Recent results on nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics

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The Keller-Segel model with degenerate diffusion

In $\mathbb{R}^N$ with $N \geq 2$, the model with degenerate diffusion is

$$\rho_t = \Delta \rho^m - \nabla \cdot (\rho \nabla (\mathcal{N} * \rho)),$$

with $m > 1$, being $\mathcal{N}$ the Newtonian kernel in $\mathbb{R}^N$.

- The nonlinear degenerate diffusion term for the 2D Keller-Segel equation avoids the blow-up phenomenon (anti-overcrowding effect). (Boi-Capasso-Morale '00, Topaz-Bertozzi-Lewis '06).
- The behaviour of solutions depends on m and on the so called critical exponent $m_c = 2 - \frac{2}{N}$:
  - for $m > m_c$, for any $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$, the solution exists globally in time and there is a uniform estimate in time of the L^∞ norm. (Sugiyama '06)
  - for $m < m_c$, there is a blow-up in finite time for an initial data with arbitrarily small mass. (Sugiyama '06)
  - for $m = m_c$ (fair competition) the behaviour of solution depends on the mass, and there is the presence of a critical mass M_c. (Blanchet-Carrillo-Laurençot '09)

The Keller-Segel model with degenerate diffusion

From now on, we will focus on the "subcrictical case" $m > 2 - \frac{2}{N}$, in which solutions exist globally in time.

Question

What about the asymptotic behaviour of solutions?

There is the existence of a free-energy functional $\mathcal{F}$ associated to the model:

$$\mathcal{F}[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho^m dx - \frac{1}{2} \int_{\mathbb{R}^N} \rho(\mathcal{N} * \rho) dx;$$

we can write the KS equation as

$$\rho_t = \nabla \cdot \left(\rho \nabla (\frac{m}{m-1} \rho^{m-1} - \mathcal{N} * \rho) \right) =: \nabla \cdot \left(\rho \nabla \left(\frac{\delta \mathcal{F}}{\delta \rho} \right) \right)$$

where $\frac{\delta \mathcal{F}}{\delta \rho} = \frac{m}{m-1} \rho^{m-1} - \mathcal{N} * \rho$.
If ρ is a solution of the KS-equation, then $\mathcal{F}[\rho]$ **decreases** in time, hence it is a Lyapunov functional.

The Keller-Segel model with degenerate diffusion

The following properties are known for the global minimizers of $\mathcal{F}$, among densities with fixed mass M:

- Existence: (Lions '84) for $N \geq 3$ and (Carrillo, Castorina, V. 2014) for $N = 2$;
- Radial symmetry (rearrangement techniques);
- Uniqueness + compact support (Lieb-Yau '87), (Kim-Yao 2012) for $N \geq 3$, (Carrillo, Castorina, V. 2014) for $N = 2$

Let ρ_M be a minimizer of $\mathcal{F}$ with mass M. Then ρ_M must be a stationary solution.

The Keller-Segel model with degenerate diffusion

Question

If $\rho_0 = \rho(0, \cdot)$ has mass M, is it always true that $\rho(\cdot, t)$ converges to (a translation of) ρ_M when $t \to \infty$?

The answer is affirmative if we have a positive answer to the following questions:

Question

Is ρ_M the unique stationary state of mass M (up to translations)?

We know the uniqueness of stationary solutions with **radial symmetry**, with fixed mass (Lieb-Yau '87), , (Kim-Yao 2014) hence the question above reduces to

Question

Is it true that every steady state is radially symmetric (up to translations)?

Stationary solutions of the Keller-Segel equation

Rewriting the KS-equation in the divergence form

$$\rho_t - \nabla \cdot \left(\rho \nabla \left(\frac{m}{m-1}\rho^{m-1} - \mathcal{N} * \rho\right)\right) = 0,$$

then any stationary solution ρ_s satisfies

$$\frac{m}{m-1}\rho_s^{m-1} - \mathcal{N} * \rho_s = C_i$$

in each connected component of $\{\rho_s > 0\}$ (C_i may be get different values in each connected component).

Stationary solutions for the degenerate aggregation-diffusion equation

Now we consider the equation with a general attractive kernel $\mathcal{K}$:

$$\rho_t = \nabla \cdot \left(\rho \nabla \left(\frac{m}{m-1} \rho^{m-1} + \mathcal{K} * \rho \right) \right),$$

where $\mathcal{K}$ is radial and strictly increasing in $|x|$. Similarly, each steady state ρ_s verifies

$$\frac{m}{m-1} \rho_s^{m-1} + \mathcal{K} * \rho_s = C_i$$

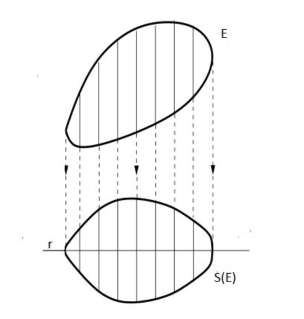
in each connected component of $\{\rho_s > 0\}$.

Theorem (Carrillo-Hittmeir-Yao, V., 2016)

Let $\rho_s \in L^1_+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ a steady state. Then ρ_s must be radially decreasing, up to translastions.

Steiner symmetrization

An element playing a decisive role in the proof of such result is the concept of **Steiner symmetrization**. If, for instance $E \subset \mathbb{R}^2$, the Steiner symmetrization of E w.r. to a line r as:



We have that $|E| = |S(E)|$, while the perimeter decreases.

Steiner symmetrization

We can define easily the Steiner symmetrization (or rearrangement) $S\rho$ of a function $\rho \in L^1(\mathbb{R}^N_+)$. We define the distribution function of ρ w.r. to $x_1 \in \mathbb{R}$ as

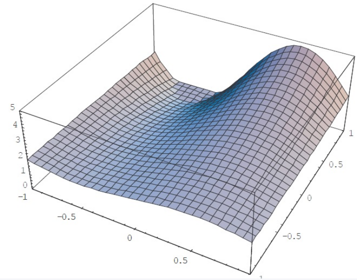
$$\mu_\rho(h, x') = \left| \left\{ x_1 \in \mathbb{R} : \rho(x_1, x') > h \right\} \right|, \quad \forall h > 0,\ x' \in \mathbb{R}^{N-1};$$

the Steiner symmetrization of ρ w.r. to x_1 is a particular function which is symmetric w.r. to the hyperplane $x_1 = 0$:

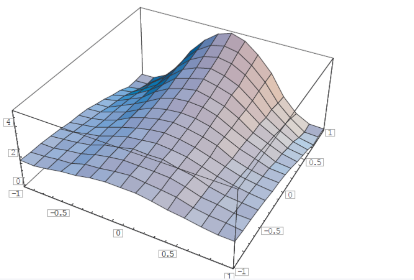
$$(S\rho)(x_1, x') = \sup \left\{ h > 0 : \mu_\rho(h, x') > 2|x_1| \right\}.$$

- For all $h > 0$, $\{(x_1, x') : (S\rho)(x_1, x') > h\}$ coincides with the Steiner symmetrization of $\{(x_1, x') : \rho(x_1, x') > h\}$ w.r. to the hyperplane $x_1 = 0$;
- In particular, ρ e $S\rho$ are equimeasurable: the L^p norms are invariant w.r. to the Steiner symmetrization;

Steiner symmetrization



ρ



$S\rho$

Continuous Steiner symmetrization of sets

If $U \subset \mathbb{R}$ open, we define its **continuous Steiner symmetrization** $M^t(U)$ for all $t \geq 0$ as:

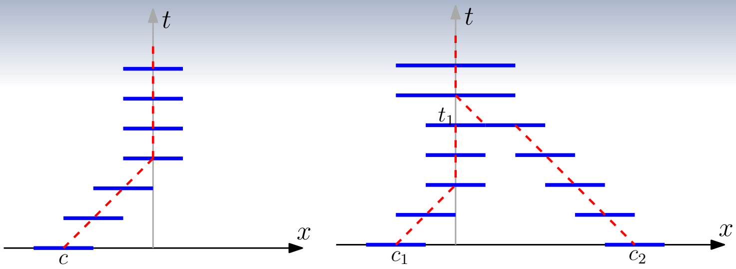
(1) If $U = (c - r, c + r)$, then

$$M^t(U) := \begin{cases} (c - t\,\mathrm{sgn}\,c - r, c - t\,\mathrm{sgn}\,c + r) & \text{per } 0 \leq t < |c|, \\ (-r, r) & \text{for } t \geq |c|. \end{cases}$$

(2) If $U = \cup_{i=1}^{N}(c_i - r_i, c_i + r_i)$ disjoint, then $M^t(U) := \cup_{i=1}^{N} M^t((c_i - r_i, c_i + r_i))$ for $0 \leq t < t_1$, where t_1 is the first time where the two intervals $M^t((c_i - r_i, c_i + r_i))$ have a common endpoint. Once it happens, we merge them into one open interval, and repeat this process starting from $t = t_1$.

(3) If $U = \cup_{i=1}^{\infty}(c_i - r_i, c_i + r_i)$ (with disjoint $(c_i - r_i, c_i + r_i)$), let $U_N = \cup_{i=1}^{N}(c_i - r_i, c_i + r_i)$ for all $N > 0$, and define $M^t(U) := \cup_{N=1}^{\infty} M^t(U_N)$.

Continuous Steiner symmetrization of sets



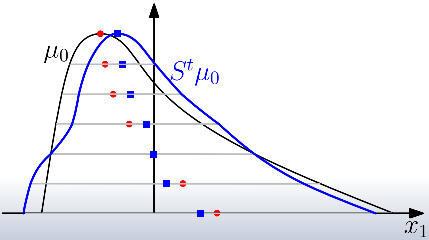
We could generalize this definition for any measurable set of $\mathbb{R}$. Some fundamental properties:

- $|M^t(U)| = |U|$ for all $t \geq 0$;
- $M^0(U) = U$ e $M^\infty(U) = S(U)$;

Continuous Steiner symmetrization of a function

Let us fix a function $\rho \in L^1(\mathbb{R}^d_+)$, define the continuous Steiner symmetrization of ρ in the direction $e_1 = (1, 0, \ldots, 0)$, through

$$(S^t\rho)(x_1, x') = \int_0^\infty \chi_{M^t(U^h_{x'})}(x_1)dh$$



- For all $h > 0$, $\left|\left\{(x_1, x') : (S^t\rho)(x_1, x') > h\right\}\right|_N = \left|\{(x_1, x') : \rho(x_1, x') > h\}\right|_N$;
- $S^0\rho = \rho$ e $S^\infty\rho = S\rho$

Symmetry of steady states: sketch of the proof

The proof relies on a contradiction argument. Let us assume that there is a steady state ρ_s that is NOT radial and decreasing after ANY translation.

- The key point yielding a contradiction is to prove that there is is a constant $c > 0$ dependant on ρ_s and $\mathcal{K}$, for which the c. Steiner symmetrization $S^t\rho_s$ is such that, for small values of t,

$$\mathcal{F}[S^t\rho_s] - \mathcal{F}[\rho_s] < -ct,$$

$$\mathcal{F}[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^N} \rho^m dx + \frac{1}{2}\int_{\mathbb{R}^N} \rho(\mathcal{K} * \rho)dx;$$

We observe that by the properties of continuous Steiner symmetrization we have

$$||S^t\rho_s||_m = ||\rho_s||_m;$$

moreover since $\mathcal{K}$ is increasing in $|x|$ one can show that

$$\int S^t\rho_s((S^t\rho_s) * \mathcal{K})dx < \int \rho_s(\rho_s * \mathcal{K})dx.$$

Uniqueness?

In principle, nothing can be said on the uniqueness of the stationary states for a **general kernel** $\mathcal{K}$: if $\mathcal{K} = -\mathcal{N}$, there is a unique stationary state with mass *M* and zero center of mass (Kim-Yao 2012).

Existence of global minimizers

It is possible to show the existence of a radially decreasing global minimizer of the energy functional

$$\mathcal{F}[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^N} \rho^m dx + \frac{1}{2}\int_{\mathbb{R}^N} \rho(\mathcal{K} * \rho) dx,$$

in the class of admissible densities

$$\mathcal{Y}_M := \left\{ \rho \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) : \, ||\rho||_1 = M, \omega(1+|x|)\,\rho(x) \in L^1(\mathbb{R}^N) \right\},$$

where we assume $\int_{\mathbb{R}^N} x\rho(x)\,dx = 0$, con $\mathcal{K}(x) = \omega(|x|)$. More precise assumptions on $\mathcal{K}$ are

(K1) $\omega'(r) > 0$ for all $r > 0$ with $\omega(1) = 0$.

(K2) $\mathcal{K}$ is not more singular than the Newtonian kernel in $\mathbb{R}^N$ close to the origin,i.e., there exists $C_w > 0$ such that $\omega'(r) \leq C_w r^{1-N}$ per $r \leq 1$.

(K3) There is some $C_w > 0$ such that $\omega'(r) \leq C_w$ for all $r > 1$.

(K4) Condition at infinity: $\lim_{r \to +\infty} \omega_+(r) = +\infty$.

Regularity of minimizers

If ρ_0 is a global minimizer, one has

- ρ_0 satisfies
$$\frac{m}{m-1}\rho_0^{m-1} + \mathcal{K} * \rho_0 = C \quad \text{q.o. in } \{\rho_0 > 0\}$$
hence it is a stationary state;
- From this equation and from the asymptotic bahavior of $\mathcal{K} * \rho_0$ one can show that ρ_0 is of compact support; moreover $\rho_0 \in L^\infty(\mathbb{R}^N)$;
- Using the locally Lipschitz regularity $W^{1,\infty}_{loc}$ of $\mathcal{K} * \rho_0$ one shows that $\rho \in C^{0,\alpha}(\mathbb{R}^N)$.

Remark: uniqueness

In general, nothing can be said on uniqueness of minimizers for general potentials, unless when $\mathcal{K} = -\mathcal{N}$.

Asymptotic behaviour in 2D

Let us consider the Keller-Segel model in 2D:

$$\rho_t = \nabla \cdot \left(\rho \nabla \left(\frac{m}{m-1}\rho^{m-1} + \mathcal{K} * \rho\right)\right) =: \nabla \cdot (\rho \nabla (h[\rho]))$$

where $\mathcal{K}(x) = \frac{1}{2\pi} \log |x|$, $N = 2$, $h = \frac{\delta \mathcal{F}}{\delta \rho} = \frac{m}{m-1}\rho^{m-1} + \mathcal{K} * \rho$.

- Let us assume that $\rho_0 \in L^\infty(\mathbb{R}^2) \cap L^1((1+|x|^2)dx)$.
- Then $\mathcal{F}$ decreases along the solution $\rho(t,x)$:

$$\frac{d}{dt}\mathcal{F}[\rho](t) = -\mathcal{D}[\rho](t)$$

then

$$\mathcal{F}[\rho](t) + \int_0^t \mathcal{D}[\rho] d\tau \leq \mathcal{F}[\rho_0]$$

with the entropy dissipation defined as

$$\mathcal{D}[\rho] = \int_{\mathbb{R}^2} \rho |\nabla h[\rho]|^2 dx.$$

Asymptotic behaviour in 2D

Theorem (Carrillo-Hittmeir-Yao-V., 2016)

For all $\rho_0 \in L^\infty(\mathbb{R}^2) \cap L^1((1+|x|^2)dx)$, for $t \to \infty$, $\rho(\cdot, t)$ converges to the unique stationary state with the same mass M and center of mass of the initial datum ρ_0 i.e., converges to

$$\rho_M^c := \rho_M(x - x_c) \qquad \text{where } x_c = \frac{1}{M}\int_{\mathbb{R}^2} x\rho_0(x)\, dx.$$

More precisely, we have

$$\lim_{t\to\infty} ||\rho(t,\cdot) - \rho_M^c||_{L^q(\mathbb{R}^2)} = 0 \qquad \text{for all } 1 \le q < \infty\,.$$

Remark: since this is obtained through a compactness argument, we do no get any rate of convergence.

What happens for $N \geq 3$?

- For $N \geq 3$, if ρ_0 is radially symmetric and of compact support, the convergence to the unique steady state is known (Kim-Yao '12), and the rate of convergence is exponential.
- For nonradial data, the asymptotic behaviour is an open problem: there are no a priori estimates avoiding that the mass escapes to infinity (no mass confinement)!

What happens when $\mathcal{K}$ is more singular?

$$\partial_t \rho = \Delta \rho^m + \nabla \cdot (\rho \nabla W_k * \rho) \quad \text{in } (0,T) \times \mathbb{R}^N,$$

The interaction is given by the the Riesz kernel
$W_k(x) := \frac{1}{k} |x|^k, \quad -N < k < 0.$

Free energy:

$$\mathcal{F}[\rho] = \mathcal{H}_m[\rho] + \mathcal{W}_k[\rho]$$

$$\mathcal{H}_m[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho^m(x)\, dx\,, \qquad \mathcal{W}_k[\rho] = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|x-y|^k}{k} \rho(x)\rho(y)\, dxdy\,.$$

Diffusion dominated regime

$\mathcal{H}_m$ and $\mathcal{W}_k$ are homogeneous by taking dilations $\rho^\lambda(x) = \lambda^N \rho(\lambda x)$

$$\mathcal{F}[\rho^\lambda] = \lambda^{N(m-1)} \mathcal{H}_m[\rho] + \lambda^{-k} \chi \mathcal{W}_k[\rho] \,.$$

Critical exponent $m_c := 1 - k/N$

- $m = m_c$: fair competition regime (critical mass)
- $m > m_c$: diffusion dominated regime $\leftarrow$ we focus on this case
- $m < m_c$: attraction dominated regime

Our results have many analogous in fair competition regime [Blanchet, Carrillo, Laurencot 2009], [Calvez, Carrillo, Hoffmann 2016, 2017]

and in case of Newtonian potential interaction [Kim, Yao 2012], [Carrillo, Castorina, Volzone 2015], [Carrillo, Hittmeir, Volzone, Yao 2016]

Stationary states

Let $\bar{\rho} \in L^1_+\left(\mathbb{R}^N\right)$, $||\bar{\rho}||_1 = M$, be a **stationary state** for the evolution equation.

- If $-N < k \leq 1 - N$, we further require $\bar{\rho} \in C^{0,\alpha}\left(\mathbb{R}^N\right)$ for some $\alpha \in (1-k-N, 1)$, implying that $\nabla W_k * \bar{\rho}$ is well defined (and bounded) as a Cauchy principal value

$$\nabla W_k * \bar{\rho}(x) := \int_{\mathbb{R}^N} \nabla W_k(x-y)\left(\bar{\rho}(y) - \bar{\rho}(x)\right) dy$$

Basic facts: if $\bar{\rho}$ is a stationary state then

- $\bar{\rho}^{m-1} \in W^{1,\infty}\left(\mathbb{R}^N\right)$.
- $\bar{\rho}(x)^{m-1} = \frac{m-1}{m}\left(C[\bar{\rho}](x) - W_k * \bar{\rho}(x)\right)_+ , \quad x \in \mathbb{R}^N$
  where $C[\bar{\rho}](x)$ is constant on each connected component of supp$(\bar{\rho})$.

Radial symmetry of stationary states

Theorem (Carrillo-Hoffmann-Mainini-V.,2017)

Stationary states are radially symmetric compactly supported.

Existence of global minimizers

Theorem

Let $k \in (-N, 0)$ *and* $m > m_c$. *There exist a minimizer of* $\mathcal{F}$ *on* $\mathcal{Y}_M := \left\{\rho \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)\,,\ ||\rho||_1 = M\,,\ \int_{\mathbb{R}^N} x\rho(x)\,dx = 0\right\}$.

- By Lions concentration-compactness, as for instance in [Kim,Yao 2012]

Properties of minimizers

Theorem

Let $k \in (-N, 0)$ and $m > m_c$. If ρ is a global minimizer of the free energy functional $\mathcal{F}$ in $\mathcal{Y}$, then ρ is radially symmetric and non-increasing, bounded, compactly supported, and

$$\rho^{m-1}(x) = \left(\frac{m-1}{m}\right) \left(D[\rho] - W_k * \rho(x)\right)_+ \quad a.e. \text{ in } \mathbb{R}^N$$

where

$$D[\rho] := 2\mathcal{F}[\rho] + \left(\frac{m-2}{m-1}\right) ||\rho||_m^m, \qquad \rho \in \mathcal{Y}_M .$$

Regularity of minimizers

Theorem

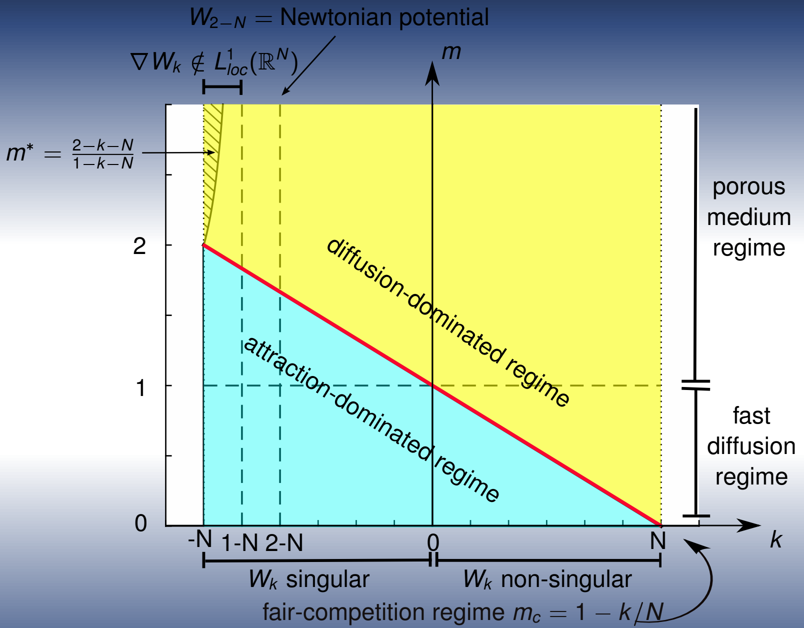
Let $k \in (-N, 0)$ and ρ a minimizer of $\mathcal{F}$ on $\mathcal{Y}_M$.

- *If $m_c < m < m^* := \frac{2-k-N}{1-k-N}$, then $\rho^{m-1} \in W^{1,\infty}(\mathbb{R}^N)$, thus $\rho \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = \min\{1, \frac{1}{m-1}\}$.*
- *If $m \geq m^*$, then $\rho^{m-1} \in C^{\alpha}(\mathbb{R}^N)$ for any $\alpha < \frac{(k+N)(m-1)}{m-2} \leq 1$.*
- *If $m \geq m_c$ and B is the interior of suppρ, then $\rho \in C^{\infty}(B)$.*

Schauder estimates for the fractional Laplacian [Ros-Oton, Serra 2016]

$$||W_k * \rho||_{C^{0,\alpha+2s}(B_{1/2}(0))} \leq c \left(||W_k * \rho||_{L^\infty(\mathbb{R}^N)} + ||\rho||_{C^{0,\alpha}(B_1(0))} \right), \quad 2s = k + N$$

Since ρ is radially symmetric decreasing, in the interior of the support the regularity of ρ is that of ρ^{m-1}. Therefore, by bootstrap we reach smoothness of ρ.



W_{2-N} = Newtonian potential
$\nabla W_k \notin L^1_{loc}(\mathbb{R}^N)$
m
$m^* = \frac{2-k-N}{1-k-N}$
2
1
0
diffusion-dominated regime
attraction-dominated regime
porous medium regime
fast diffusion regime
-N 1-N 2-N
0
N
k
W_k singular
W_k non-singular
fair-competition regime $m_c = 1 - k/N$

Open problems

- Uniqueness of minimizers. Up to now we have a proof only for $N = 1$.
- Long time asymptotics for the evolution equation
- Characterization of self-similar profiles

Thank you for your attention!