Blow-up solutions for a mean field equation on sphere and torus

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Mean field equation on closed surfaces:

$$\Delta_g u + \rho \left(\frac{he^u}{\int_M he^u} - \frac{1}{|M|} \right) = 0.$$

Caglioti, Lions, Marchioro, Pulvirenti (1992, 1995): Euler flows.

Chanillo, Kiessling (1994) and Kiessling (1993): planar and spherical Onsager vortex theories.

Limiting case when $\epsilon \to 0$ of the Chern-Simons equation:

$$\Delta u + \frac{1}{\epsilon^2} e^u (1 - e^u) = 4\pi \sum_{i=1}^{N} \delta_{q_i}.$$

The equation is often posed on $\mathbb{R}^2$ or on a flat torus.

Jackiw, Weinberg (1990) and Hong et al. (1990): the relativistic abelian Chern-Simons gauge field theory.

Kao, Lee (1994) and Dunne (1995): the non-Abelian theory.

Blow-up phenomena

Brezis and Merle (1991): If sequence of solutions to the prescribed Gauss curvature problem blow up, then $h_n e^{u_n} \to \sum \alpha_i \delta_{p_i}$.

Li and Shafrir (1994): $\alpha_i = 8\pi$.

Li (1999) proposed to compute the Leray-Schauder degree of MFE and showed that the degree is independent of the choice of h.

The solution set of the MFE is compact unless $\rho = 8\pi m$.

Regular case: $h > 0$.

Chen and Lin (2002, 2003): $d_\rho = C^m_{m-\chi(M)}$ if $\rho \in (8\pi m, 8\pi(m+1))$.

Singular case: $h = 0$ **on some discrete set**.

$$\Delta_g u + \rho\left(\frac{Ke^u}{\int_M Ke^u dH^2} - \frac{1}{|M|}\right) = 4\pi \sum_{j=1}^{N} \beta_j \left(\delta_{q_i} - \frac{1}{|M|}\right).$$

Chen and Lin (2015): d_ρ depends on $\chi(M)$ and β_j's.

Existence of blow-up solutions

Chen and Lin (2003): blow-up occurs at a non-degenerate critical point P of f_h with $l(P) \neq 0$ where

$$f_h(p_1, \cdots, p_m) = \sum_{j=1}^{m} (\log h(p_j) + 4\pi R(p_j, p_j)) + 8\pi \sum_{i<j} G(p_i, p_j)$$

and

$$l(P) = \sum_{j=1}^{m} \left[\Delta \log h(p_j) + 8\pi m - 2K(p_j)\right] h(p_j) e^{G_j^*(p_j)}$$

$(G_j^*(p_j) = 8\pi(\sum_{l \neq j} G(p_j, p_l) + R(p_j, p_j)))$.

Esposito and Figueroa (2014) extended the existence result to “stable” critical point, the singular case and the case when $l(P) = 0$ but $D(P) \neq 0$.

MFE on a flat torus:

$$\Delta u + \rho \left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|} \right) = 0,$$

T is a flat torus with a rectangular fundamental domain. The corresponding f_h is no longer a morse function. Actually, f_h is translation invariant.

Obviously, $u \equiv C$ is a solution.

Non-trivial solutions

Struwe and Tarantello (1998): mountain pass solutions exist when $\rho \in (8\pi, \lambda_1(T)|T|)$.

Ricciardi and Tarantello (1998): 1d solutions exist if and only if $\rho > \lambda_1(T)|T|$.

Uniqueness and symmetry

The fundamental domain of T is any parallelogram.

Lin and Lucia (2006): Trivial solutions are the only solutions if $\rho \leq \min\{8\pi, 32\frac{l^2}{|T|}\}$;

Sequence of solutions must blow up if $\frac{l^2}{|T|} \geq \frac{\pi}{4}$ as $\rho \to 8\pi$.

Here l denotes the length of the shortest geodesic.

Rectangular torus.

Lin and Lucia (2007): 1d symmetryness of Steiner symmetric solutions ($\rho \leq 8\pi$).

Gui and Moradifam (2016): Sharp uniqueness result of trivial solutions if $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$; evenly symmetryness ($\rho \leq 16\pi$ when we have two critical points: one at the origin and another on the edge of the fundamental domain).

MFE on $\mathbb{S}^2$:

$$\Delta u + \rho \left(\frac{e^u}{\int_{\mathbb{S}^2} e^u} - \frac{1}{4\pi} \right) = 0.$$

f_h is invariant under orthogonal transformations.

Axially symmetric solutions.

Lin (2000): the equation only admits trivial solutions if $\rho < 8\pi$;

axially symmetric solutions blow up at two antipodal points;

axially symmetric blow-up solutions are unique above 16π hence symmetric about the great circle.

Gui and Wei (2000) and Lin (2000): the only axially symmetric solutions (mass center at 0) are trivial solutions if $\rho \leq 16\pi$.

Dolbeault, Esteban and Tarantello (2009): multiplicity of axially symmetric solutions.

General case.

Gui and Moradifam (2016) developed a new tool named "sphere covering inequality" to extend the uniqueness result to $\rho \leq 16\pi$.

Shi, Sun, Tian and Wei (2017): only admits trivial solution when $\rho = 24\pi$.

Let $V_\lambda = \ln\left(\frac{8\lambda^2}{(\lambda^2+|x|^2)^2}\right)$ be the family of solutions of the Liouville equation:

$$\Delta V_\lambda + e^{V_\lambda} = 0.$$

Chen and Li (1991): classification of solutions to the Liouville equation.

$$U_{\lambda,p}(y) = V_\lambda(x) + 2\ln\left(1+|x|^2\right) - \ln 4,$$

$$\int_{\mathbb{S}^2} e^{U_{\lambda,p}} = 8\pi.$$

Singular MFE.

Chen, Lin and Wang (2004): the Green's function $G(z)$ of a rectangular torus T is evenly symmetric about both axes;

$G(z)$ has only three critical points which are half-periods.

Lin and Wang (2010): the singular equation on $T[\tau]$ with $\rho = 8\pi$ and a singular source at the origin has solutions if and only if $G(z)$ has an extra pair of critical points other than half-periods. ($T[\tau] = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$.)

Chen, Kuo, Lin and Wang (2018) studied the region that $G(z)$ has five critical points and the region that $G(z)$ has three critical points.

Regular MFE. Kazdan and Warner (1974), Chang and Yang (1987),Chang, Yang and Gursky (1993), Ding, Jost, Li and Wang (1997, 1998), Cabre, Lucia and Sanchon (2005), Lucia (2006, 2007), Djadli (2008), Malchiodi (2008), De Marchis (2008, 2010, 2011)...

Singular MFE. Bartolucci and Tarantello (2002, 2017), Prajapat and Tarantello (2001), Eremenko (2004), Bartolucci, Chen, Lin and Tarantello (2004), Tarantello (2004, 2005), Esposito (2005), Bartolucci and Montefusco (2006, 2007), Bartolucci and Lin (2009, 2012), Zhang (2009), del Pino, Esposito and Musso (2010, 2012), Bartolucci, De Marchis and Malchiodi (2011), Bartolucci, Lin and Tarantello (2011), Carlotto and Malchiodi (2011, 2012), Chen and Lin (2011), Malchiodi and Ruiz (2011), Bartolucci and Malchiodi (2013), Carlotto (2013), D'Aprile (2013), Chai, Lin and Wang (2015), De Marchis and Lopez-Soriano (2016), Kuo and Lin (2016), Lin and Wang (2017), Chen, Kuo and Lin (2017)...

Nonlocal equation. Da Lio, Martinazzi and Riviere (2015), Da Lio and Martinazzi (2017).

Cosmic strings. Chen, Guo and Spirn (2012), Poliakovski and Tarantello (2012), Bartolucci and Castorina (2016), Tarantello (2017)...

C-S equation. Spruck and Yang (1992, 1995), Caffarelli and Yang (1995), Tarantello (1996, 2007), Ding, Jost, Li and Wang (1998, 1999, 2001), Nolasco and Taratello (1998, 1999), Chae and Imanuvilov (2000, 2002), Chan, Fu and Lin (2002), Nolasco (2003), Choe (2005, 2007, 2009), Choe and Kim (2008), Lin and Yan (2010, 2013, 2017), Choe, Kim and Lin (2011), del Pino, Esposito, Figueroa and Musso (2015).

Systems. Nolasco and Tarantello (2000), Jost and Wang (2001, 2002), Lucia and Nolasco (2002), Jost, Lin and Wang (2006), Lin, Wei and Ye (2012), Lin, Wei and Zhao (2012), Lin and Yan (2013), Han, Lin, Tarantello and Yang (2014), Battaglia, Jevnikar, Malchiodi(2015), Lin, Wei and Zhang (2015, 2016), Battaglia and Malchiodi (2016)...

Theorem (Cheng, Gui and Hu). For simplicity, let's take the fundamental domain of T to be a unit square. MFE with $h \equiv 1$ has a sequence u_λ of blow-up solutions which blow up at $\xi_1 = 0$ and ξ_2 (any of the three half periods) as $\rho \to 16\pi$ with the following properties:

(1) $\rho - 16\pi = (C(l(\xi_1, \xi_2)) + o(1))\lambda^2 \ln \frac{1}{\lambda}$,

(2) $u_\lambda(z) = u_\lambda(-z) = u_\lambda(\overline{z})$,

(3) $\frac{\rho}{\int_T e^{u_\lambda}} e^{u_\lambda} \to 8\pi(\delta_{\xi_1} + \delta_{\xi_2})$.

Remark 1. Blow-up rates at the two blow-up points are identical, i.e. $u_\lambda(z + \omega_k/2) = u(z)$ for some $k \in \{1, 2, 3\}$. Thus, it also implies the existence of blow-up solutions as $\rho \to 8\pi$.

Remark 2. The result can be extended to any rectangular torus and even to any flat torus with parallelogrammic fundamental domain. In the latter case, the solutions we construct are evenly symmetric, i.e. $u(z) = u(-z)$.

Theorem (Gui and Hu). MFE with $h \equiv 1$ on $\mathbb{S}^2$ has a sequence u_λ of blow-up solutions which blow up at ξ_1, ξ_2, ξ_3 and ξ_4 as $\rho \to 32\pi$. The four blow-up points form a regular tetrahedron. u_λ satisfies the following:

(1) $\rho - 32\pi = (C(l(\xi_1, \cdots, \xi_4)) + o(1))\lambda^2 \ln \frac{1}{\lambda}$,

(2) u_λ possesses tetrahedral symmetry,

(3) $\frac{\rho}{\int_{\mathbb{S}^2} e^{u_\lambda}} e^{u_\lambda} \to 8\pi \sum_{j=1}^{4} \delta_{\xi_j}$.

Remark 1. Other families of solutions can be similarly constructed with blow-up points at the vertices of equilateral triangles on a great circle or various inscribed platonic solids (cubes, octahedrons, icosahedrons and dodecahedrons). All of these solutions are non-axially symmetric.

Remark 2. Patterns minimizing $\sum_{j \neq k} G(x_j, x_k)$: tetrahedral configuration, octahedral configuration and regular icosahedral configuration. One may also construct solutions with the “twistered cuboid” configuration.

The proofs rely on a Lyapunov-type reduction. Let us consider the torus case.

Approximate solution. Let $-\Delta w_{\lambda,i} = e^{V_{\lambda,i}}\eta_{R_0,i} - m_i$ and $\int_T w_{\lambda,i} = 0$. Define $\overline{w_\lambda} = 2\ln\lambda + \ln 8 - 8\pi R - 8\pi G(\xi_1 - \xi_2)$. Let $w_\lambda = \sum_{i=1}^{2} w_{\lambda,i} + \overline{w_\lambda}$ be the approximation solution.

The linearized operator. Expand $S_\rho(w_\lambda + \phi)$ as $S_\rho(w_\lambda) + S'_\rho(w_\lambda)(\phi) + N(\phi)$. Let $\mathcal{L}(u) = \Delta u + \frac{\rho}{\int_T e^{w_\lambda}} e^{w_\lambda} u$. Then $S'_\rho(w_\lambda)(u) = \mathcal{L}\left(u - \frac{\int_T e^{w_\lambda} u}{\int_T e^{w_\lambda}}\right)$. Let $L(u) = \lambda^2 \mathcal{L}(u)$. Blow up the torus T to T_λ, then $L(u) \to$ the linearized operator associated to the Liouville equation.

The key step is to show the invertibility of L among the space

$$\mathcal{C}_* = \{u \in L^\infty | u(z) = u(-z) = u(\overline{z}), u \perp Z_{0,j}\chi_{R_1,j}\}.$$

Define $||u||_* = \sup_{z\in T_\lambda} \left(\sum_{j=1}^{2} (1 + |z - \xi'_j|)^{-3} + \lambda^2 \right)^{-1} |u(z)|.$

We adopt the same technique introduced by del Pino, Kowalczyk and Musso (2005) to prove an a prior estimate of the problem $L(\phi) = h$ and $\int_{T_\lambda} \chi_{R_1,j} Z_{i,j} \phi = 0$:
$||\phi||_\infty \leq C||h||_*$.
(1) Construct a positive supersolution V.

(2) Prove the claim: $||\phi||_\infty \leq C[||\phi||_i + ||h||_*]$ by suitable barrier functions.

(3) Use the claim to show that $\sup_{|z-\xi_j'|<R_2'} |\phi_n| \geq \kappa > 0$ for some index j. Then blow up the torus and shifted the center to ξ_j', then ϕ_n converges uniformly to a bounded solution ϕ of problem $\Delta u + \frac{8}{(1+|z|^2)^2} u = 0$. Contradicts the orthogonal conditions.
A priori estimate implies the invertibility of the linearized operator.

Uniqueness of bubbling solutions.

Lin and Yan proved the uniqueness of the bubbling solutions to CS equation that blow up at a non-degenerate critical point $\mathbf{q}$ of a potential function for both CS-type bubbling and MF-type bubbling while an extra condition on a quantity $D(q)$ is required in the MF case. They considered the normalized difference $\xi_n = (u_n^{(1)} - u_n^{(2)})/||u_n^{(1)} - u_n^{(2)}||_\infty$. By a suitable scaling, ξ_n converge to a solution ξ of the linearized problem associated to the Liouville equation in MF-type bubbling. Hence $\xi = \sum_{j=0}^{2} b_j Z_j$. They used various kinds of Pohozaev identities to show $b_j = 0$.

Bartolucci, Jevnikar, Lee and Yang showed the uniqueness of blow-up solutions to MFE by assuming that at least one of $l(\mathbf{q})$ and $D(\mathbf{q})$ is non-zero. The critical point here is also required to be non-degenerate.

We can show that the solutions constructed for MFE on torus are unique among the class of evenly symmetric solutions.