The sphere covering inequality and its applications

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Isoperimetric inequalities

Suppose $\Omega \subset \mathbb{R}^2$, then

$$L^2(\partial\Omega) \geq 4\pi A(\Omega)$$

Equality holds if and only if Ω is a disk.

Similar inequalities hold for high dimensions.

$$|\partial\Omega|^n \leq S_n|\Omega|^{n-1}$$

where $S_n = |S^{n-1}|^n/|B_1|^{n-1} = n^n\omega_n$, S^{n-1} and B_1 are the unit sphere and ball in R^n respectively.

Levy's Isoperimetric inequalities on spheres (1919)

On the standard unit sphere with the metric induced from the flat metric of $\mathbb{R}^3$,

$$L^2(\partial\Omega) \geq A(\Omega)\big(4\pi - A(\Omega)\big)$$

If the sphere has radius R, then

$$L^2(\partial\Omega) \geq A(\Omega)\big(4\pi R^2 - A(\Omega)\big)/R^2$$

Alexandrov-Bol's inequality (1941)

In general, we can identify a sphere with $\mathbb{R}^2$ by the stereographic projection, and equip it with a metric conformal to the flat metric of $\mathbb{R}^2$, i.e., $ds^2 = e^{2v}(dx_1^2 + dx_2^2)$.
Assume v satisfies

$$\Delta v + K(x)e^{2v} \geq 0, \quad \mathbb{R}^2,$$

with the Gaussian curvature $k \leq 1$. Then

$$\Big(\int_{\partial\Omega} e^v ds\Big)^2 \geq \Big(\int_\Omega e^{2v}\Big)\Big(4\pi - \int_\Omega e^{2v}\Big)$$

Slightly different equation

Set $u = 2v + \ln 2$

$$\Delta u + e^u \geq 0, \quad \mathbb{R}^2$$

Then

$$\Big(\int_{\partial\Omega} e^{u/2}\Big)^2 \geq \frac{1}{2}\Big(\int_{\Omega} e^u\Big)\Big(8\pi - \int_{\Omega} e^u\Big)$$

We may think that this is the Levy's isoperimetric inequality on the sphere with radius $\sqrt{2}$ and the gaussian curvature $1/2$ in $\mathbb{R}^3$.

Sobolev inequalities (1938)

Given $u \in H_0^1(\Omega) \subset \mathbb{R}^n$. We have

$$||u||_{L^p(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}$$

for $0 \leq p \leq \frac{2n}{n-2}$.

Question: Is $H_0^1 \subset L^\infty$? **NO!**

Moser-Trudinger inequality concerns the borderline case $n = 2$.

Moser-Trudinger inequality (1971)

Let S^2 be the unit sphere and for $u \in H^1(S^2)$.

$$J_\alpha(u) = \frac{\alpha}{4}\int_{S^2} |\nabla u|^2 d\omega + \int_{S^2} u d\omega - \log \int_{S^2} e^u d\omega \geq C > -\infty,$$

if and only if $\alpha \geq 1$, where the volume form $d\omega$ is normalized so that $\int_{S^2} d\omega = 1$.

Aubin's Result (1979) and Onofri Inequality (1982)

Aubin observed that for $\alpha \geq \frac{1}{2}$,

$$J_\alpha(u) \geq C > -\infty$$

for

$$u \in \mathcal{M} := \{u \in H^1(S^2) : \int_{S^2} e^u x_i = 0, \quad i = 1, 2, 3\},$$

Onofri showed for $\alpha \geq 1$

$$J_\alpha(u) \geq 0;$$

Chang and Yang conjecture (1987)

Chang and Yang showed that for α close to 1 the best constant again is equal to zero. They proposed the following conjecture.

Conjecture A. For $\alpha \geq \frac{1}{2}$,

$$\inf_{u \in \mathcal{M}} J_\alpha(u) = 0.$$

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Indeed, they showed that the minimizer u exists and satisfies

$$\frac{\alpha}{2}\Delta u + \frac{e^u}{\int_{S^2} e^u d\omega} - 1 = 0 \quad \text{on} \quad S^2 \tag{1}$$

by showing

$$\mu_i = 0, \quad i = 1, 2, 3.$$

in the Euler-Lagrange equations

$$\frac{\alpha}{2}\Delta u + \frac{e^u}{\int_{S^2} e^u d\omega} - 1 = \sum_{i=1}^{i=3} \mu_i x_i e^u \quad \text{on} \quad S^2$$

Mean field Equationc on S^2

Let $\alpha = \frac{8\pi}{\rho}$

$$\Delta u + \rho\Big(\frac{e^u}{\int_{S^2} e^u d\omega} - 1\Big) = 0 \quad \text{on} \quad S^2 \tag{2}$$

Many Results by
Brezis, Merle, Caglioti, Lions Marchioro, Pulvirenti, Y.Y. Li, Shafir, Chanillo, Kiessling, Chang, Chen, Lin, Lucia, Cabre, Bartolucci, Tarantello, De Marchis, Malchiodi, ...

One Important Result (Brezis, Merle, Y.Y. Li): Blow up of solutions happens only when $\rho \to 8\pi m$. The solution sets are compact in C^2 for a compact set of ρ in $\cup_{m=1}^{\infty}\big(8\pi m, 8\pi(m+1)\big)$.

Other applications

If the metric $g = e^{2u} g_0$ has Gaussian curvature $K(x)$, then

$$\Delta u + K(x)e^{2u} = 1 \quad \text{on} \quad S^2.$$

Navier-Stokes equations

$$\Delta u + (u \cdot \nabla)u = \nabla p \quad div(u) = 0 \quad \text{on} \quad R^3$$

scale under $u \to \lambda u(\lambda x)$. What are the solutions that are invariant under scaling? Explicit examples are the Landau solutions. Anything else?
Sverak (2009): **NO!**. Proof: For $x \in S^2$ decompose $u = T(x) + xN(x)$, where T is tangent to S^2. After some work, one can show that $T = \nabla \varphi$ and

$$\Delta \varphi + 2e^{\varphi} = 2 \quad \text{on} \quad S^2.$$

Earlier results on conjecture A

Axially symmetric functions:
Feldman, Froese, Ghoussoub and Gui (1998)

$$\alpha > \frac{16}{25} - \epsilon$$

Gui and Wei, and independently Lin (2000)

$$\alpha \geq \frac{1}{2}$$

Non-axially symmetric functions:
Ghoussoub and Lin (2010)

$$\alpha \geq \frac{2}{3} - \epsilon$$

Strategies of Proof

For axially symmetric functions, to show (1) has only solution $u \equiv C$.

For general functions, to show solutions to (1) are axially symmetric.

Equations on $\mathbb{R}^2$

Let Π be the stereographic projection $S^2 \to \mathbb{R}^2$ with respect to the north pole $N = (1, 0, 0)$:

$$\Pi := \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).$$

Suppose u is a solution of (1) and let

$$v = u(\Pi^{-1}(y)) - \frac{2}{\alpha} \ln(1 + |y|^2) + \ln(\frac{8}{\alpha}), \tag{3}$$

then v satisfies

$$\Delta v + (1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^v = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{4}$$

and

$$\int_{R^2} (1 + |y|^2)^{2(\frac{1}{\alpha} - 1)} e^v dy = \frac{8\pi}{\alpha}. \tag{5}$$

General Equations

Consider in general the equation

$$\Delta v + (1+|y|^2)^l e^v = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{6}$$

and

$$\int_{R^2} (1+|y|^2)^l e^v dy = 2\pi(2l+4). \tag{7}$$

Are solutions to (6) and (7) radially symmetric?

For $l = 0$: Chen and Li (1991)

For $-2 < l < 0$: Chanillo and Kiessling (1994)

Conjecture B. For $0 < l \leq 2$, solutions to (6) and (7) must be radially symmetric.

$0 < l \leq 1$: Ghoussoub and Lin (2010)

Existence of non-radial solutions

Lin (2000): For $2 < l \neq (k-1)(k+2)$, where $k \geq 2$ there is a non radial solution.

Example of Chanillo-Kiessling (1994)
Consider

$$\Delta u + 8k^2 r^{2(k-1)} e^u = 0, \quad \mathbb{R}^2$$

with

$$\int_{\mathbb{R}^2} 8k^2 r^{2(k-1)} e^u = 8\pi k.$$

There exists a non-radial solution (with explicit formula) for any integer $k \geq 2$.

Theorem (Gui and M., 2015)

Both Conejcture A and B hold true.

Conjecture A.
For $\alpha \geq \frac{1}{2}$,

$$\inf_{u \in \mathcal{M}} J_\alpha(u) = 0.$$

Conjecture B.
For $0 < l \leq 2$, solutions to (6) and (7) must be radially symmetric.
Note

$$l = 2(\frac{1}{\alpha} - 1).$$

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4π lower bound

Theorem (Lin and Lucia, 2007)

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain and $w \in C^2(\overline{\Omega})$ satisfying

$$\Delta w + e^w > 0$$

in $\overline{\Omega}$ and $\int_\Omega e^w \leq 8\pi$.
Consider an open set $\omega \subset \Omega$ such that $\lambda_{1,w}(\omega) \leq 0$, where the first eigenvalue of the linearized operator $\Delta + e^w$

$$\lambda_{1,w}(\omega) := \inf_{\phi \in H_0^1(\omega)} \Big(\int_\omega |\nabla \phi|^2 - \int_\omega \phi^2 e^w \Big) \leq 0.$$

Then $\int_\omega e^w > 4\pi$.

Radial symmetry

Just to show

$$\varphi(x,y) = y\frac{\partial v}{\partial x} - x\frac{\partial v}{\partial y} \equiv 0,$$

where v is defined by (3).

Now let $w := \ln((1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^v)$.

$$\Delta\varphi + e^w\varphi = 0 \ \text{ in } \ \mathbb{R}^2.$$

Given v be even, then

$$\frac{8\pi}{\alpha} = \int_{\mathbb{R}^2}(1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^v dy = \sum_{i=1}^{4}\int_{\Omega_i} e^w > 4\pi = 16\pi.$$

This implies $\alpha < \frac{1}{2}$ which is a contradiction.

New 8π lower bound

Theorem (Gui and M., 2015)

Let Ω be a simply connected subset of R^2 and assume $w_i \in C^2(\overline{\Omega})$, $i = 1, 2$ satisfy

$$\Delta w_i + e^{w_i} = f_i(y), \tag{8}$$

where $f_2 \geq f_1 \geq 0$ in Ω.
Suppose $\omega \subset \Omega$ and $w_2 > w_1$ in ω and $w_2 = w_1$ on $\partial\omega$, then

$$\int_\omega e^{w_1} + e^{w_2} dy \geq 8\pi. \tag{9}$$

Furthermore if $f_1 \not\equiv 0$ or $f_2 \not\equiv f_1$ in ω, then $\int_\omega e^{w_1} + e^{w_2} dy > 8\pi$.

Even symmetry of solutions

Suppose

$$\Delta v_1 + (1+|y|^2)^{2(\frac{1}{\alpha}-1)} e^{v_1} = 0 \quad \text{in} \quad \mathbb{R}^2.$$

and let $v_2(x,y) = v_1(x,-y)$. Define

$$w_i := \ln((1+|y|^2)^{2(\frac{1}{\alpha}-1)} e^{v_i}), i = 1, 2.$$

Then

$$\Delta w_i + e^{w_i} = \frac{8(\frac{1}{\alpha}-1)}{(1+|y|^2)^2} \geq 0 \quad \text{in} \quad \mathbb{R}^2, i = 1, 2.$$

$$\begin{aligned} 2 \times \frac{8\pi}{\alpha} &= \int_{\mathbb{R}^2} (1+|y|^2)^{2(\frac{1}{\alpha}-1)} e^{v_1} dx + \int_{\mathbb{R}^2} (1+|y|^2)^{2(\frac{1}{\alpha}-1)} e^{v_2} dx \\ &\geq \sum_{i=1}^{4} \int_{\Omega_i} e^{w_1} + e^{w_2} dx > 4 \times 8\pi. \end{aligned}$$

Hence $\alpha < \frac{1}{2}$.

Proof of 8π lower bound: An Example

For $\lambda > 0$ define U_λ by

$$U_\lambda := -2\ln(1 + \frac{\lambda^2|y|^2}{8}) + 2\ln(\lambda) \qquad (10)$$

Proposition

Let $\lambda_2 > \lambda_1$, and U_{λ_1} and U_{λ_2} be radial solutions of the equation

$$\Delta u + e^u = 0$$

with $U_{\lambda_2} > U_{\lambda_1}$ in B_R and $U_{\lambda_1} = U_{\lambda_2}$ on ∂B_R, for some $R > 0$. Then

$$\int_{B_R} (e^{U_{\lambda_1}} + e^{U_{\lambda_2}})dy = 8\pi.$$

Bol's inequality for radial weak subsolutions

Proposition (Gui and M., 2015)

Let B_R be the ball of radius R in $\mathbb{R}^2$ $u \in C^1(\overline{B_R})$ be a strictly decreasing radial function satisfying

$$\int_{\partial B_r} |\nabla u| ds \leq \int_{B_r} e^u dy \ \text{ for all } \ r \in (0, R), \ \text{ and } \ \int_{B_R} e^u \leq 8\pi.$$

Then the following inequality holds

$$\left(\int_{\partial B_R} e^{\frac{u}{2}} \right)^2 \geq \frac{1}{2} \left(\int_{B_R} e^u \right) \left(8\pi - \int_{B_R} e^u \right). \qquad (11)$$

Moreover if $\int_{\partial B_r} |\nabla u| ds < \int_{B_r} e^u dy$ for some $r \in (0, R)$, then the inequality in (11) is strict.

Integral comparison for subsolution

Lemma (Gui and M., 2015)

Assume that $\psi \in C^1(\overline{B_R})$ *is an strictly decreasing radial function and satisfies*

$$\int_{\partial B_r} |\nabla \psi| \le \int_{B_r} e^{\psi} \tag{12}$$

for all $r \in (0, R)$ *and* $\psi = U_{\lambda_1} = U_{\lambda_2}$ *for some* $\lambda_2 > \lambda_1$ *on* ∂B_R*, for some* $R > 0$*. Then*

$$\int_{B_R} e^{\psi} \le \int_{B_R} e^{U_{\lambda_1}} \quad \text{or} \quad \int_{B_R} e^{\psi} \ge \int_{B_R} e^{U_{\lambda_2}}. \tag{13}$$

Moreover if the inequality in (12) is strict for some $r \in (0, R)$*, then the inequalities in (13) are also strict.*

Rearrangment arguments

Suppose that $w \in C^2(\overline{\Omega})$ satisfies

$$\Delta w + e^w \geq 0.$$

Then any function $\phi \in C^2(\overline{\Omega})$ can be equimeasurably rearranged with respect to the measures $e^w dy$ and $e^{U_\lambda} dy$. More precisely, for $t > \min_{x \in \Omega} \phi$ define

$$\Omega_t := \{\phi > t\} \subset\subset \Omega,$$

and define Ω_t^* be the ball centered at origin in $\mathbb{R}^2$ such that

$$\int_{\Omega_t^*} e^{U_\lambda} dy = \int_{\Omega_t} e^w dy.$$

Rearrangment arguments

Then $\phi^* : \Omega^* \to \mathbb{R}$ defined by $\phi^*(x) := \sup\{t \in \mathbb{R} : x \in \Omega_t^*\}$ provides an equimeasurable rearrangement of ϕ with respect to the measure $e^w dy$ and $e^{U_\lambda} dy$, i.e.

$$\int_{\{\varphi^* > t\}} e^{U_\lambda} dy = \int_{\{\phi > t\}} e^w dy, \quad \forall t > \min_{x \in \Omega} \phi. \tag{14}$$

Moreover we have

$$\int_{\{\phi = t\}} |\nabla \phi| ds \geq \int_{\{\phi^* = t\}} |\nabla \varphi^*| ds. \tag{15}$$

Continued: The Proof of 8π Bound

$$\Delta(w_2 - w_1) + e^{w_2} - e^{w_1} = f_2 - f_1 \geq 0.$$

$$\int_\Omega e^{w_1} = \int_{B_1} e^{U_{\lambda_1}}. \tag{16}$$

Let φ be the symmetrization of $w_2 - w_1$ with respect to the measures $e^{w_1}dy$ and $e^{U_{\lambda_1}}dy$. Then

$$\begin{aligned}
\int_{\{\varphi=t\}} |\nabla\varphi| &\leq \int_{\{w_2-w_1=t\}} |\nabla(w_2 - w_1)| \\
&\leq \int_{\Omega_t} \left(e^{w_2} - e^{w_1}\right) \\
&= \int_{\{\varphi>t\}} e^{U_{\lambda_1}+\varphi} - \int_{\{\varphi>t\}} e^{U_{\lambda_1}} \\
&= \int_{\{\varphi>t\}} e^{U_{\lambda_1}+\varphi} - \int_{\{\varphi=t\}} |\nabla U_{\lambda_1}|,
\end{aligned}$$

Continued: The Proof of 8π Bound

Hence

$$\int_{\{\varphi=t\}} |\nabla(\varphi + U_{\lambda_1})| \leq \int_{\varphi>t} e^{(\varphi+U_{\lambda_1})} dy \tag{18}$$

for all $t > 0$.

$$\int_{\partial B_r} |\nabla(\varphi + U_{\lambda_1})| \leq \int_{B_r} e^{(\varphi+U_{\lambda_1})} dy. \tag{19}$$

Since $\psi = U_{\lambda_1} + \varphi > U_{\lambda_1}$,

$$\int_{B_1} e^{U_{\lambda_1}+\varphi} dx \geq \int_{B_1} e^{U_{\lambda_2}}.$$

Hence

$$\int_{\Omega} e^{w_1} + e^{w_2} dx = \int_{B_1} e^{U_{\lambda_1}} + e^{U_{\lambda_2}+\varphi} dx \geq \int_{B_1} e^{U_{\lambda_1}} + e^{U_{\lambda_2}} dx = 8\pi.$$

A Mean Field equation with singularity on S^2

Consider the mean field equation

$$\Delta_g u + \lambda \left(\frac{e^u}{\int_{S^2} e^u d\omega} - \frac{1}{4\pi} \right) = 4\pi(\delta(P) - \frac{1}{4\pi}) \quad \text{on} \quad S^2, \quad (20)$$

with

$$\lambda = 4\pi(3 + \alpha)$$

Existence: It admits a solution if and only if $\alpha \in (-1, 1)$.

Axial symmetry

Axial Symmetry: D. Bartolucci, C.S. Lin, and G. Tarantello in *Comm. Pure Appl. Math.* 64 (2011), no. 12, 1677-1730.

Main result: There exists $\delta > 0$ such that for $\alpha \in (1-\delta, 1)$ all solutions to equation (20) is axially symmetric about the direction $\overrightarrow{OP}$.

Question C. Are all solutions of (20) axially symmetric about $\overrightarrow{OP}$ for every $\alpha \in (-1, 1)$?

Theorem (Gui, M. (2015))

For every $\alpha \in (-1, 1)$ the solution to equation (20) is unique and axially symmetric about $\overrightarrow{OP}$.

Mean field equations for the spherical Onsager vortex

Consider the following equation

$$\Delta_g u(x) + \frac{\exp(\alpha u(x) - \gamma\langle n, x\rangle)}{\int_{S^2} \exp(\alpha u(x) - \gamma\langle n, x\rangle) d\omega} - \frac{1}{4\pi} = 0 \quad \text{on} \quad S^2. \quad (21)$$

with

$$\int_{S^2} u d\omega = 0.$$

C.S. Lin (2000): If $\alpha < 8\pi$, then for $\gamma \geq 0$ the solution to equation (21) is unique and axially symmetric with respect to n.

Conjecture D Let $\gamma \geq 0$ and $\alpha \leq 16\pi$. Then every solution u of (21) is axially symmetric with respect to n.

Axial symmetry of spherical Onsager vortex

Theorem (Gui and M., 2015)

Suppose $8\pi < \alpha \leq 16\pi$ *and*

$$0 \leq \gamma \leq \frac{\alpha}{8\pi} - 1. \tag{22}$$

Then every solution of (21) is axially symmetric with respect to n.

A mean field equation on flat torus

Consider the mean field equation on a flat torus with fundamental domain

$$\Omega_\epsilon = [-\frac{1}{\epsilon}, \frac{1}{\epsilon}] \times [-1, 1]$$

$$\Delta v + \rho\left(\frac{e^v}{\int_{\Omega_\epsilon} e^v} - \frac{1}{|\Omega_\epsilon|}\right) = 0, \quad (x, y) \in \Omega_\epsilon. \tag{23}$$

Earlier results on flat torus

Cabré, Lucia, and Sanchón (2005): If

$$\rho \leq \rho^* := \frac{16\pi^3}{\pi^2 + \frac{2}{R_\epsilon^2} + \sqrt{(\pi^2 + \frac{2}{R_\epsilon^2})^2 - \frac{8\pi^3}{|T_\epsilon|}}} \leq 0.879 \times 8\pi,$$

then every solutions are one-dimensional. Here R_ϵ is the maximum conformal radius of the rectangle T_ϵ.
Lin and Lucia (2006) proved that the constant are the unique solutions if

$$\rho \leq \begin{cases} 8\pi & \text{if } \epsilon \geq \frac{\pi}{4} \\ 32\epsilon & \text{if } \epsilon \leq \frac{\pi}{4}. \end{cases}$$

The optimal results was conjectured to be $\rho \leq \min\{8\pi, 4\pi^2\epsilon\}$.
Note: $32\epsilon < 4\pi^2\epsilon \simeq 39.47\epsilon$.

Sharp result on flat torus

Theorem (Gui, M. (2016))

Assume that $v \in C^2(\Omega)$ *is a period solution of* (23). *Then* u *must depend only on* x *if* $\rho \leq 8\pi$. *In particular,* u *must be constant if* $\rho \leq \min\{8\pi, 4\pi^2\epsilon\}$.

Thank You!