

SCENARIO OPTIMIZATION: THE PERFORMANCE-RISK TRADEOFF

Speaker: **Simone Garatti**

(Politecnico di Milano, Italy – email: simone.garatti@polimi.it)

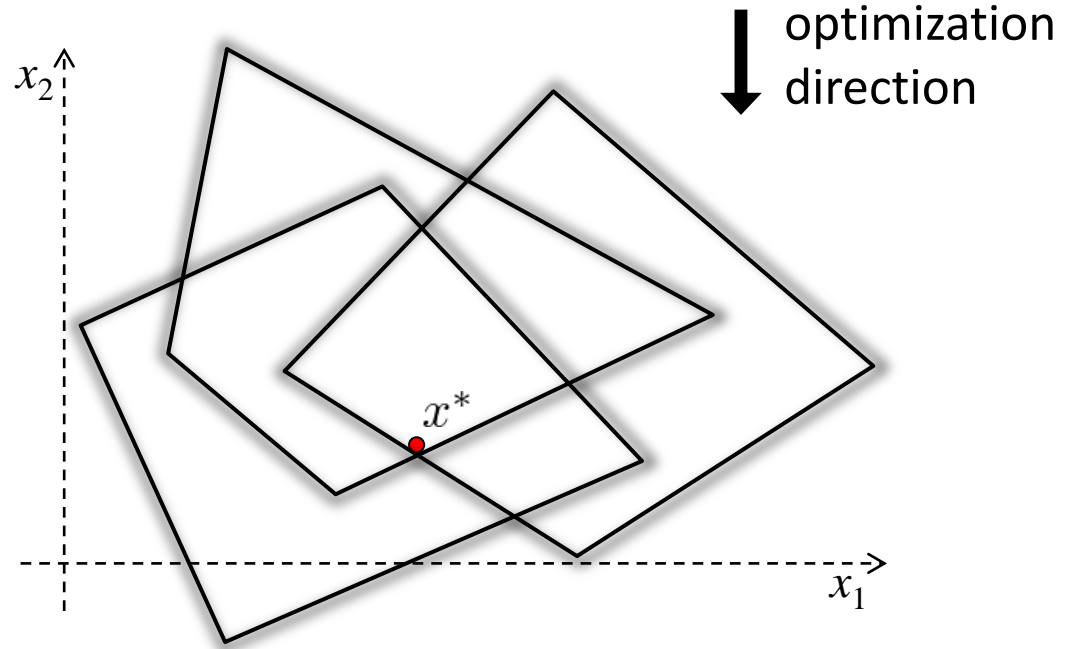
in collaboration with: **Marco C. Campi**

(University of Brescia, Italy – email: marco.campi@unibs.it)

Scenario optimization

$$\begin{aligned} \min_x \quad & c(x) \\ \text{s.t.} \quad & x \in \mathcal{X}_{\delta_i} \\ & i = 1, \dots, N \end{aligned}$$

solution: x^*



$$\text{scenarios (data)} \left\{ \begin{array}{l} \delta_1 \rightarrow \mathcal{X}_{\delta_1} \\ \delta_2 \rightarrow \mathcal{X}_{\delta_2} \\ \vdots \\ \delta_N \rightarrow \mathcal{X}_{\delta_N} \end{array} \right.$$

Problem ingredients

Cost function: $c(x)$

Family of constraint sets: \mathcal{X}_δ δ stochastic parameter
 $(\Delta, \mathcal{F}, \mathbb{P})$ **unknown**

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performance $c(x)$

vs.

risk $V(x) = \mathbb{P} \{ \delta \in \Delta : x \notin \mathcal{X}_\delta \}$

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Cost function: $c(x)$

Family of constraint sets: \mathcal{X}_δ δ stochastic parameter
 $(\Delta, \mathcal{F}, \mathbb{P})$ **unknown**

For any given x in the optimization domain

performance $c(x)$ \longrightarrow **accessible**

vs.

risk $V(x) = \mathbb{P}\{\delta \in \Delta : x \notin \mathcal{X}_\delta\}$ \longrightarrow **not accessible**

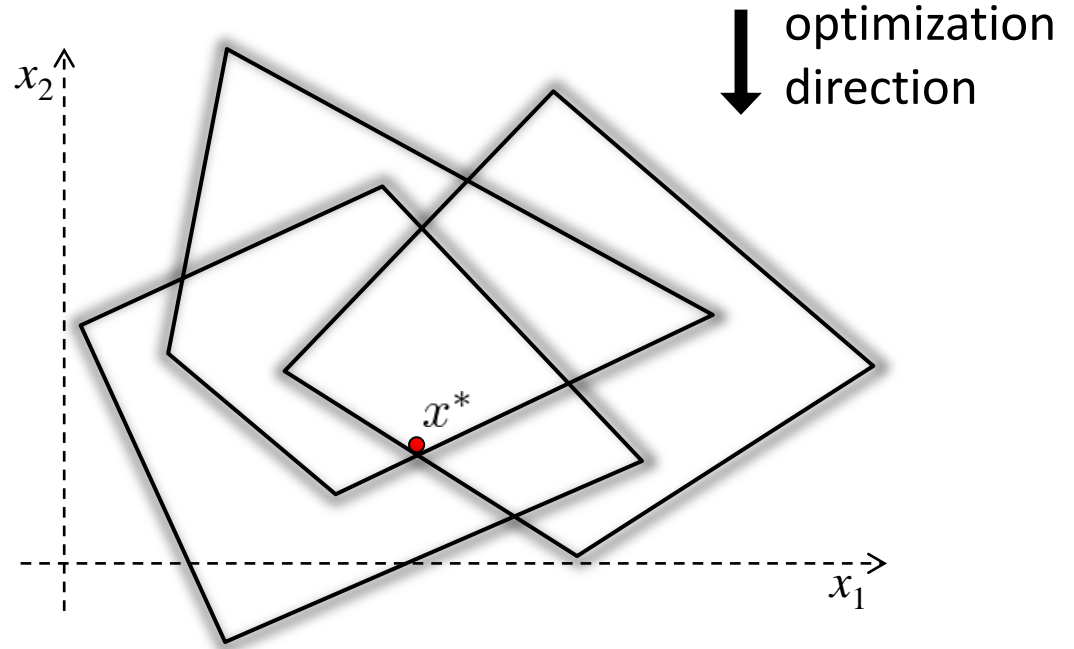
Scenario optimization theory

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solution: x^*

performance $c(x^*)$

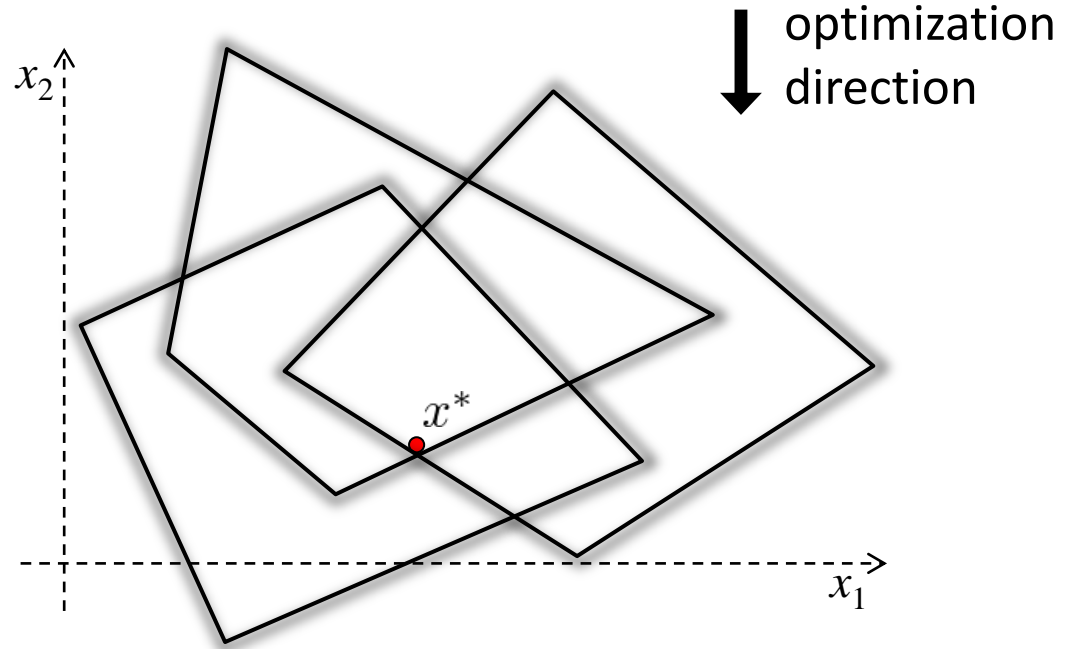
risk $V(x^*)$



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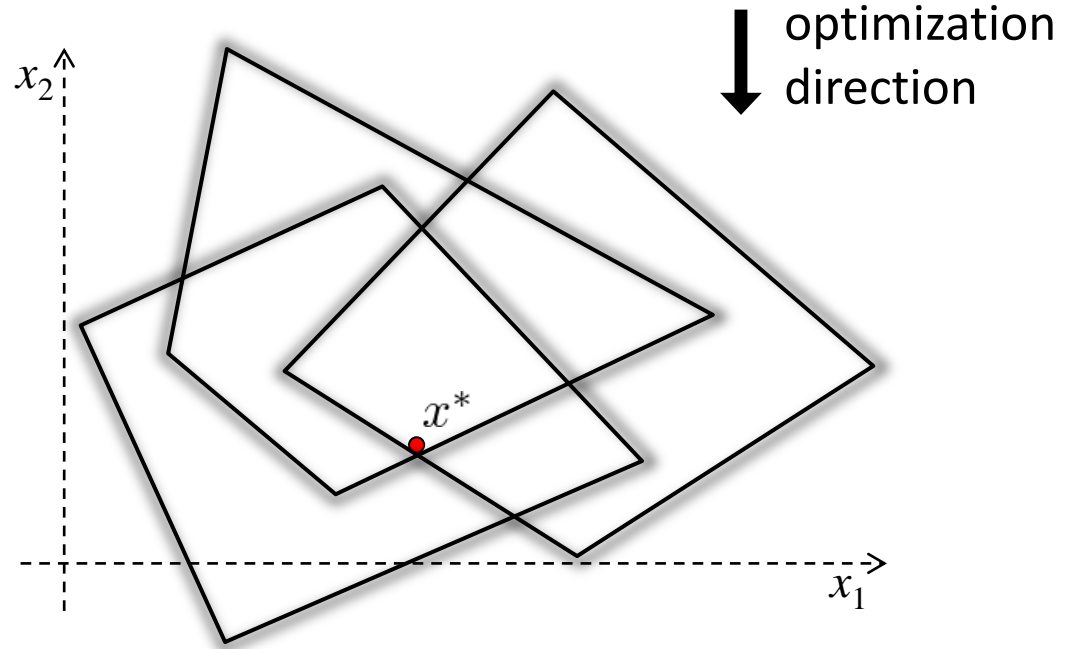
performance $c(x^*)$ known

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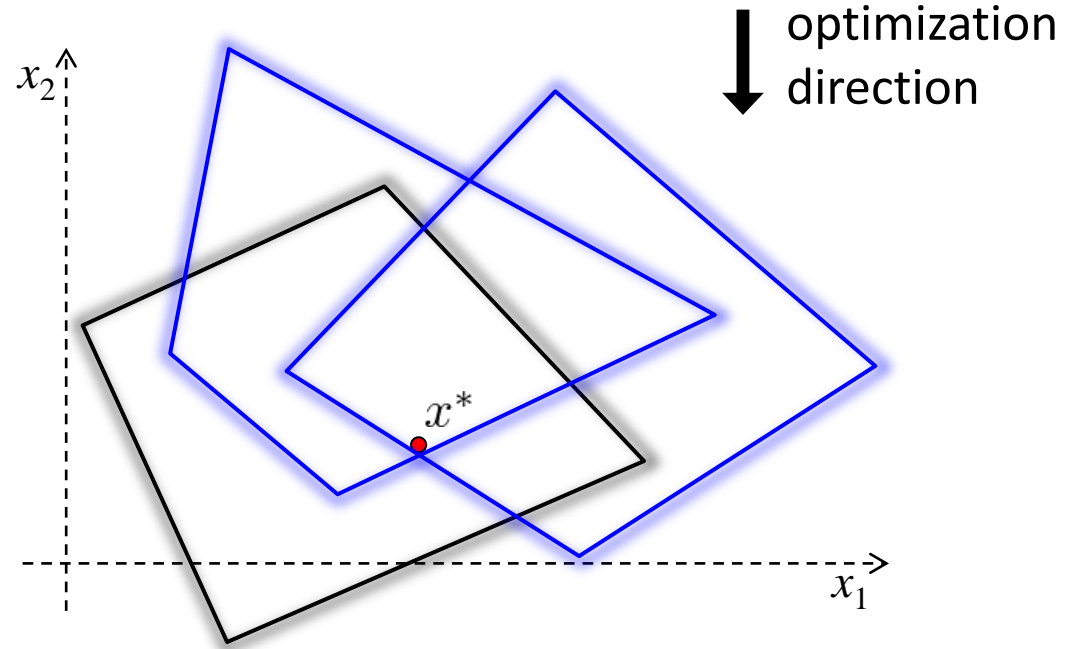
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risk $V(x^*)$ can be tightly estimated from s^*

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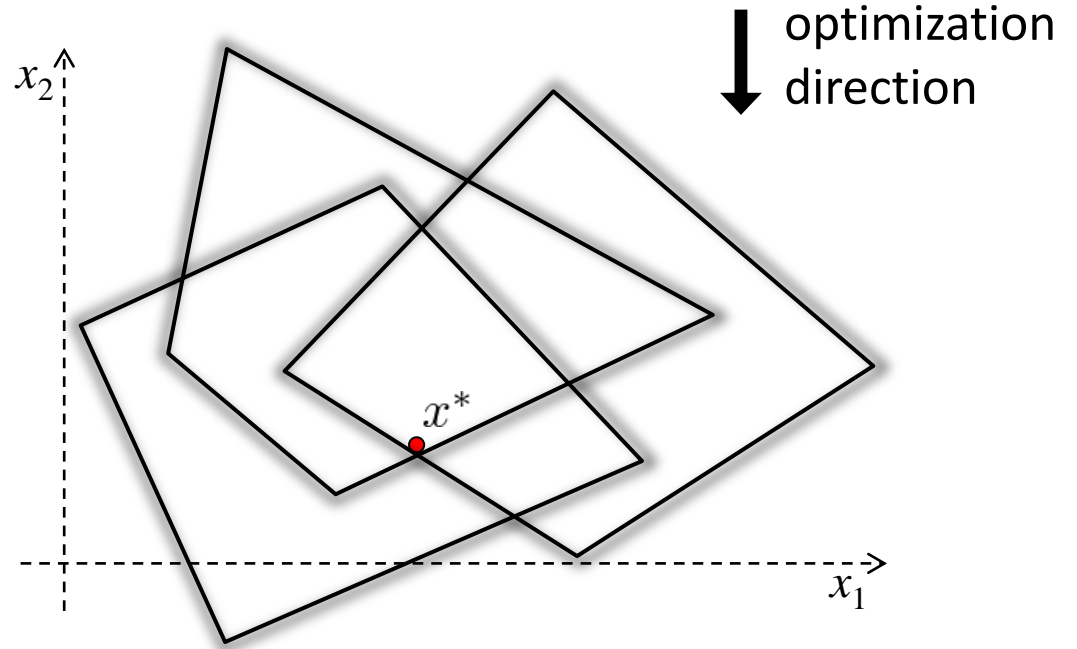
complexity

risk $V(x^*)$ can be tightly estimated from s^*

Quality certification

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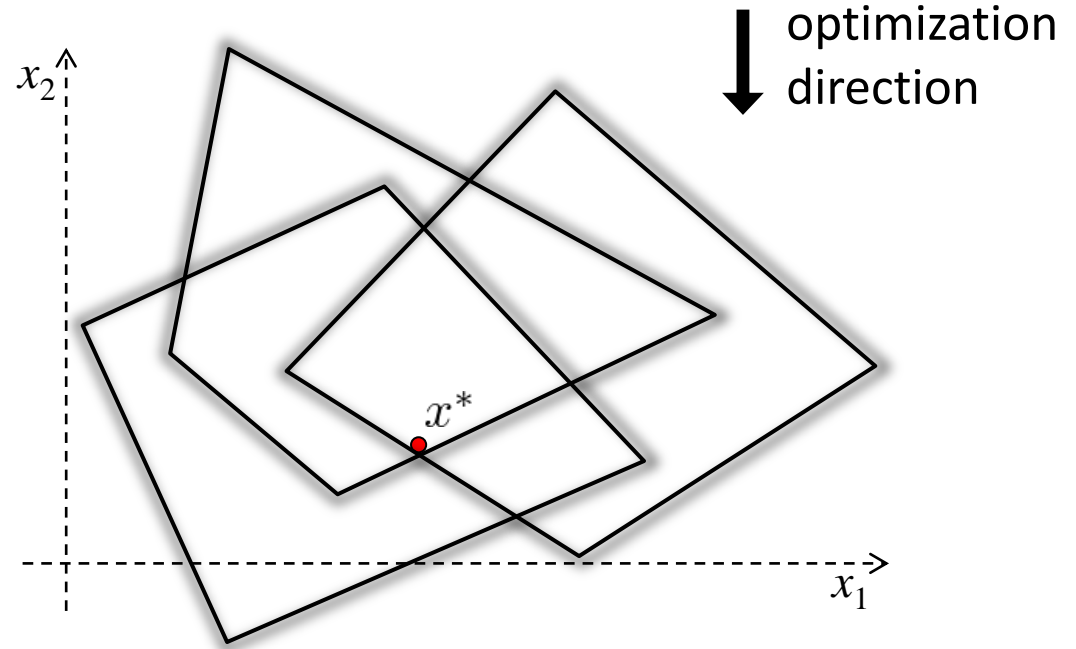
**solution quality
assessed via**

$$c(x^*), \hat{V}(s^*)$$

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solution: x^*





**solution quality
assessed via**

$$c(x^*), \hat{V}(s^*)$$

**solution quality may be not
satisfactory:**

$c(x^*)$ **is too big**

Take-home message

- ❑ It is possible to introduce many alternative **scenario-based** schemes  several “solutions” $x_1^*, x_2^*, x_3^*, \dots$ each attaining a different performance
- ❑ Scenario theory as a tool that allows one to **evaluate the risk of each solution**
- ❑ Quantitative comparison in terms of performance (known) and risk (**estimated**)  select the “best” solution for the problem at hand

Mathematical tool: scenario decision-making (1/2)

uncertainty domain $(\Delta, \mathcal{F}, \mathbb{P})$

 **scenarios** $(\delta_1, \delta_2, \dots, \delta_N)$

decision space \mathcal{Z}

scenario-based decision $M_N : (\delta_1, \delta_2, \dots, \delta_N) \rightarrow z^*$

support set: $(\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_k})$ such that

i. $M_k(\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_k}) = M_N(\delta_1, \delta_2, \dots, \delta_N)$ **ii.** smallest

s^* = size of the support set = **complexity**

Mathematical tool: scenario decision-making (2/2)

$$\delta \rightarrow \mathcal{Z}_\delta \subseteq \mathcal{Z} \quad \longrightarrow \quad V(z^*) = \mathbb{P}\{\delta \in \Delta : z^* \notin \mathcal{Z}_\delta\}$$

risk

scenario theory:

$V(z^*)$ can be estimated from s^*

$V(z^*) \leq \widehat{V}(s^*)$ with confidence $1 - \beta$ where
 $\widehat{V}(k)$, $k = 0, 1, \dots$, is the solution to equation

$$\frac{\beta}{N+1} \sum_{m=k}^N \binom{m}{k} (1-v)^{m-k} - \binom{N}{k} (1-v)^{N-k} = 0$$

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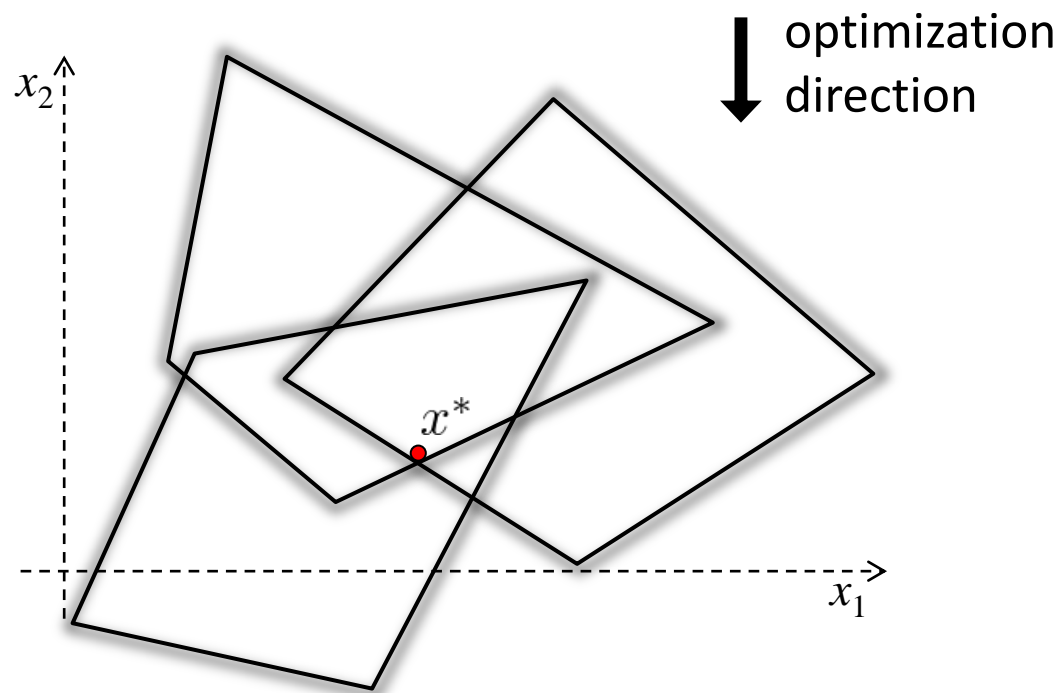
$$\frac{\beta}{N+1} \sum_{m=k}^N \binom{m}{k} (1-v)^{m-k} - \binom{N}{k} (1-v)^{N-k} = 0$$

same theorem as in Marco's presentation

Scenario optimization with constraints relaxation

$$\begin{aligned} \min_{x \in \mathbb{R}^d} \quad & c(x) \\ \text{s.t.} \quad & f(x, \delta_i) \leq 0, \quad i = 1, \dots, N \end{aligned}$$

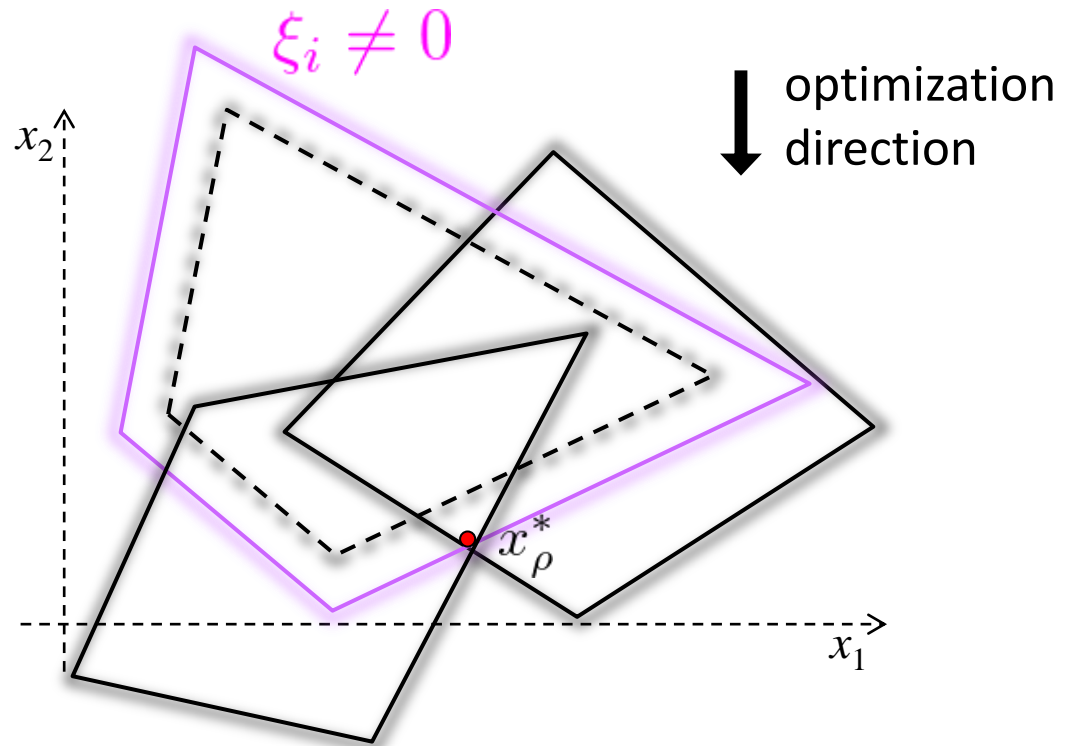
solution: x^*



Scenario optimization with constraints relaxation

$$\begin{aligned} \min_{x \in \mathbb{R}^d, \xi \geq 0} \quad & c(x) + \rho \sum_{i=1}^N \xi_i \\ \text{s.t.} \quad & f(x, \delta_i) \leq \xi_i, \quad i = 1, \dots, N \end{aligned}$$

decision: x_ρ^* , $\{\xi_i^* : \xi_i^* \neq 0\}$



Scenario optimization with constraints relaxation

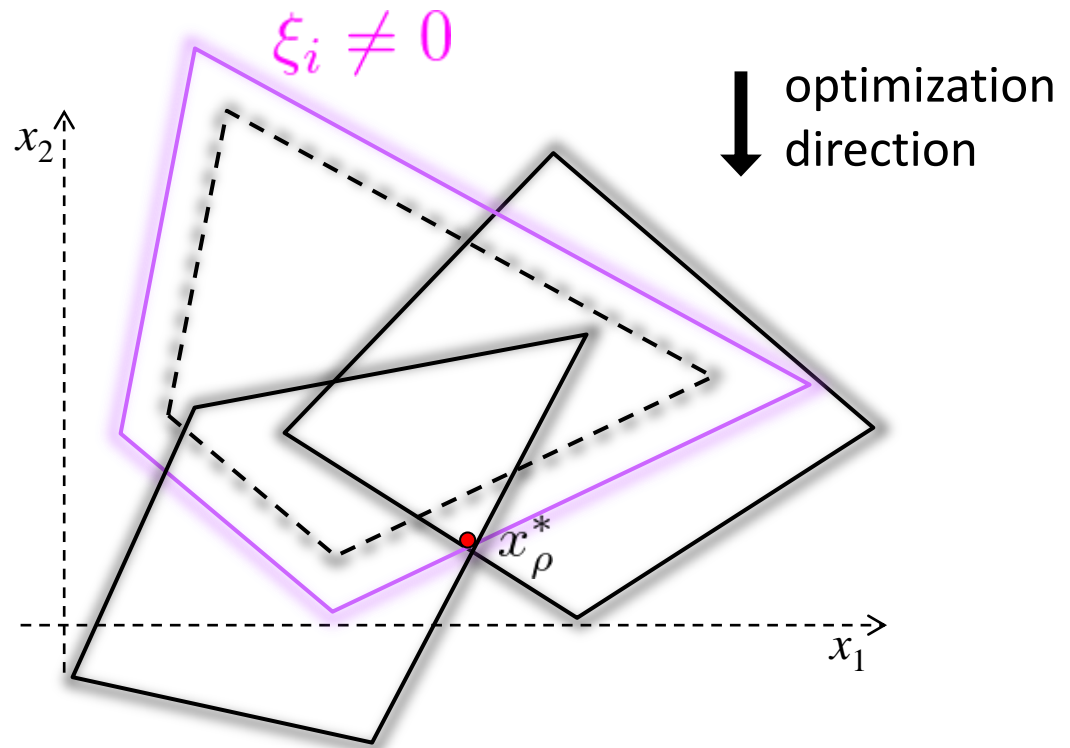
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decision: x_ρ^* , $\{\xi_i^* : \xi_i^* \neq 0\}$

$c(x_\rho^*)$ known

$V(x_\rho^*) =$

$$\mathbb{P}\{\delta \in \Delta : f(x_\rho^*, \delta) > 0\}$$



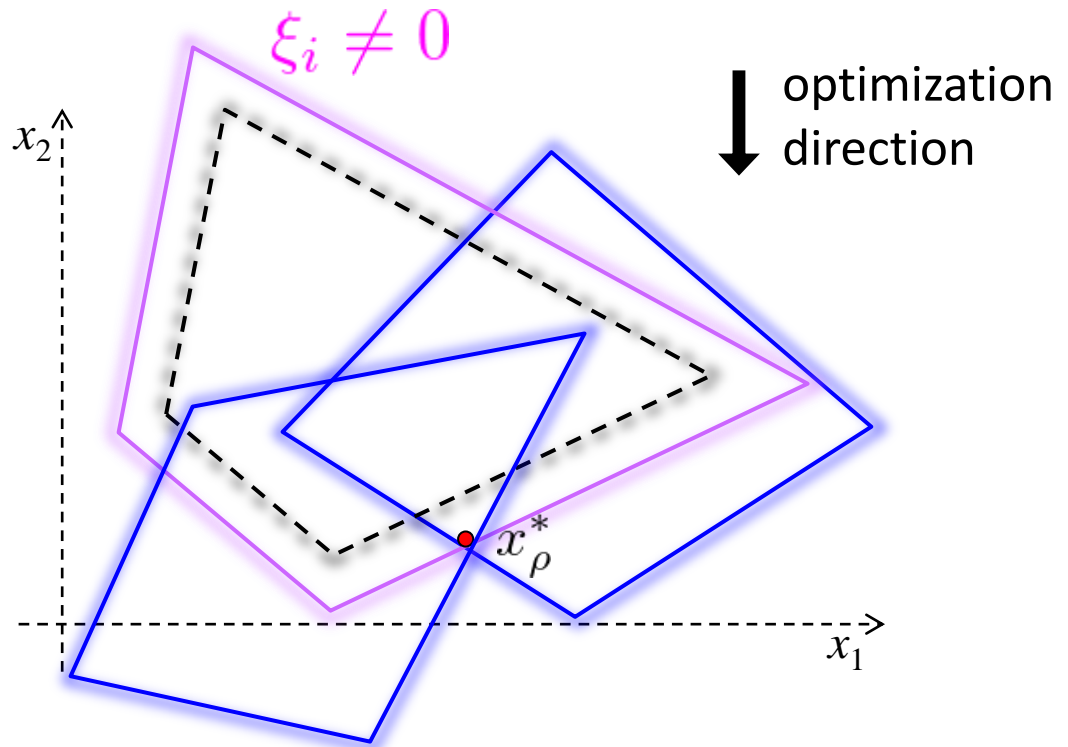
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complexity: s_ρ^*

active



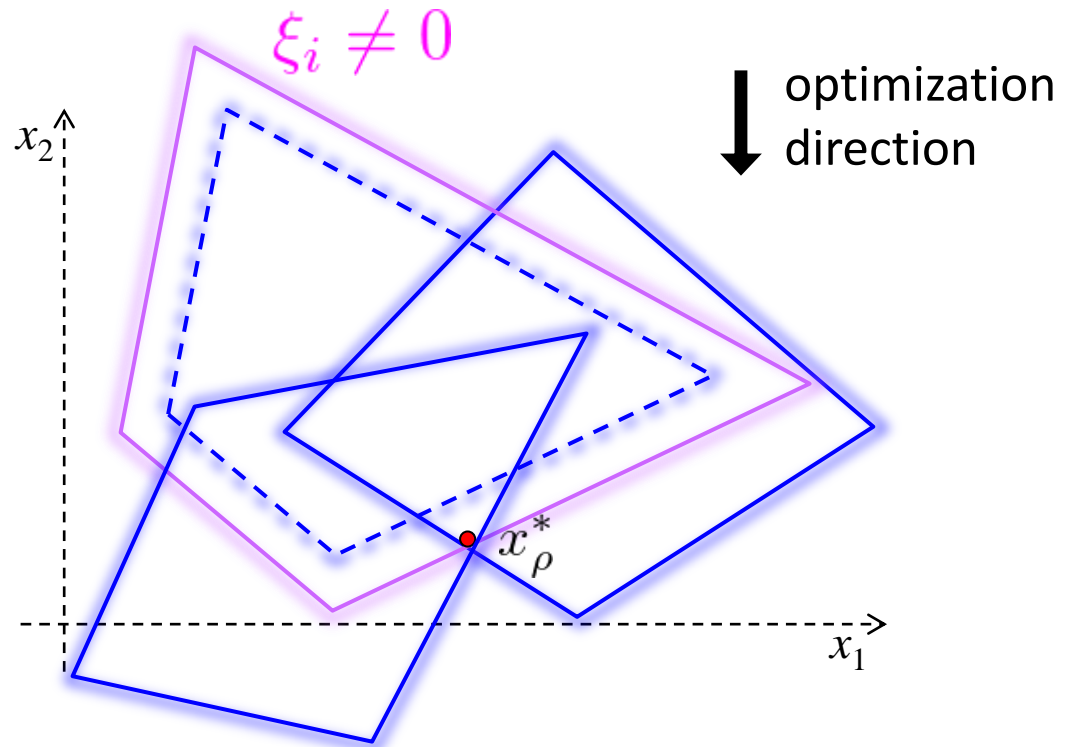
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decision: x_ρ^* , $\{\xi_i^* : \xi_i^* \neq 0\}$

complexity: s_ρ^*

active + violated



Main theorem

N = no. of scenarios

choose $\beta \in (0, 1)$ (**confidence parameter**)

let $\widehat{V}(k)$, $k = 0, 1, \dots$, be the solution in $(0, 1)$ to equation

$$\frac{\beta}{N+1} \sum_{m=k}^N \binom{m}{k} (1-v)^{m-k} - \binom{N}{k} (1-v)^{N-k} = 0$$

with confidence $1 - \beta$ it holds that $V(x_{\rho}^*) \leq \widehat{V}(s_{\rho}^*)$

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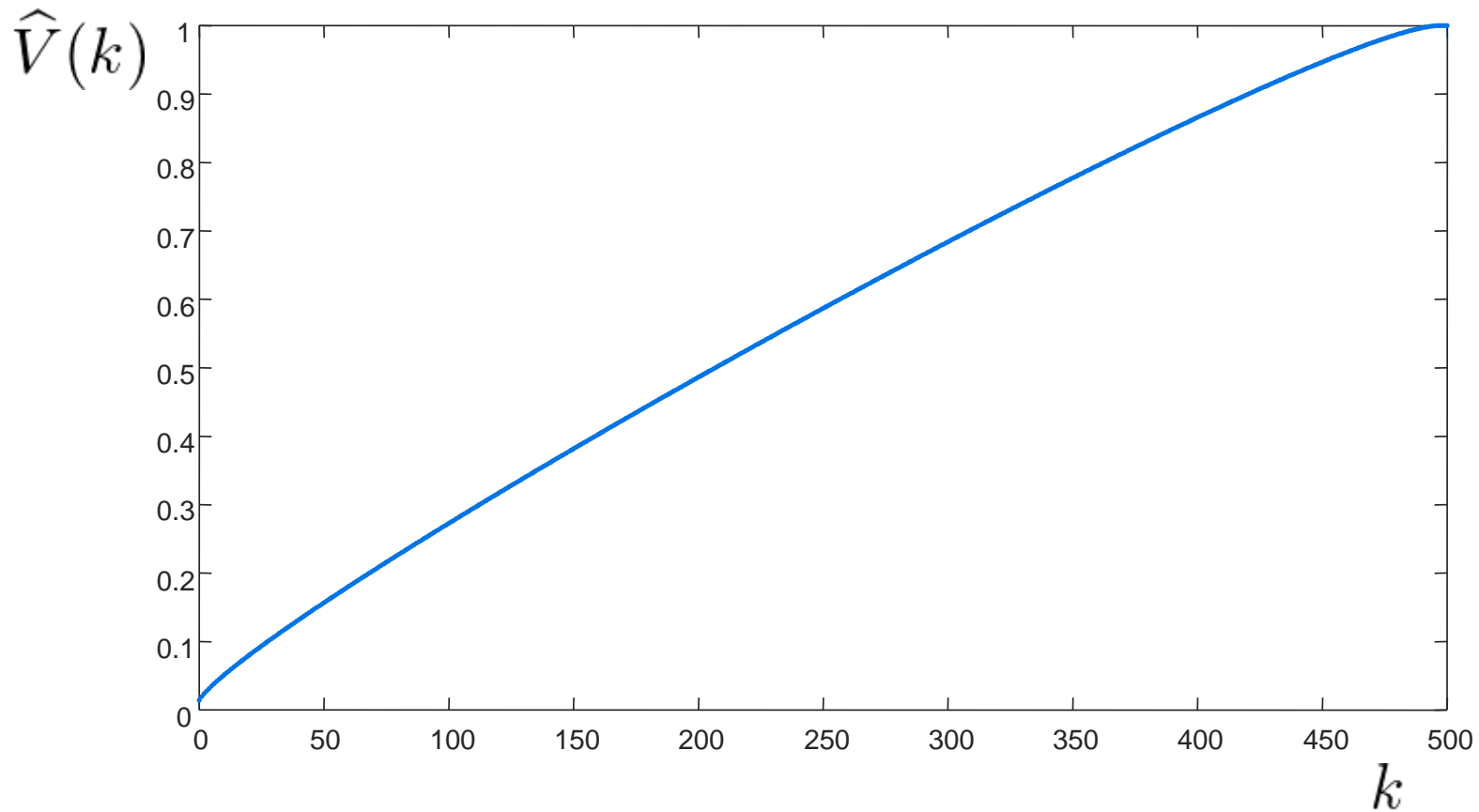
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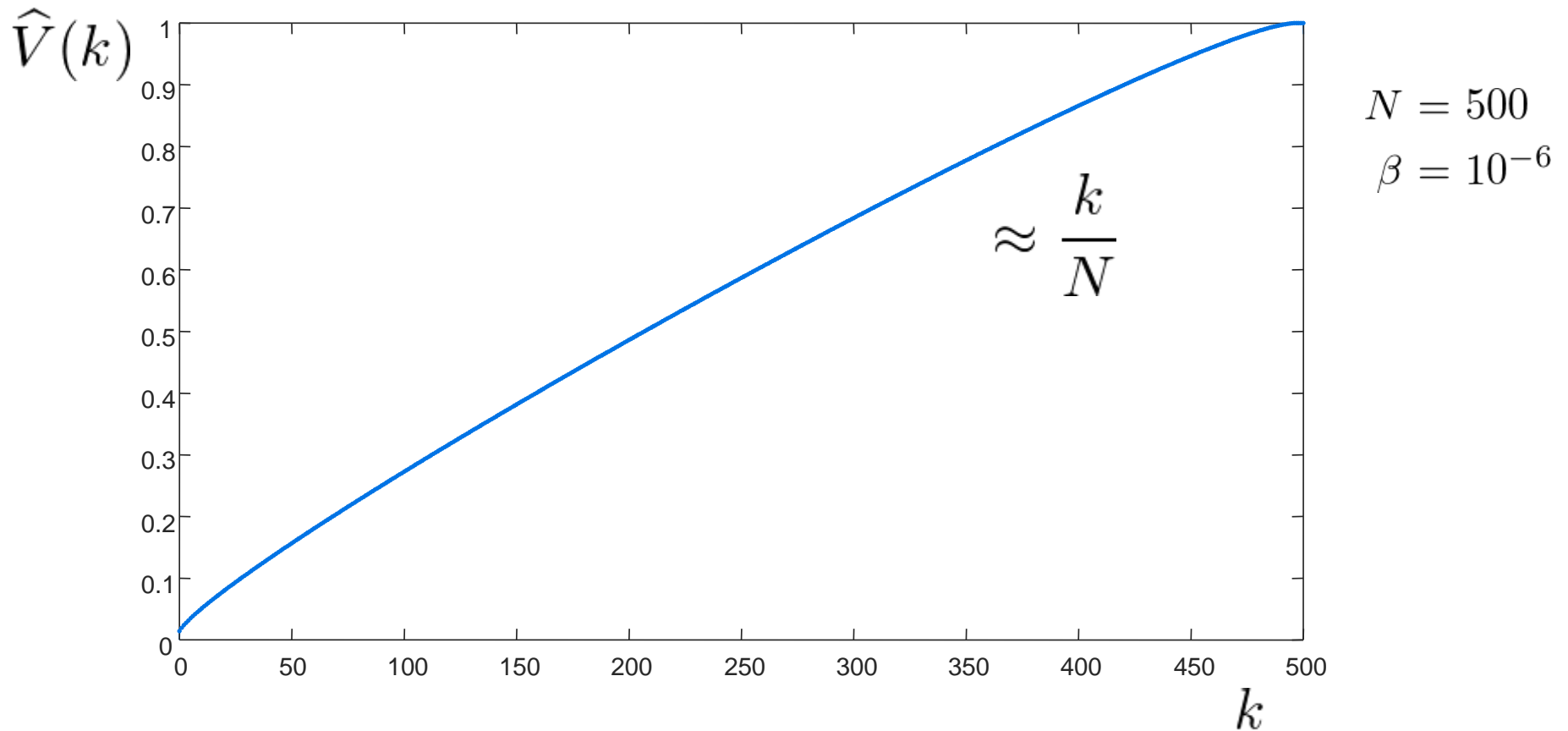
For all problems in the world,

risk $V(x_{\rho}^*)$ can be assessed through $\widehat{V}(s_{\rho}^*)$

Main theorem (cont'd)



Main theorem (cont'd)



$$\widehat{V}(s_{\rho}^*) \approx \frac{s_{\rho}^*}{N} \approx \frac{\text{active}}{N} + \frac{\text{violated}}{N}$$

empirical violation

Risk vs. performance tradeoff

$$\begin{aligned} \min_{x \in \mathbb{R}^d, \xi \geq 0} \quad & c(x) + \rho \sum_{i=1}^N \xi_i \\ \text{s.t.} \quad & f(x, \delta_i) \leq \xi_i, \quad i = 1, \dots, N \end{aligned}$$

solution: x_ρ^* ← tunable parameter

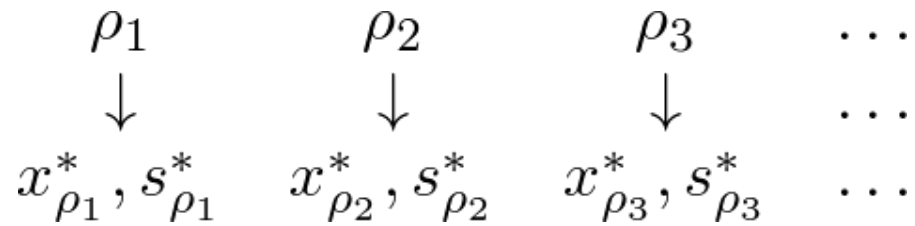
as $\rho \rightarrow 0$

cost $c(x_\rho^*)$ decreasing

risk $\widehat{V}(s_\rho^*)$ increasing (trend)

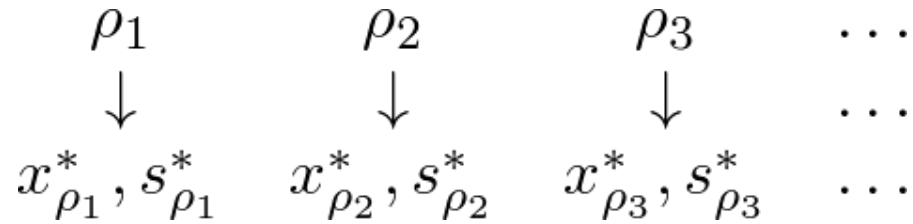
} risk vs. performance tradeoff

Risk vs. performance tradeoff

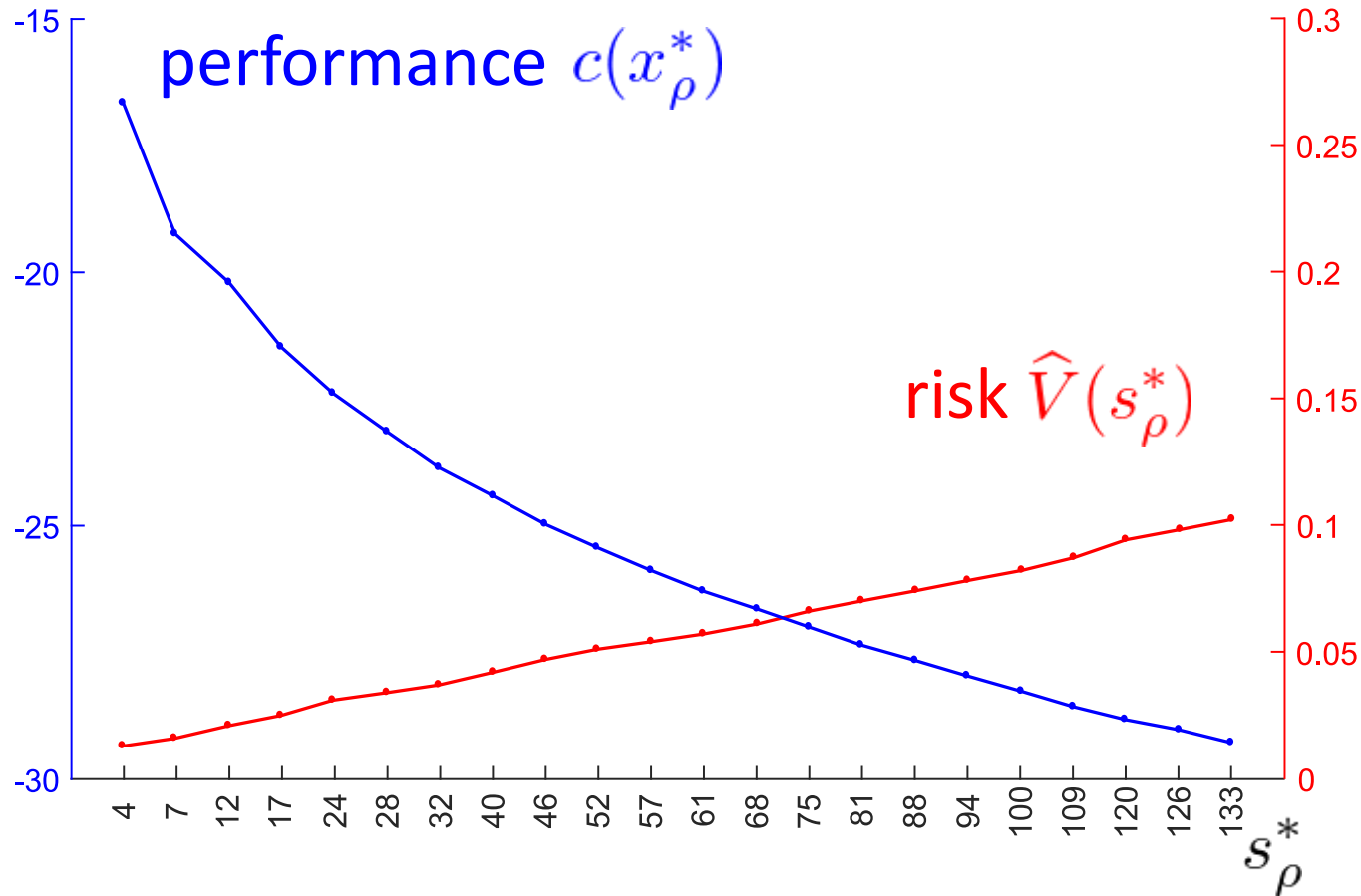


quantitative comparison
via $c(x_{\rho}^*)$ and $\widehat{V}(s_{\rho}^*)$

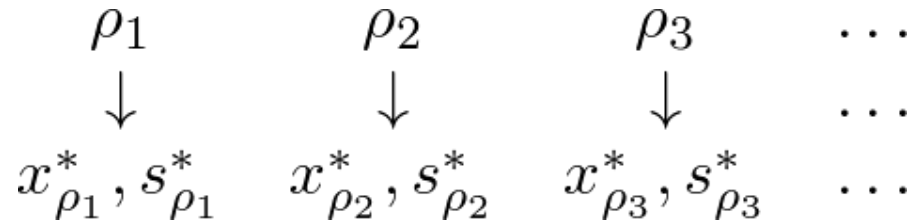
Risk vs. performance tradeoff



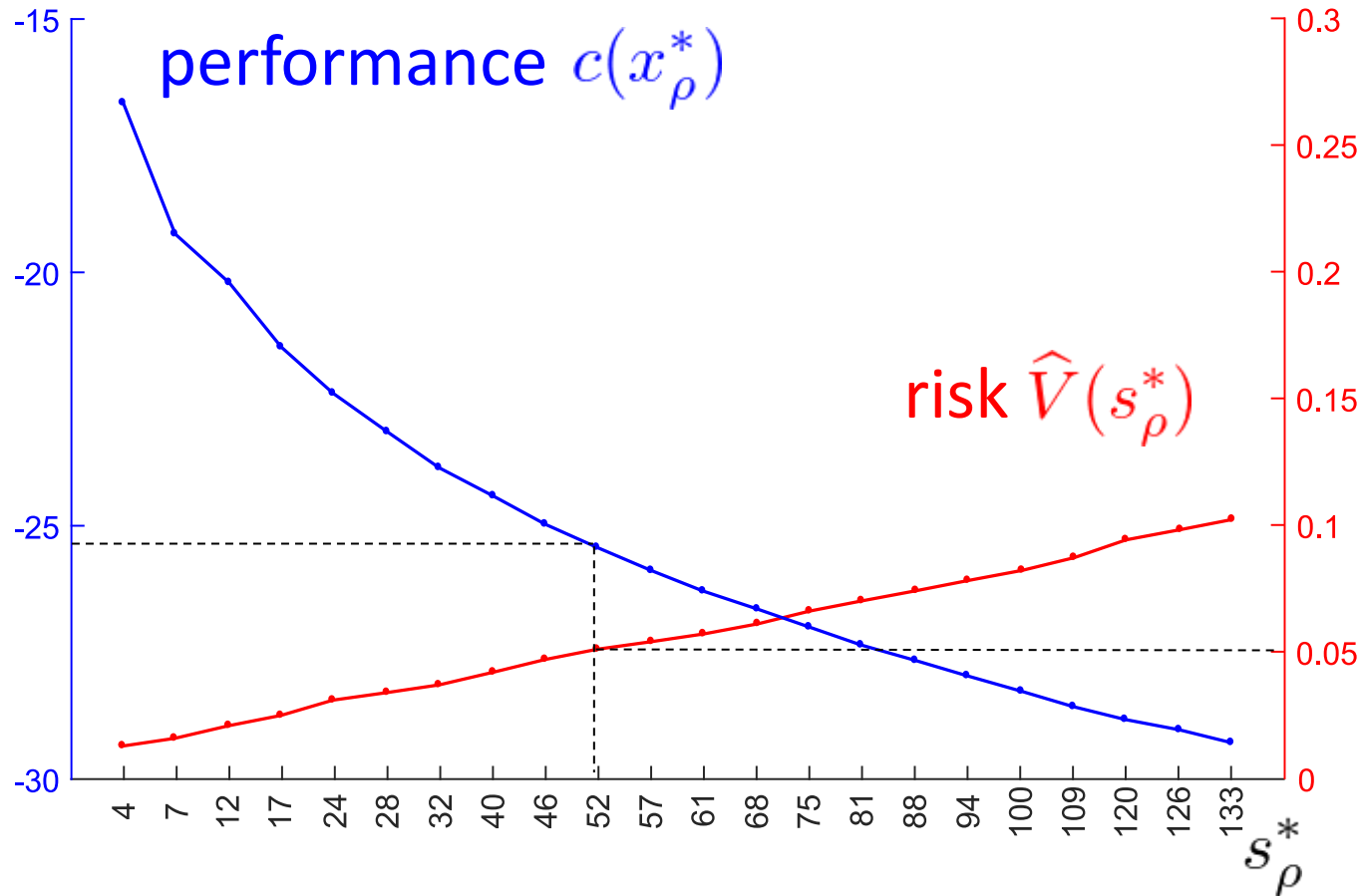
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Risk vs. performance tradeoff



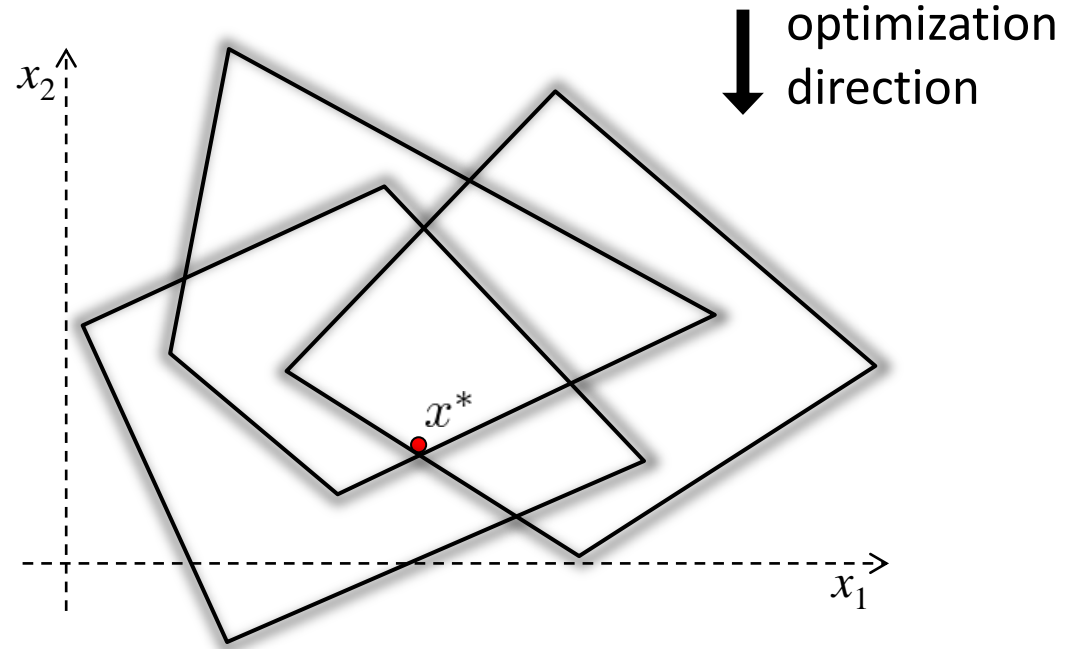
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The other side of the coin

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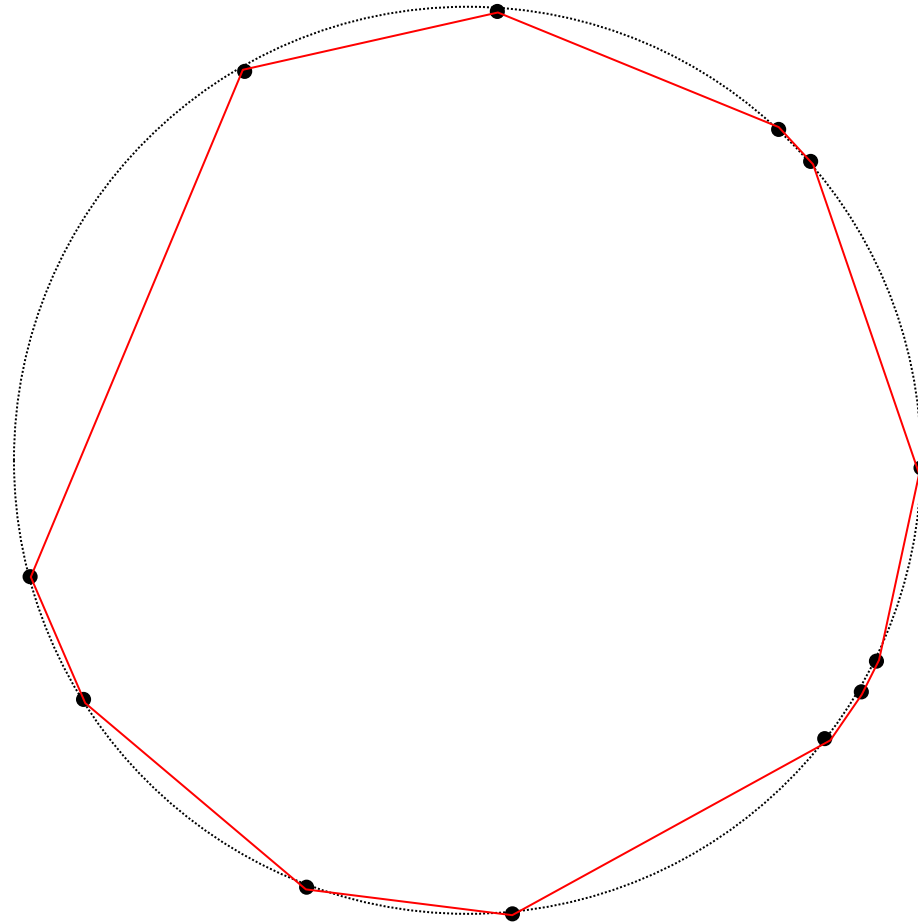
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**solution quality may be not
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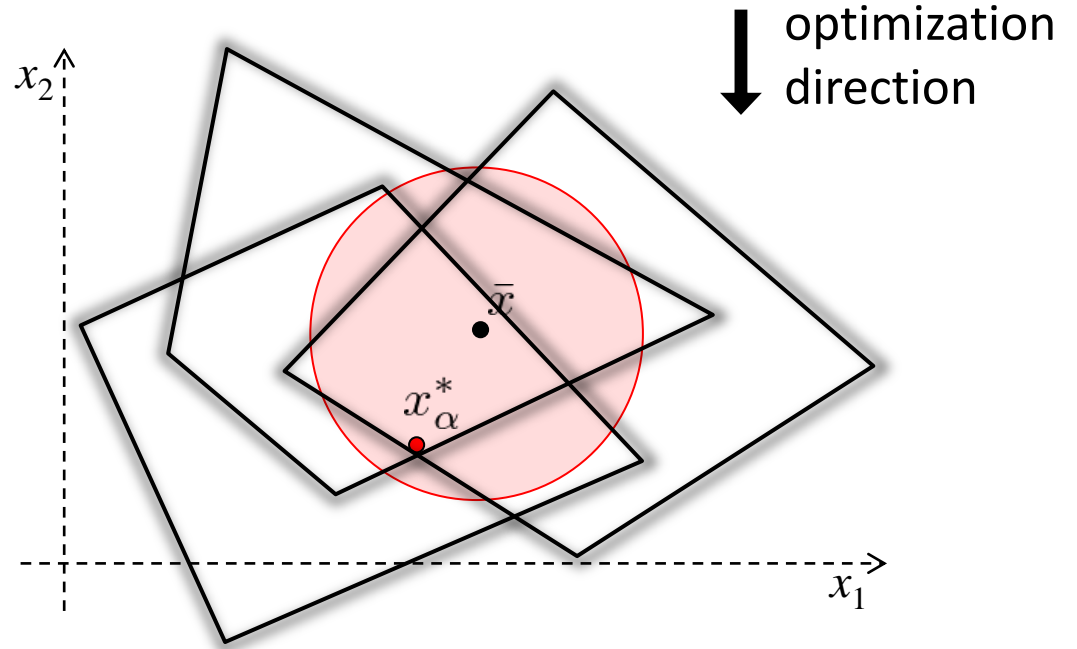
$$\hat{V}(s^*) \text{ is too big}$$

The other side of the coin (example)



Scenario optimization with regularization

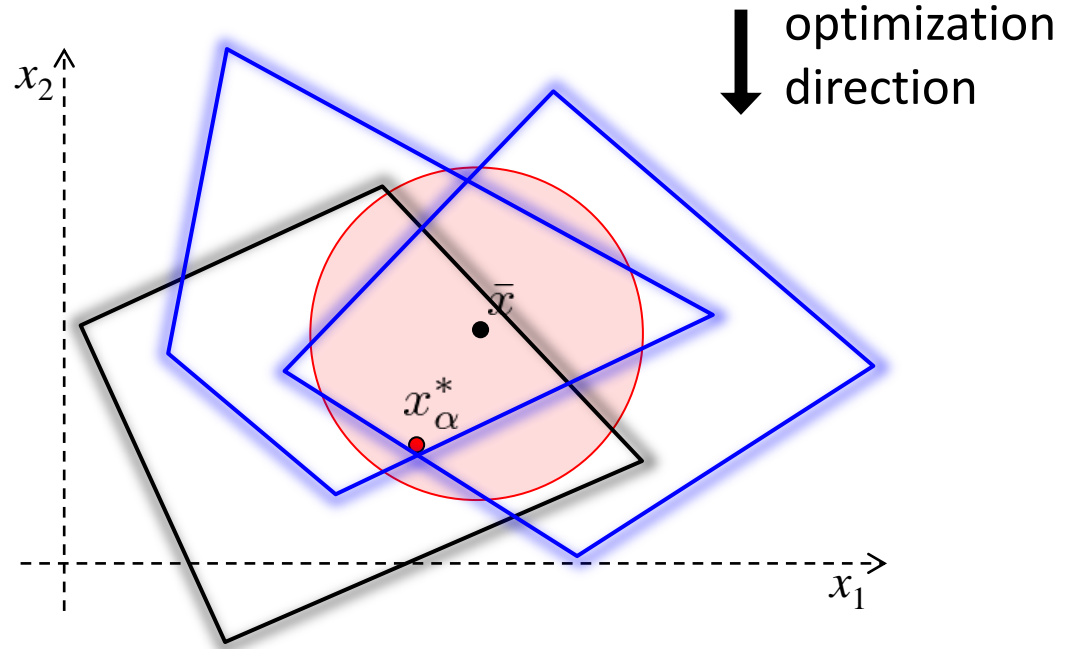
$$\begin{aligned} \min_x \quad & c(x) \\ \text{s.t.} \quad & x \in \mathcal{X}_{\delta_i} \\ & i = 1, \dots, N \\ & \|x - \bar{x}\| \leq \alpha \end{aligned}$$



solution: x_α^*

Scenario optimization with regularization

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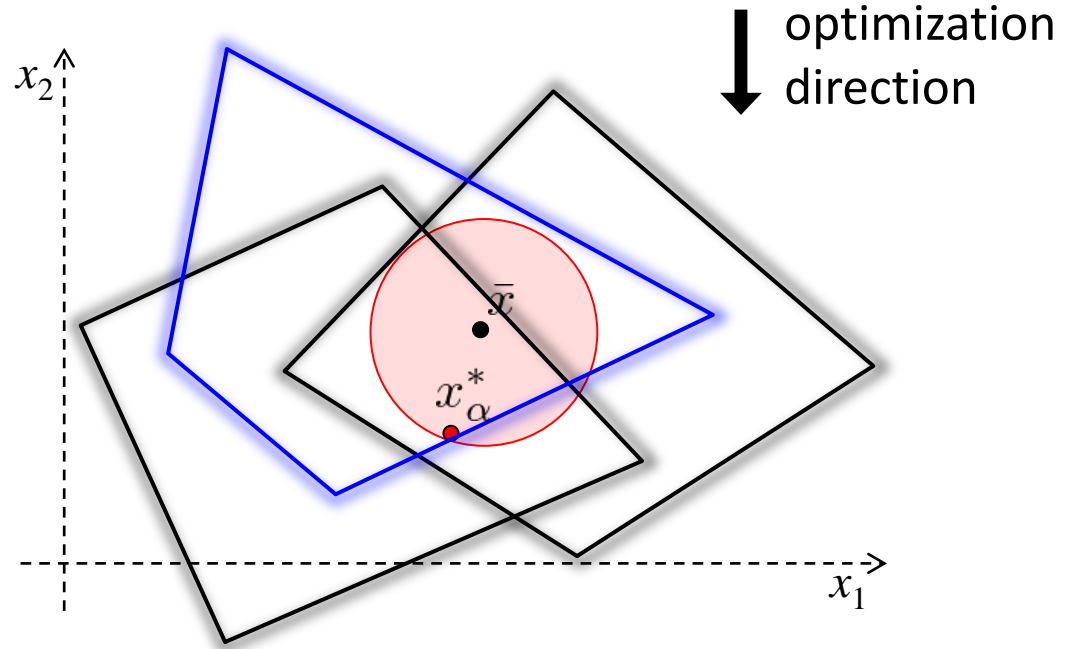
solution: x_{α}^*

complexity: S_{α}^*

(support set)

Scenario optimization with regularization

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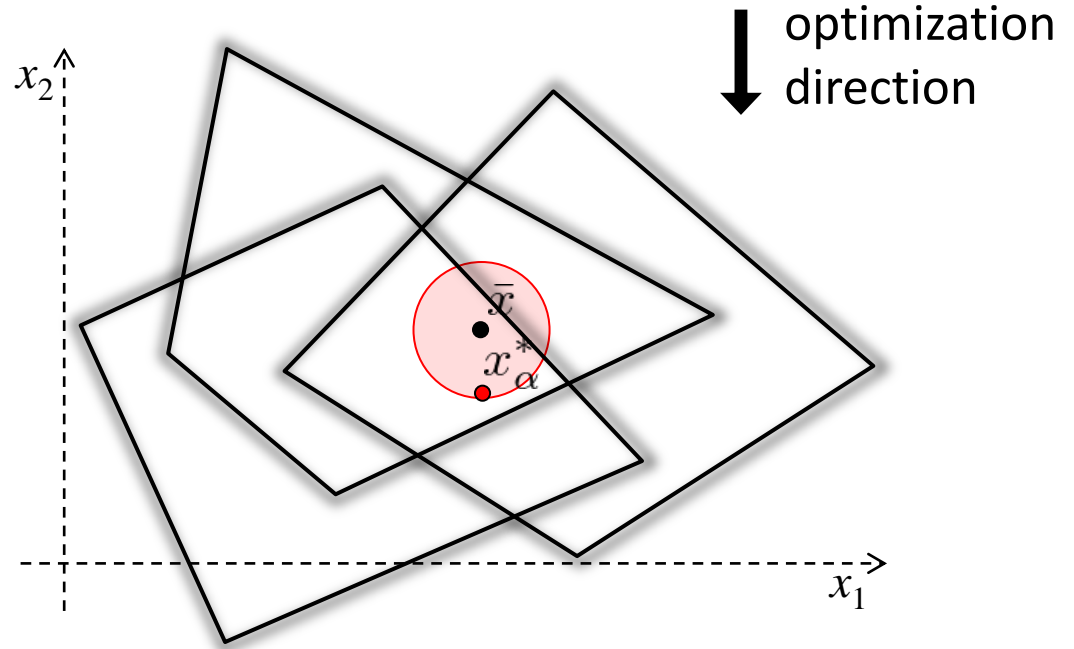
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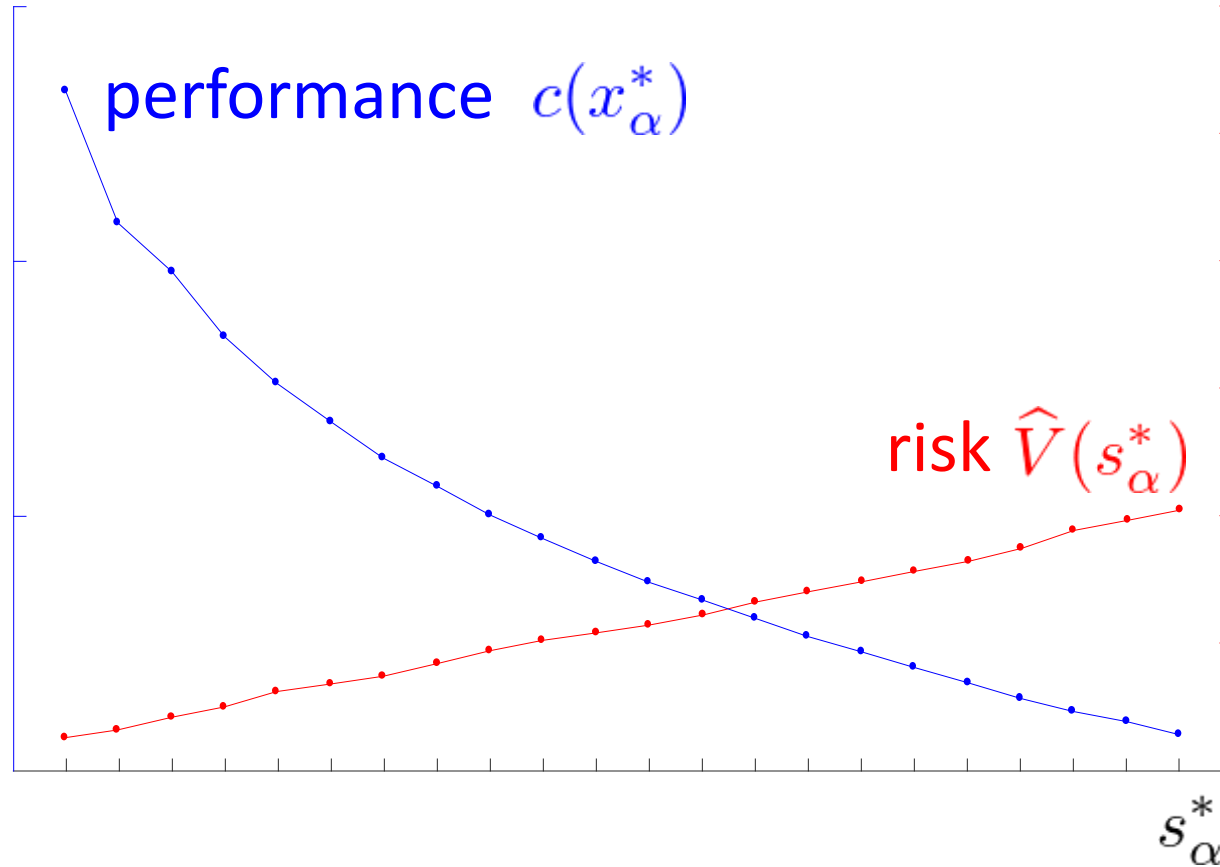
(support set)

as $\alpha \rightarrow 0$

cost $c(x_{\alpha}^*)$ **increasing**

risk $\widehat{V}(s_{\alpha}^*)$ **decreasing**

Scenario optimization with regularization



Conclusions

- ❑ Scenario optimization extended to scenario decision-making: a very general setup
- ❑ Alternative (**tunable**) schemes to obtain many alternative “solutions” (many other schemes exist, many others have to be discovered)
- ❑ Scenario theory: for each solution the risk (**invisible**) can be estimated from the complexity (**visible**)
- ❑ The risk estimate along with the performance allows the user to perform a quantitative comparison among the obtained solutions and to choose the one that is best suited for the problem at hand

Thank you !