Prescribed Scalar Curvature in the AE Setting

David Maxwell
(joint with James Dilts, UCSD)

University of Alaska Fairbanks

BIRS
May 18, 2018

Overview

Problem to Solve

Given an asymptotically Euclidean manifold (M^n, g) with $n \geq 3$ and a desired scalar curvature R' (decaying suitably at infinity), is there a conformally related asymptotically Euclidean metric g' with scalar curvature $R[g'] = R'$?

Overview

Problem to Solve

Given an asymptotically Euclidean manifold (M^n, g) with $n \geq 3$ and a desired **non-positive** scalar curvature R' (decaying suitably at infinity), is there a conformally related, asymptotically Euclidean metric g' with scalar curvature $R[g'] = R'$?

Motivation

Initial data for the Cauchy problem in general relativity:

- Riemannian manifold (M^3, h)
- Second fundamental form K (i.e. a symmetric $(0,2)$-tensor)

$$R[h] - |K|^2 + \operatorname{tr} K^2 = 2\rho \qquad \text{[Hamiltonian constraint]}$$

$$-\operatorname{div}(K - \operatorname{tr} K\, g) = j \qquad \text{[momentum constraint]}$$

Motivation

Initial data for the Cauchy problem in general relativity:

- Riemannian manifold (M^3, h)
- Second fundamental form K (i.e. a symmetric $(0,2)$-tensor)

$$R[h] - |K|^2 + \operatorname{tr} K^2 = 2\rho \qquad \text{[Hamiltonian constraint]}$$

$$-\operatorname{div}(K - \operatorname{tr} K\, g) = j \qquad \text{[momentum constraint]}$$

- ρ: observed energy density
- j: observed momentum density

Motivation

Initial data for the Cauchy problem in general relativity:

- Riemannian manifold (M^3, h)
- Second fundamental form K (i.e. a symmetric $(0,2)$-tensor)

$$R[h] - |K|^2 + \operatorname{tr} K^2 = 2\rho \qquad \text{[Hamiltonian constraint]}$$
$$-\operatorname{div}(K - \operatorname{tr} K\, g) = j \qquad \text{[momentum constraint]}$$

- ρ: observed energy density
- j: observed momentum density
- Analogous to $\operatorname{div} E = \rho$ in electromagnetism.
- Underdetermined: 4 equations for 12 unknowns.

The Conformal Method

Seed data:

- g: a metric determining the conformal class of the solution metric.
- σ: a symmetric, trace-free, divergence-free $(0,2)$-tensor.
- τ: a mean curvature
- N: a positive function

Unknowns:

- ϕ: a conformal factor
- W: a vector field.

Seek solution:

- $h = \phi^4 g$
- $K = \phi^{-2}\left(\sigma + \frac{1}{N}\mathbb{D}\,W\right) + \frac{\tau}{3}h$

Notation: $\mathbb{D}$ = conformal Killing operator.

The Conformal Method

Substitute

- $h = \phi^4 g$
- $K = \phi^{-2}\left(\sigma + \frac{1}{N}\,\mathbb{D}\,W\right) + \frac{\tau}{3} h$

into the constraint equations to yield (in three dimensions, vacuum case):

$$-8\Delta\phi + R[g]\phi - \left|\sigma + \frac{1}{N}\,\mathbb{D}\,W\right|^2 \phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 0$$

$$\mathbb{D}^*\left[\frac{1}{N}\,\mathbb{D}\,W\right] + \frac{2}{3}\phi^6\,d\tau = 0.$$

The Conformal Method

If τ is constant

$$\frac{1}{2}\mathbb{D}^*\left[\frac{1}{N}\mathbb{D}\,W\right]+\frac{2}{3}\phi^6\,d\tau=0;$$

implies $\mathbb{D}\,W=0$. All that remains is the Lichnerowicz equation

$$-8\Delta_g\phi+R[g]\phi-|\sigma|_g^2\,\phi^{-7}+\frac{2}{3}\tau^2\phi^5=0.$$

If, in addition, $\sigma\equiv 0$

$$-8\Delta_g\phi+R[g]\phi+\frac{2}{3}\tau^2\phi^5=0.$$

So this is the Yamabe problem (in the easy case of non-positive scalar curvature).

Lichnerowicz Equation (Compact Setting)

When does

$$-8\Delta_g\phi + R[g]\phi - \eta^2\phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 0$$

admit a solution?

Scalar curvature of $h = \phi^4 g$ is

$$R[h] = \phi^{12}\eta^2 - \frac{2}{3}\tau^2.$$

Lichnerowicz Equation (Compact Setting)

When does

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} + \frac{2}{3}\tau^2 \phi^5 = 0$$

admit a solution?

Scalar curvature of $h = \phi^4 g$ is

$$R[h] = \phi^{12}\eta^2 - \frac{2}{3}\tau^2.$$

- $Y_g > 0$: solveable iff $\eta \not\equiv 0$
- $Y_g = 0$: solveable iff either
  - $\eta \not\equiv 0$ and $\tau \not\equiv 0$, or
  - $\eta \equiv 0$ and $\tau \equiv 0$

Lichnerowicz Equation (Compact Setting)

When does

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} + \frac{2}{3}\tau^2 \phi^5 = 0$$

admit a solution?

Scalar curvature of $h = \phi^4 g$ is

$$R[h] = \phi^{12}\eta^2 - \frac{2}{3}\tau^2.$$

- $Y_g > 0$: solveable iff $\eta \not\equiv 0$
- $Y_g = 0$: solveable iff either
  - $\eta \not\equiv 0$ and $\tau \not\equiv 0$, or
  - $\eta \equiv 0$ and $\tau \equiv 0$
- $Y_g < 0$: If τ is constant, solveable iff $\tau \neq 0$

Lichnerowicz Equation (Compact Setting)

When does

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} + \frac{2}{3}\tau^2 \phi^5 = 0$$

admit a solution?

Scalar curvature of $h = \phi^4 g$ is

$$R[h] = \phi^{12}\eta^2 - \frac{2}{3}\tau^2.$$

- $Y_g > 0$: solveable iff $\eta \not\equiv 0$
- $Y_g = 0$: solveable iff either
  - $\eta \not\equiv 0$ and $\tau \not\equiv 0$, or
  - $\eta \equiv 0$ and $\tau \equiv 0$
- $Y_g < 0$: [M '06] solvable iff g is conformally related to a metric h with $R[h] = -\tau^2$.

Lichnerowicz Equation (Compact Setting)

When does

$$-8\Delta_g\phi + R[g]\phi - \eta^2\phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 0$$

admit a solution?

Scalar curvature of $h = \phi^4 g$ is

$$R[h] = \phi^{12}\eta^2 - \frac{2}{3}\tau^2.$$

- $Y_g > 0$: solveable iff $\eta \not\equiv 0$
- $Y_g = 0$: solveable iff either
  - $\eta \not\equiv 0$ and $\tau \not\equiv 0$, or
  - $\eta \equiv 0$ and $\tau \equiv 0$
- $Y_g < 0$: [M '06] solvable iff g is conformally related to a metric h with $R[h] = -\tau^2$.
- The prescribed non-positive scalar curvature problem on a Yamabe negative compact manifold is solved: [Rauzy '95]

Lichnerowicz Equation (AE Setting)

Solve

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} + \frac{2}{3}\tau^2 \phi^5 = 0$$

with the additional condition that $\phi - 1$ suitably decays at infinity so $\phi^4 g$ is again AE.

- [Dilts and Isenberg '16] this problem is solvable iff g is conformally related to an AE metric h with $R[h] = -\tau^2$.

Lichnerowicz Equation (AE Setting)

Solve

$$-8\Delta_g\phi + R[g]\phi - \eta^2\phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 0$$

with the additional condition that $\phi - 1$ suitably decays at infinity so $\phi^4 g$ is again AE.

- [Dilts and Isenberg '16] this problem is solvable iff g is conformally related to an AE metric h with $R[h] = -\tau^2$.
- Is there a generalization of [Rauzy '95] to AE manifolds?

CMC Conformal Method (AE Version)

CMC + Asymptoticaly Euclidean implies $\tau \equiv 0$.

Solve

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} = 0$$

with suitable decay on $\phi - 1$.

Resulting scalar curvature:

$$R[\phi^4 g] = \phi^{12}\eta^2 \geq 0$$

So, morally, g must be something like Yamabe positive.

CMC Conformal Method (AE Version)

CMC + Asymptoticaly Euclidean implies $\tau \equiv 0$.

Solve

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} = 0$$

with suitable decay on $\phi - 1$.

Resulting scalar curvature:

$$R[\phi^4 g] = \phi^{12} \eta^2 \geq 0$$

So, morally, g must be something like Yamabe positive.

But Yamabe positive might not mean what you think it means...

Lowering Scalar Curvature Is Easy

g asymptotically Euclidean, $R[g] \geq R'$, R' with suitable decay.
Writing $\phi = 1 + u$ we wish to solve

$$-8\Delta u + Ru = R'(1+u)^{-7} - R$$

where u decays at infinity.

Lowering Scalar Curvature Is Easy

g asymptotically Euclidean, $R[g] \geq R'$, R' with suitable decay. Writing $\phi = 1 + u$ we wish to solve

$$-8\Delta u + Ru = R'(1+u)^{-7} - R$$

where u decays at infinity.

Assuming $R' \leq 0$:

1. Solve $-a\Delta v + (R - R')v = R' - R$, v decaying at infinity.
2. A homotopy & maximum principle argument shows $0 < 1 + v \leq 1$.
3. $-a\Delta v + Rv = R'(1+v) - R \leq R'(1+v)^{-7} - R$
4. So $u = 0$ is a supersolution, and $u = v$ is a subsolution yielding a solution $v \leq u \leq 0$.

Lowering Scalar Curvature Is Easy

g asymptotically Euclidean, $R[g] \geq R'$, R' with suitable decay. Writing $\phi = 1 + u$ we wish to solve

$$-8\Delta u + Ru = R'(1+u)^{-7} - R$$

where u decays at infinity.

Assuming $R' \leq 0$:

1. Solve $-a\Delta v + (R - R')v = R' - R$, v decaying at infinity.
2. A homotopy & maximum principle argument shows $0 < 1 + v \leq 1$.
3. $-a\Delta v + Rv = R'(1+v) - R \leq R'(1+v)^{-7} - R$
4. So $u = 0$ is a supersolution, and $u = v$ is a subsolution yielding a solution $v \leq u \leq 0$.

General case via barriers: $\max(R', 0) \leq R' \leq R$.

CMC Conformal Method (AE Version)

Claim: The CMC Lichnerowicz equation

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} = 0$$

is solvable if and only if g is conformally equivalent to a scalar flat AE metric.

CMC Conformal Method (AE Version)

Claim: The CMC Lichnerowicz equation

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} = 0$$

is solvable if and only if g is conformally equivalent to a scalar flat AE metric.

If solvable

$$R[\phi^4 g] = \phi^{12}\eta^2 \geq 0$$

can be lowered to zero.

CMC Conformal Method (AE Version)

Claim: The CMC Lichnerowicz equation

$$-8\Delta_g \phi + R[g]\phi - \eta^2 \phi^{-7} = 0$$

is solvable if and only if g is conformally equivalent to a scalar flat AE metric.

If solvable

$$R[\phi^4 g] = \phi^{12}\eta^2 \geq 0$$

can be lowered to zero.

Conversely, to solve

$$-8\Delta_g u = \hat{\eta}^2 (1+u)^{-7}$$

observe zero is a subsolution and

$$-8\Delta_g v = \hat{\eta}^2$$

yields $v \geq 0$, a supersolution.

Yamabe Invariant

When is an AE metric conformally equivalent to an AE scalar flat metric?

Yamabe Invariant

When is an AE metric conformally equivalent to an AE scalar flat metric?

- [Cantor-Brill '82]: If and only if

$$\int a|\nabla u|^2 + Ru^2 \, dV_g > 0 \text{ if } u \in C_c^\infty(M), u \not\equiv 0$$

$$a = 4(n-1)/(n-2)$$

Yamabe Invariant

When is an AE metric conformally equivalent to an AE scalar flat metric?

- [Cantor-Brill '82]: If and only if

$$\int a|\nabla u|^2 + Ru^2 \, dV_g > 0 \text{ if } u \in C_c^\infty(M), u \not\equiv 0$$

$$a = 4(n-1)/(n-2)$$

- [M '06]: Define

$$Y_g(M) = \inf\{Q_g(u) : u \in C_c^\infty(M), u \not\equiv 0\}$$

$$Q_g(u) = \frac{\int a|\nabla u|^2 + R[g]u^2}{\|u\|_{2^*}^2}; \qquad 2^* = \frac{2n}{n-2}$$

$R' \equiv 0$ is possible if and only if $Y_g(M) > 0$

Yamabe Invariant

When is an AE metric conformally equivalent to an AE scalar flat metric?

- [Cantor-Brill '82]: If and only if

$$\int a|\nabla u|^2 + Ru^2 \, dV_g > 0 \text{ if } u \in C_c^\infty(M), u \not\equiv 0$$

$$a = 4(n-1)/(n-2)$$

- [M '06]: Define

$$Y_g(M) = \inf\{Q_g(u) : u \in C_c^\infty(M), u \not\equiv 0\}$$

$$Q_g(u) = \frac{\int a|\nabla u|^2 + R[g]u^2}{\|u\|_{2^*}^2}; \qquad 2^* = \frac{2n}{n-2}$$

$R' \equiv 0$ is possible if and only if $Y_g(M) > 0$

- [Friedrich '11]: Counterexample showing [Cantor-Brill '82] $\not\Longrightarrow$ [M '06]

Problem to Solve

Given an asymptotically Euclidean manifold (M^n, g) with $n \geq 3$ and a desired **non-positive** scalar curvature R' (decaying suitably at infinity), is there a conformally related, asymptotically Euclidean metric g' with scalar curvature $R[g'] = R'$?

- If $Y_g(M) > 0$ then we can transform to scalar flat and then to $R' < 0$.
- If $Y_g(M) \leq 0$, rasing $R' \leq 0$ up to zero is the challenge. E.g., how big can the zero set be?
- Can we characterize AE metrics with $Y_g \leq 0$?

Rauzy's Condition

Start with (M^n, g), compact, $g \in Y_-$. We wish to conformally transform to h, $R[h] = R' \leq 0$, R' smooth.

1. Conformally transform to $\hat{g}$ with $R[\hat{g}]$ a negative constant.
2. Compute

$$\mu_{R'} = \inf\left\{ \frac{\|\nabla u\|_{2,\hat{g}}^2}{\|u\|_{2,\hat{g}}^2} : u \in W^{1,2}, u \geq 0, \int R'u = 0 \right\}.$$

3. The desired conformal transformation is possible if and only if

$$a\mu_{R'} \geq -R_{\hat{g}}$$

Rauzy Simplified

Start with (M^n, g), compact, $g \in Y_-$. We wish to conformally transform to h, $R[h] = R' \leq 0$, R' smooth. Let $V = \{R' = 0\}$.

1. Conformally transform to $\hat{g}$ with $R[\hat{g}]$ a negative constant.
2. Compute

$$\lambda_{\hat{g}}(V) = \inf \left\{ \frac{\int a|\nabla u|^2_{\hat{g}} + R[\hat{g}]u^2 \, dV_{\hat{g}}}{\|u\|^2_{2,\hat{g}}} : u \in W^{1,2}, u \not\equiv 0, u|_{V^c} = 0 \right\}$$

3. The conformal transformation is possible iff $\lambda_{\hat{g}}(V) > 0$

Rauzy Simplified

Start with (M^n, g), compact, $g \in Y_-$. We wish to conformally transform to h, $R[h] = R' \le 0$, R' smooth. Let $V = \{R' = 0\}$.

1. Conformally transform to $\hat{g}$ with $R[\hat{g}]$ a negative constant.
2. Compute

$$\lambda_{\hat{g}}(V) = \inf\left\{\frac{\int a|\nabla u|^2_{\hat{g}} + R[\hat{g}]u^2 \, dV_{\hat{g}}}{\|u\|^2_{2,\hat{g}}} : u \in W^{1,2}, u \not\equiv 0, u|_{V^c} = 0\right\}$$

3. The conformal transformation is possible iff $\lambda_{\hat{g}}(V) > 0$

Morally, the first Dirichlet eigenvalue of $-a\Delta + R[\hat{g}]$ on V needs to be positive.

Weighted Spaces

Norm in $W^{k,p}_\delta(M)$:

$$\|u\|_{W^{k,p}_\delta(M)} := \sum_{j=0}^{k} \left\| \rho^{-\delta-\frac{n}{p}+j} \, |\nabla^j u| \right\|_{L^p(M)} < \infty$$

- $u \in W^{k,p}_\delta$ 'implies' $O(\rho^\delta)$ growth at infinity, ∇u is $O(\rho^{\delta-1})$, etc.
- $\delta < 0$ implies decay.
- Compact Sobolev embedding requires extra decay
- $\delta^* = \dfrac{2-n}{2}$ is special.
    - $W^{1,2}_{\delta^*}$ norm is equivalent to $\|\nabla u\|_2$
    - $L^{2^*}_{\delta^*}$ norm is exactly the L^{2^*} norm.
    - $\Delta : W^{2,p}_\delta(\mathbb{R}^n) \to W^{0,p}_{\delta-2}(\mathbb{R}^n)$ is an isomorphism if $\delta \in (2\delta^*, 0)$.

Weighted Spaces

Norm in $W^{k,p}_\delta(M)$:

$$\|u\|_{W^{k,p}_\delta(M)} := \sum_{j=0}^{k} \left\| \rho^{-\delta-\frac{n}{p}+j} \, |\nabla^j u| \right\|_{L^p(M)} < \infty$$

- $u \in W^{k,p}_\delta$ 'implies' $O(\rho^\delta)$ growth at infinity, ∇u is $O(\rho^{\delta-1})$, etc.
- $\delta < 0$ implies decay.
- Compact Sobolev embedding requires extra decay
- $\delta^* = \dfrac{2-n}{2}$ is special.
  - $W^{1,2}_{\delta^*}$ norm is equivalent to $\|\nabla u\|_2$
  - $L^{2^*}_{\delta^*}$ norm is exactly the L^{2^*} norm.
  - $\Delta : W^{2,p}_\delta(\mathbb{R}^n) \to W^{0,p}_{\delta-2}(\mathbb{R}^n)$ is an isomorphism if $\delta \in (2\delta^*, 0)$.

Metrics of class $W^{2,p}_\tau$ $(\tau < 0,\ p > n/2)$: $g - g_{\text{Euc}} \in W^{2,p}_\tau$.

Yamabe Invariant of a Measurable Set

Suppose:

(M, g) asymptotically Euclidean of class $W^{2,p}_{\tau}$

$V \subseteq M$ measurable

Define:

$$Y_g(V) = \inf \left\{ Q_g(u) : u \in W^{1,2}_{\delta^*}, u \not\equiv 0, u|_{V^c} = 0 \right\}$$

$$Q_g(u) = \frac{\int a|\nabla u|^2 + R[g]u^2}{\|u\|^2_{2^*}}$$

Yamabe Invariant of a Measurable Set

Suppose:

(M, g) asymptotically Euclidean of class $W^{2,p}_\tau$

$V \subseteq M$ measurable

Define:

$$Y_g(V) = \inf\left\{Q_g(u) : u \in W^{1,2}_{\delta^*}, u \not\equiv 0, u|_{V^c} = 0\right\}$$
$$Q_g(u) = \frac{\int a|\nabla u|^2 + R[g]u^2}{\|u\|^2_{2^*}}$$

Properties:

- $V_1 \subseteq V_2 \implies Y_g(V_1) \geq Y_g(V_2)$
- $Y_g(V) \geq Y_g(M) > -\infty$
- If $g' = \Phi^2 g$ with $\Phi - 1 \in W^{2,p}_\tau$ then $Y_g(V) = Y_{g'}(V)$

Yamabe Invariant of a Measurable Set

Suppose:

(M, g) asymptotically Euclidean of class $W^{2,p}_{\tau}$

$V \subseteq M$ measurable

Define:

$$Y_g(V) = \inf\left\{Q_g(u) : u \in W^{1,2}_{\delta^*}, u \not\equiv 0, u|_{V^c} = 0\right\}$$

$$Q_g(u) = \frac{\int a|\nabla u|^2 + R[g]u^2}{\|u\|^2_{2^*}}$$

Properties:

- $V_1 \subseteq V_2 \implies Y_g(V_1) \geq Y_g(V_2)$
- $Y_g(V) \geq Y_g(M) > -\infty$
- If $g' = \Phi^2 g$ with $\Phi - 1 \in W^{2,p}_{\tau}$ then $Y_g(V) = Y_{g'}(V)$
- Otherwise recalcitrant.

Weighted First Eigenvalues

Suppose:

(M, g) asymptotically Euclidean of class $W^{2,p}_{\tau}$

$V \subseteq M$ measurable

For $\delta > \delta^*$, define

$$\lambda_{g,\delta}(V) = \inf \left\{ J_{g,\delta}(u) : u \in W^{1,2}_{\delta^*}, u \not\equiv 0, u|_{V^c} = 0 \right\}$$

$$J_{g,\delta}(u) = \frac{\int a|\nabla u|^2 + R[g]u^2 dV_g}{\|u\|^2_{2,\delta}}.$$

Weighted First Eigenvalues

Suppose:

(M, g) asymptotically Euclidean of class $W^{2,p}_\tau$

$V \subseteq M$ measurable

For $\delta > \delta^*$, define

$$\lambda_{g,\delta}(V) = \inf\left\{J_{g,\delta}(u) : u \in W^{1,2}_{\delta^*}, u \not\equiv 0, u|_{V^c} = 0\right\}$$
$$J_{g,\delta}(u) = \frac{\int a|\nabla u|^2 + R[g]u^2 dV_g}{\|u\|^2_{2,\delta}}.$$

- Value is not especially meaningful. Depends on choice of weight function, e.g.

Weighted First Eigenvalues

Suppose:

(M, g) asymptotically Euclidean of class $W^{2,p}_{\tau}$

$V \subseteq M$ measurable

For $\delta > \delta^*$, define

$$\lambda_{g,\delta}(V) = \inf \left\{ J_{g,\delta}(u) : u \in W^{1,2}_{\delta^*}, u \not\equiv 0, u|_{V^c} = 0 \right\}$$
$$J_{g,\delta}(u) = \frac{\int a|\nabla u|^2 + R[g]u^2 dV_g}{\|u\|^2_{2,\delta}}.$$

- Value is not especially meaningful. Depends on choice of weight function, e.g.
- But $Y_g(V)$ has the same sign as $\lambda_{g,\delta}(V)$ for all $\delta > \delta^*$.

Properites of Weighted First Eigenvalues

Monotonicity

$V_1 \subseteq V_2 \implies \lambda_{g,\delta}(V_1) \geq \lambda_{g,\delta}(V_2)$

(Limited) Strict Monotonicity

If Ω is connected and open and if $E \subseteq \Omega$ has positive measure,
$\lambda_{g,\delta}(\Omega \setminus E) > \lambda_{g,\delta}(\Omega)$

Continuity from Above

$V_k \searrow V \implies \lambda_{g,\delta}(V_k) \to \lambda_{g,\delta}(V)$

Limited Continuity from Below

$V_k \nearrow V$ "nicely" $\implies \lambda_{g,\delta}(V_k) \to \lambda_{g,\delta}(V)$

Minimizers Exist They morally solve
$-a\Delta u + Ru = \lambda_{g,\delta}(V)\rho^{-2\delta-n}u$

Small Sets are Yamabe Positive

$$\int_V \rho^{-n-\epsilon} dV_g < C_\epsilon \implies Y_g(V) > 0.$$

Main Result

Theorem ([Dilts, M '16])

Let g be an AE metric of class $W^{2,p}_\tau$, with $2 - n < \tau < 0$ and $p > n/2$. Suppose $R' \in L^p_{\tau-2}$ with $R' \leq 0$. The following are equivalent.

1. *There is a positive function ϕ with $\phi - 1 \in W^{2,p}_\tau$ such that the scalar curvature of $\phi^{2^*-2}g$ is R'.*
2. *The set $\{R' = 0\}$ is Yamabe positive with respect to g.*

Easy Direction

The set $V = \{R = 0\}$ is Yamabe positive:

1. Pick $\delta > \delta^*$. Let $u \in W^{1,2}_{\delta^*}$, $u \not\equiv 0$ be a minimizer of $J_{2,\delta}$ among the functions that vanish on V^c.
2. Since $Ru^2 = 0$,

$$\lambda_{g,\delta}(V) = a \frac{\int |\nabla u|^2 \, dV_g}{\|u\|_2^2} \geq 0.$$

3. If λ_g were zero, u would be constant, and hence zero a.e. But it isn't.

Prelude to Hard Direction

1. Opt to solve first for an R' that is bounded below and equal to zero in a neighbourhood of infinity. But only adjust on a small enough neighborhood of infinity such that the zero set of R' is still Yamabe positive. (Continuity from above for λ_g).
2. Make an initial conformal change to a scalar curvature that equals zero in a neighbourhood of infinity. (Ad hoc construction. Uses small neighbourhoods of infinity are Yamabe positive).

Outline of Proof

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2 \right] - \frac{q}{2} \int R'(1+u)^q$$

1. Coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $\|u\|_{2,\delta} > K$ implies $F_q(u) \ge B$.
2. Existence of subcritical minimizers u_q, uniformly bounded in $W^{1,2}_{\delta^*}$. Uses $R' \le 0$ and uniform L^2_δ bounds.
3. On compact sets u_q is uniformly bounded in L^M for some $M > 2^*$.
4. Bootstrap to uniform bounds in $W^{2,p}_\sigma$ for each $\sigma \in (2-n, 0)$.
5. Minimizer subsequence converges strongly in $W^{1,2}_{\delta^*}$ and uniformly on compact sets to a $W^{2,p}_\sigma$ solution of

$$-a\Delta u + R(1+u) = R'(1+u)^{2^*-1}.$$

Outline of Proof

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2\right] - \frac{q}{2}\int R'(1+u)^q$$

1. Coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $||u||_{2,\delta} > K$ implies $F_q(u) \ge B$.
2. Existence of subcritical minimizers u_q, uniformly bounded in $W^{1,2}_{\delta^*}$. Uses $R' \le 0$ and uniform L^2_δ bounds.
3. On compact sets u_q is uniformly bounded in L^M for some $M > 2^*$. Oops Rauzy.
4. Bootstrap to uniform bounds in $W^{2,p}_\sigma$ for each $\sigma \in (2-n, 0)$.
5. Minimizer subsequence converges strongly in $W^{1,2}_{\delta^*}$ and uniformly on compact sets to a $W^{2,p}_\sigma$ solution of

$$-a\Delta u + R(1+u) = R'(1+u)^{2^*-1}.$$

Outline of Proof

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2\right] - \frac{q}{2} \int R'(1+u)^q$$

1. Coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $\|u\|_{2,\delta} > K$ implies $F_q(u) \ge B$.
2. Existence of subcritical minimizers u_q, uniformly bounded in $W^{1,2}_{\delta^*}$. Uses $R' \le 0$ and uniform L^2_δ bounds.
3. On compact sets u_q is uniformly bounded in L^M for some $M > 2^*$. Oops Rauzy. Oops us.
4. Bootstrap to uniform bounds in $W^{2,p}_\sigma$ for each $\sigma \in (2-n, 0)$.
5. Minimizer subsequence converges strongly in $W^{1,2}_{\delta^*}$ and uniformly on compact sets to a $W^{2,p}_\sigma$ solution of

$$-a\Delta u + R(1+u) = R'(1+u)^{2^*-1}.$$

Outline of Proof

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2\right] - \frac{q}{2}\int R'(1+u)^q$$

1. Coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $\|u\|_{2,\delta} > K$ implies $F_q(u) \ge B$.
2. Existence of subcritical minimizers u_q, uniformly bounded in $W^{1,2}_{\delta^*}$. Uses $R' \le 0$ and uniform L^2_δ bounds.
3. On compact sets u_q is uniformly bounded in L^M for some $M > 2^*$. Oops Rauzy. Oops us. Thanks Rafe.
4. Bootstrap to uniform bounds in $W^{2,p}_\sigma$ for each $\sigma \in (2-n, 0)$.
5. Minimizer subsequence converges strongly in $W^{1,2}_{\delta^*}$ and uniformly on compact sets to a $W^{2,p}_\sigma$ solution of

$$-a\Delta u + R(1+u) = R'(1+u)^{2^*-1}.$$

Role of the Yamabe Positive Assumption

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2 \right] - \frac{q}{2} \int R'(1+u)^q$$

Want coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $\|u\|_{2,\delta} > K$ implies $F_q(u) \ge B$.

Role of the Yamabe Positive Assumption

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2 \right] - \frac{q}{2} \int R'(1+u)^q$$

Want coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $||u||_{2,\delta} > K$ implies $F_q(u) \ge B$.

$$A_\eta = \left\{ u \in W^{1,2}_{\delta^*} : \int |R'| u^2 \, dV_g \le \eta ||u||^2_{2,\delta} \int |R'| \, dV_g \right\}$$

Role of the Yamabe Positive Assumption

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2\right] - \frac{q}{2}\int R'(1+u)^q$$

Want coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $||u||_{2,\delta} > K$ implies $F_q(u) \ge B$.

$$A_\eta = \left\{u \in W^{1,2}_{\delta^*} : \int |R'|u^2 \, dV_g \le \eta ||u||^2_{2,\delta} \int |R'| \, dV_g\right\}$$

Given $\mathcal{L} \in (0, \lambda_{g,\delta}(V))$, can find η_0 so $u \in A_{\eta_0}$ implies

$$\int a|\nabla u|^2 + Ru^2 \, dV_g \ge \mathcal{L}||u||^2_{2,\delta}$$

For $u \in A_{\eta_0}$, $F_q(u)$ grows faster than $(\mathcal{L}/2)||u||^2_{2,\delta}$.

Role of the Yamabe Positive Assumption

Subcritical functionals ($2 \le q < 2^*$):

$$F_q(u) = \int \left[a|\nabla u|^2 + R(1+u)^2\right] - \frac{q}{2}\int R'(1+u)^q$$

Want coercivity: Given $B > 0$ and $\delta > \delta^*$, there is a bound K, independent of q, such that $\|u\|_{2,\delta} > K$ implies $F_q(u) \ge B$.

$$A_\eta = \left\{u \in W^{1,2}_{\delta^*} : \int |R'|u^2\, dV_g \le \eta \|u\|^2_{2,\delta} \int |R'|\, dV_g\right\}$$

Given $\mathcal{L} \in (0, \lambda_{g,\delta}(V))$, can find η_0 so $u \in A_{\eta_0}$ implies

$$\int a|\nabla u|^2 + Ru^2\, dV_g \ge \mathcal{L}\|u\|^2_{2,\delta}$$

For $u \in A_{\eta_0}$, $F_q(u)$ grows faster than $(\mathcal{L}/2)\|u\|^2_{2,\delta}$.

For $u \notin A_{\eta_0}$, $F_q(u)$ grows faster than $C(\eta_0, \|R'\|_1)\|u\|^q_{2,\delta}$.

Characterization of Yamabe Classes

$\mathcal{R}_{\le 0} = \{R \in L^p_{\tau-2} : R \le 0\}$

- $Y_g(M) > 0$ if and only if for each $R \in \mathcal{R}_{\le 0}$ we can conformally change to an AE metric with scalar curvature R.
- $Y_g(M) = 0$ if and only if for each $R \in \mathcal{R}_{\le 0} \setminus 0$ we can conformally change to an AE metric with scalar curvature R, and $R \equiv 0$ is unattainable. (Limited strict monotonicity of $\lambda_g(V)$).
- $Y_g(M) < 0$ if and only if there is an $R \in \mathcal{R}_{\le 0}$, $R \not\equiv 0$, that is unattainable via a conformal transformation. (Limited continuity from below for $\lambda_g(V)$)

Characterization of AE Yamabe Classes

The compactification of a smooth AE metric need not be smooth. But:

Theorem

Suppose $p > n/2$ and g is an AE metric of class $W^{2,p}_\tau$ where

$$\tau = \frac{n}{p} - 2.$$

Then there is a smooth conformal factor ϕ, decaying like r^{2-n} at infinity, such that $\bar{g} = \phi^{2^*-2} g$ is a $W^{2,p}$ metric on $\bar{M}$.

Conversely, a $W^{2,p}$ metric with $p > n/2$ on $\bar{M}$ admits a conformal change to an AE metric on M of class $W^{2,p}_\tau$ with

$$\tau = \frac{n}{p} - 2.$$

Characterization of AE Yamabe Classes

Proposition

If (g, M) and $(\bar{g}, \bar{M})$ are related as in the previous theorem, the Yamabe invariants are the same.

1. Find an approximate minimizer u for M: $Q_g(u) < Y_g(M) + \epsilon$.
2. Find a compactly supported approximate, $\hat{u}$, so $Q_g(\hat{u}) < Y_g(M) + 2\epsilon$.
3. Since $Q_{\bar{g}}(\bar{\phi}\hat{u}) = Q_g(\hat{u})$, $Y_{\bar{g}} \leq Y_g(M) + 2\epsilon$.
4. So $Y_{\bar{g}} \leq Y_g(M)$.
5. Now reverse.

Characterization of AE Yamabe Classes

Proposition (Dilts-M '16)

An asymptotically Euclidean manifold (M, g) *of class* $W^{2,p}_{\tau}$ *with* $p > n/2$ *and* $\tau < 0$ *is Yamabe positive/negative/null if and only if it admits a* $W^{2,q}$ *conformal compactification of the same class for some* $q > n/2$.