Remark: The issue of asymptotical stability will be more delicate.

Orbital stability

- To prove the Morse index result, we use a continuation argument.
- By nondegeneracy, the Morse index is invariant along the family of periodic solution. Hence the Morse index of lump is equal to one, since that of the 1D solution is one.
- Let u_c be the family of lumps with speed c. Let

$$d(c) := \int\int \left(\frac{1}{2} (\partial_x u_c)^2 - u_c^3 + \frac{1}{2} \left(\partial_y \partial_x^{-1} u_c \right)^2 + \frac{1}{2} c u_c^2 \right) dxdy.$$

  Then $d'(c) = \frac{1}{2}\sqrt{c} \int\int u_1^2(x, y)\, dxdy$. Hence $d''(c) > 0$.
- Orbital stability then essentially follows from the classical result of Grillakis-Shatah-Strauss-1987: The energy E_1 is locally minimized in the hypersurface $\left\{ \phi : \int\int \phi^2 = \text{costant} \right\}$.

Proof of Nondegeneracy of Lump Solution Q–Bilinear form of the KP-I equation

Introduce the τ function:

$$u = 2\partial_x^2 \left(\ln \tau\right).$$

KP-I can be written in the bilinear form:

$$\left(D_x D_t + D_x^4 - D_y^2\right) \tau \cdot \tau = 0.$$

D is the bilinear derivative operator:

$$D_s D_t f \cdot g = \left[\left(\partial_s - \partial_{s'}\right)\left(\partial_t - \partial_{t'}\right)\right]\left(f\left(s,t\right) g\left(s',t'\right)\right)|_{s'=s,t'=t}.$$

For instance, $D_x f \cdot g = \partial_x f g - f \partial_x g$.

$$D_x D_y f \cdot g = \partial_x \partial_y f g - \partial_x f \partial_y g - \partial_y f \partial_x g + f \partial_x \partial_y g.$$

Special solutions

Let

$$\tau_0 = 1,$$
$$\tau_1 = x + iy + \sqrt{3},$$
$$\tau_2 = x^2 + y^2 + 3.$$

Then $\tau_i(x - t, y)$ are solutions to the KP-I equation in bilinear form.

The solution corresponding to τ_0 is the trivial one. The solution corresponding to τ_1 is complex valued. The solution τ_2 corresponds to the lump solution Q.

Proof of nondegeneracy for lump solution-Backlund Transformation

Our key idea of the proof is to use that the fact that some special solutions of KP-I can be connected through Backlund transformation.

A bilinear identity:

$$
\begin{aligned}
&\frac{1}{2}\left[\left(D_x D_t + D_x^4 - D_y^2\right) f \cdot f\right] gg - \frac{1}{2}\left[\left(D_x D_t + D_x^4 - D_y^2\right) g \cdot g\right] ff \\
&= D_x\left[\left(D_t - \sqrt{3} i \mu D_y + D_x^3 - \sqrt{3} i D_x D_y\right) f \cdot g\right] \cdot (fg) \\
&+ 3D_x\left[\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right) f \cdot g\right] \cdot \left(D_x g \cdot f\right) \\
&+ \sqrt{3} i D_y\left[\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}} i D_y\right) f \cdot g\right] \cdot (fg).
\end{aligned}
$$

Backlund transformation of lump

Recall $\tau_0 = 1, \tau_1 = x + yi + \sqrt{3}, \tau_2 = x^2 + y^2 + 3.$
The Backlund transformation between τ_0 and τ_1 :

$$\begin{cases} \left(D_x^2 + \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} iD_y\right) \tau_0 \cdot \tau_1 = 0, \\ \left(-D_x - iD_y + D_x^3 - \sqrt{3} iD_x D_y\right) \tau_0 \cdot \tau_1 = 0. \end{cases}$$

The Backlund transformation between τ_1 and τ_2 :

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} iD_y\right) \tau_1 \cdot \tau_2 = 0, \\ \left(-D_x + iD_y + D_x^3 - \sqrt{3} iD_x D_y\right) \tau_1 \cdot \tau_2 = 0. \end{cases}$$

Backlund transformation of y-periodic solutions

Let $\Lambda_0 = 1$,

$$\Lambda_1 = \exp\left(\frac{1}{2}k\left(x - biy - t\right)\right) + r\exp\left(-\frac{1}{2}k\left(x - biy - t\right)\right),$$

where r is an explicit constant determined by k.

$$\Lambda_2 = \Gamma_k = \cosh\left(k\left(x - t\right)\right) + \sqrt{\frac{1-4k^2}{1-k^2}}\cos\left(kby\right).$$

The Backlund transformation between Λ_1 and Λ_2 is

$$\begin{cases} \left(D_x^2 + \frac{b}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\Lambda_1 \cdot \Lambda_2 = \frac{k^2}{4}\Lambda_1\Lambda_2, \\ \left(D_t + \frac{3k^2}{4}D_x - biD_y + D_x^3 - \sqrt{3}iD_xD_y - \frac{\sqrt{3}k^2b}{4}\right)\Lambda_1 \cdot \Lambda_2 = 0. \end{cases}$$

Similarly for Λ_0, Λ_1.

Linearized Backlund transformation

To prove the nondegeneracy of the lump, we linearize the transformation between τ_0 and τ_1

$$\begin{cases} \left(D_x^2 + \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y\right) \tau_0 \cdot \tau_1 = 0, \\ \left(-D_x - i D_y + D_x^3 - \sqrt{3} i D_x D_y\right) \tau_0 \cdot \tau_1 = 0. \end{cases}$$

We get

$$\begin{cases} L_1\phi = G_1\eta, \\ M_1\phi = N_1\eta. \end{cases}$$

Here

$$L_1\phi = \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\phi\cdot\tau_1,$$
$$M_1\phi = \left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\phi\cdot\tau_1,$$

$$G_1\eta = -\left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\tau_0\cdot\eta,$$
$$N_1\eta = -\left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_0\cdot\eta.$$

Transform the kernel to a simpler operator

Lemma

Let η be a solution of the linearized bilinear KP-I equation at τ_1 :

$$-D_x^2\eta \cdot \tau_1 + D_x^4\eta \cdot \tau_1 - D_y^2\eta \cdot \tau_1 = 0.$$

Suppose η satisfies

$$|\eta| + (1+r)\,|\partial_x \eta| + (1+r)\,|\partial_y \eta| \leq C\,(1+r)^{\frac{5}{2}}\,.$$

Then the linearized Backlund transformation between τ_0 and τ_1 has a solution ϕ with

$$|\phi| + |\partial_x \phi| + |\partial_y \phi| \leq C\,(1+r)^{\frac{5}{2}}\,.$$

Sketch of the proof of the Lemma

Step 1. Insert the equation $L_1\phi = G_1\eta$ into $M_1\phi = N_1\eta$, we get the inhomogeneous third order ODE:

$$4\partial_x^3\phi\tau_1 + \left(2\sqrt{3}\tau_1 - 12\right)\partial_x^2\phi + \left(-4\sqrt{3} + \frac{12}{\tau_1}\right)\partial_x\phi = F_1.$$

Here

$$\begin{aligned} F_1 &= 3\partial_x\left(G_1\eta\right) + \sqrt{3}G_1\eta + N_1\eta - \frac{6}{\tau_1}G_1\eta \\ &= -2\partial_x^3\eta + 2\sqrt{3}i\partial_x\partial_y\eta - \frac{6}{\tau_1}G_1\eta. \end{aligned}$$

For each fixed y, the homogeneous equation has solutions $\xi_0 = 1$,

$$\xi_1 := \frac{1}{2}\tau_1^2 - \frac{\sqrt{3}}{6}\tau_1^3,$$

and

$$\xi_2 := \left(\frac{\sqrt{3}}{2}\tau_1 + 1\right) e^{-\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{4}yi}.$$

Solve the inhomogeneous equation, we get a solution w_0, for each fixed y.

Solve the first equation $L_1\phi = G_1\eta$(Involving derivatives of y)

Step 2. Define

$$\Phi_0(x,y) := L_1\phi - G_1\eta,$$

$$\Phi_1 = \partial_x\Phi_0, \Phi_2 = \partial_x^2\Phi_0.$$

Note that Φ_i depends on the function ϕ.
Consider the system of equations

$$\begin{cases} \Phi_0(x,y) = 0, \\ \Phi_1(x,y) = 0, \quad \text{for } x = 1. \\ \Phi_2(x,y) = 0, \end{cases}$$

We seek a solution ϕ in the form $w_0 + w_1$, where

$$w_1(x,y) = \rho_0(y)\xi_0(x,y) + \rho_1(y)\xi_1(x,y) + \rho_2(y)\xi_2(x,y).$$

This is a system of ODE for ρ and can be solved.

Step 3. Prove $\Phi_0 = 0$ in $\mathbb{R}^2$. That is, the equation $L_1\phi = G_1\eta$ is satisfied for all x.

This follows from the identity:

$$\partial_x^3\Phi_0 = \left(-\frac{\sqrt{3}}{2} + \frac{6}{\tau_1}\right)\partial_x^2\Phi_0 + \frac{1}{\tau_1}\left(2\sqrt{3} - \frac{15}{\tau_1}\right)\partial_x\Phi_0 + \frac{1}{\tau_1^2}\left(\frac{15}{\tau_1} - 2\sqrt{3}\right)\Phi_0.$$

This is a third order ODE for Φ_0, initial value at $x = 1$ is zero.

Linearized Backlund transformation between τ_1 and τ_2

The linearization is

$$\begin{cases} L_2\phi = G_2\eta, \\ M_2\phi = N_2\eta. \end{cases} \tag{1}$$

Here

$$L_2\phi = \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\phi\cdot\tau_2,$$

$$M_2\phi = \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\phi\cdot\tau_2,$$

and

$$G_2\eta = -\left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\tau_1\cdot\eta,$$

$$N_2\eta = -\left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_1\cdot\eta.$$

Similar as the τ_0, τ_1 case, we have

Lemma

Let η be a function solving the linearized bilinear KP-I equation at τ_2 :

$$-D_x^2\eta \cdot \tau_2 + D_x^4\eta \cdot \tau_2 = D_y^2\eta \cdot \tau_2.$$

Suppose

$$|\eta| + (1+r)\,|\partial_x \eta| + (1+r)\,|\partial_y \eta| \leq C\,(1+r)^{\frac{5}{2}}\,.$$

Then the linearized system has a solution ϕ with

$$|\phi| + |\partial_x \phi| + |\partial_y \phi| \leq C\,(1+r)^{\frac{5}{2}}\,.$$

Proof of the nondegeneracy of lump

Suppose η satisfies

$$-D_x^2\eta \cdot \tau_2 + D_x^4\eta \cdot \tau_2 = D_y^2\eta \cdot \tau_2.$$

Case 1. $G_2\eta_2 = N_2\eta_2 = 0.$
Then $\eta_2 = c_1(x + yi) + c_2\tau_2.$

Case 2. $G_2\eta_2 \neq 0$ or $N_2\eta_2 \neq 0.$
Then using the linearized Backlund transformation, there exists a solution η_1 of the equation

$$\left(D_x^2 - D_x^4 + D_y^2\right)\eta_1 \cdot \tau_1 = 0,$$

satisfying suitable growth estimate.

Proof continued

Subcase 1. $G_1\eta_1 = N_1\eta_1 = 0$.
In this case, we can show

$$\eta_1 = a_1 + a_2\tau_1$$

(In the kernel of the linearized operator around τ_1).
Accordingly,

$$\eta_2 = c_1\partial_x\tau_2 + c_2\partial_y\tau_2 + c_3\tau_2.$$

Subcase 2. $G_1\eta_1 \neq 0$ or $N_1\eta_1 \neq 0$.
In this case, using the linearized Backlund transformation, we get a solution η_0 of

$$\left(D_x^2 - D_x^4 + D_y^2\right)\eta_0 \cdot \tau_0 = 0,$$

satisfying

$$|\eta_0| + |\partial_x\eta_0| + |\partial_y\eta_0| \leq C\left(1+r\right)^{\frac{5}{2}}.$$

Then η_0 is harmonic:

$$\partial_x^2\eta_0 + \partial_y^2\eta_0 = 0.$$

and can be written as

$$\eta_0 = c_1 + c_2x + c_3y + c_4\left(x^2 - y^2\right) + c_5xy.$$

We can prove(after tedious computation and using the linearized Backlund transformation again) that $c_2 = c_3 = c_4 = c_5 = 0$. Then

$$\eta_2 = a_1 \partial_x \tau_2 + a_2 \partial_y \tau_2 + a_3 \tau_2.$$

This finishes the proof.
Remark: The proof of nondegeneracy of periodic solutions is similar (more complicated computations).

Open Questions

$$-ic\partial_x U = \Delta U + U\left(1 - |U|^2\right) \quad \text{in} \quad \mathbb{R}^2.$$

- $c \sim 0$: multi-vortex solutions (roots of Adler-Moser polynomials)
- $c \sim \sqrt{2}$: multi-bump solutions of KP-I
- Question1: nondegeneracy of multi-bump solutions of KP-I?
- Question 2: multi-bump solutions to trvaelling wave GP?
- Question 3: are theses two branches connected?

2+1 Toda lattice

$$\Delta q_n = 4e^{q_{n-1}-q_n} - 4e^{q_n-q_{n+1}}, \text{ in } \mathbb{R}^2, n \in \mathbb{Z}.$$

Lump solution(Ablowitz-Villarroel-1998):

$$Q_n(x,y) := \ln \frac{\frac{1}{4} + \left(n - 1 + 2\sqrt{2}x\right)^2 + 4y^2}{\frac{1}{4} + \left(n + 2\sqrt{2}x\right)^2 + 4y^2}.$$

Nondegeneracy of the lump

Theorem (Liu-Wei-2017)

Let $\{\phi_n\}$ *be a solution of the linearized Toda lattice:*

$$\Delta\phi_n = e^{Q_{n-1}-Q_n}(\phi_{n-1}-\phi_n) - e^{Q_n-Q_{n+1}}(\phi_n-\phi_{n+1}), n \in \mathbb{Z}.$$

Suppose $\phi_{n+1}(x) = \phi_n\left(x + \frac{1}{2\sqrt{2}}\right)$ *and* ϕ_n *is smooth and decaying at infinity:*

$$\phi_n(x, y) \to 0, \text{ as } x^2 + y^2 \to +\infty.$$

Then $\phi_n = c_1\partial_x Q_n + c_2\partial_y Q_n$.

Remark:

- More complicated than the KP-I case. Analyze the Fourier transform of the linearized Backlund transformation systems.
- Applying the nondegeneracy result of Toda lattice yields the existence of solutions to Allen-Cahn equation in $\mathbb{R}^3$ with infinitely many ends

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^3$$