Asymptotical analysis of a weighted very fast diffusion equation arising in quantization of measures

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Summary

The quantization problem

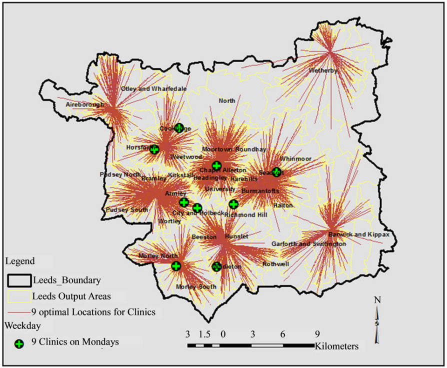
- overview
- the gradient flow approach

Very fast diffusion equations

- exponential convergence to equilibrium for smooth solutions
- existence of weak solutions via JKO scheme
- uniqueness/stability of weak solutions

An example of quantization problem

Question: what is the "optimal" way to locate N clinics in a region in order to meet the demand of the population?



- Notion of "optimality"
- Locations $\rightsquigarrow x^i$
- Masses $\rightsquigarrow m_i$

Setup of the problem

Let ρ be a probability density on a domain $\Omega \subset \mathbb{R}^d$.

Quantization problem: fixed $N \in \mathbb{N}$, find the best approximation of ρ by an atomic measure $\sum_i m_i \delta_{x^i}$ supported on at most N points in Ω.

Fix N points $x^1, \ldots, x^N \in \Omega$, and minimize

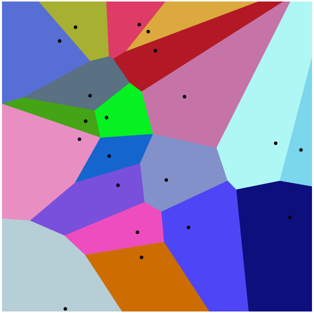
$$\inf \left\{ W_r\Big(\rho, \sum_i m_i \delta_{x^i}\Big)^r \,:\, m_1, \ldots, m_N \geq 0,\ \sum_i m_i = 1 \right\}.$$

Voronoi diagrams

Given $x^1, \ldots, x^N$, best choice of m_i is via the *Voronoi tessellation* of $x^1, \ldots, x^N$:

$$m_i := \int_{V(x^i)} \rho(y)dy$$

$$V(x^i) := \{y \in \Omega : |y - x^i| \leq |y - x^j| \qquad \text{for all } j \neq i\}$$



With the optimal choice

$$m_i = \int_{V(x^i)} \rho(y) dy$$

it holds

$$W_r\left(\rho, \sum_i m_i \delta_{x^i}\right)^r = F_{N,r}(x^1, \ldots, x^N),$$

where

$$F_{N,r}(x^1, \ldots, x^N) := \int_\Omega \min_{1 \le i \le N} |x^i - y|^r \rho(y)\, dy$$

Goal: Minimize $F_{N,r}$ to find the optimal configuration for $x^1, \ldots, x^N$

Optimal location/Quantization problem: Bourne, Bouchitté, Brancolini, Bucklew, Buttazzo, Fejes Tóth, Graf, Jimenez, Luschgy, Mosconi, Pagès, Peletier, Rajesh, Santambrogio, Stepanov, Theil, Tilli, Wise...

A dynamical approach (Caglioti-Golse-I., M3AS 2015, Ann. IHP 2017)

Given N points $x_0^1, \dots, x_0^N \in \mathbb{R}^d$, consider their evolution under the gradient flow generated by $F_{N,r}$

$$\begin{cases} \left(\dot{x}^1(t), \dots, \dot{x}^N(t)\right) = -\nabla F_{N,r}\left(x^1(t), \dots, x^N(t)\right) \\ \left(x^1(0), \dots, x^N(0)\right) = (x_0^1, \dots, x_0^N) \end{cases}$$

Idea: embed $(\mathbb{R}^d)^N$ in $L^2(\mathbb{R}^d)$, and find a continuous functional $\mathcal{F}[X]$, $X \in L^2(\mathbb{R}^d)$, whose GF well approximate the above ODE for $N \gg 1$.

L^2-GF for $\mathcal{F}[X]$: the 1d case

Let $\Omega = [0,1]$, $X = X(t,\theta)$ with $\theta \in [0,1]$.

The L^2-GF for $\mathcal{F}[X]$ is the following parabolic equation

$$\begin{aligned}\partial_t X(t,\theta) = (r+1)\partial_\theta\big(\rho(X(t,\theta))|\partial_\theta X(t,\theta)|^{r-1}\partial_\theta X(t,\theta)\big)\\ - \rho'(X(t,\theta))|\partial_\theta X(t,\theta)|^{r+1}\end{aligned}$$

with Dirichlet boundary condition

$$X(t,0) = 0, \qquad X(t,1) = 1.$$

From Lagrangian to Eulerian

Define $f \equiv f(t,x)$ by

$$f(t,x)\,dx = X(t,\cdot)_{\#}d\theta \Leftrightarrow f(t,X(t,\theta)) = \frac{1}{\partial_\theta X(t,\theta)}$$

Then

$$\begin{cases} \partial_t f = -r\,\partial_x\left(f\partial_x\left(\frac{\rho}{f^{r+1}}\right)\right), \quad x \in \mathbb{R} \\ f(t,x+1) = f(t,x) \end{cases}$$

Remark: if $\rho \equiv 1$ the Eulerian equation becomes

$$\partial_t f = -(r+1)\,\partial_x^2\left(f^{-r}\right)$$

which is an equation of very fast diffusion type.

Weighted very fast diffusion equations

Given $r > 0$ and $0 < \lambda \leq \rho \leq 1/\lambda$, we want to study the very fast diffusion equation

$$\partial_t f = -r \, \mathrm{div}_x \left(f \nabla_x \left(\frac{\rho}{f^{r+1}} \right) \right)$$

on the d-dimensional torus $\mathbb{T}^d$, with $f \geq 0$.

Goals:

1) Existence;

2) Uniqueness;

3) Asymptotic behavior.

Fast/Very fast diffusion: Bonforte, Carlen, Carrillo, Daskalopulous, Del Pino, Denzler, Dolbeault, Esteban, Grillo, Loss, McCann, Muratori, Nazaret, Rodriguez, Slepčev, Vázquez...

The smooth case

We assume:

(1) $0 < \lambda \leq \rho \leq 1/\lambda$;

(2) ρ is smooth enough (say $\rho \in C^{2,\alpha}(\mathbb{T}^d)$);

(3) $0 < a_0 \leq f(0) \leq A_0$.

In this case, local in time existence and uniqueness of smooth solutions is guaranteed by parabolic regularity theory.

Main questions:

(1) preservation of non-degeneracy;

(2) convergence to equilibrium.

Comparison principle

Remark: Let $m := \rho^{1/(r+1)}$. Then $c\,m(x)$ is a solution for all $c \geq 0$.

Lemma (Caglioti - Golse - I., M3AS 2015)

Let $0 < \lambda \leq \rho \leq 1/\lambda$, $\rho \in C^{2,\alpha}(\mathbb{T}^d)$.
If $c > 0$, *then*

$$\frac{d}{dt}\int_{\mathbb{T}^d}\big(f(t,x) - c\,m(x)\big)_+\,dx \leq 0,$$

$$\frac{d}{dt}\int_{\mathbb{T}^d}\big(f(t,x) - c\,m(x)\big)_-\,dx \leq 0.$$

By the lemma,

$$0 < c_0 m(x) \leq f(0,x) \leq C_0 m(x) \quad \Rightarrow \quad 0 < c_0 m(x) \leq f(t,x) \leq C_0 m(x).$$

Therefore, if $0 < \lambda \leq \rho \leq 1/\lambda$ and $0 < a_0 \leq f(0,x) \leq A_0$,

$$0 < b_0 \leq f(t,x) \leq B_0 \qquad \forall\, t \geq 0.$$

Exponential convergence

Theorem (I., DCDS 2017)

Let $\rho \in C^{2,\alpha}(\mathbb{T}^d)$, $0 < \lambda \leq \rho \leq 1/\lambda$, $0 < a_0 \leq f(0) \leq A_0$, $\int_{\mathbb{T}^d} f(0) = 1$. *Define*

$$\gamma := \frac{1}{\int_0^1 \rho(y)^{1/(r+1)}\, dy}.$$

Then

$$\|f(t) - \gamma\, \rho^{1/(r+1)}\|_{L^2(\mathbb{T}^d)} \leq C_0 e^{-c_0 t},$$

where c_0, C_0 *depend on* λ, a_0, A_0 *only.*

Idea of the proof (1).

Recall that $0 < b_0 \leq f(t) \leq B_0$ for all $t > 0$ (comparison principle).
Let

$$\mathcal{F}[f] := \int_{\mathbb{T}^d} \frac{\rho}{f^r}.$$

Then our PDE is the gradient flow with respect to W_2 of $\mathcal{F}$.
Set

$$F(x,s) := \frac{\rho(x)}{s^r},$$

and

$$G(x,s) := F(x,s) - F\big(x, \gamma\,\rho(x)^{1/(r+1)}\big) - F'\big(x, \gamma\,\rho(x)^{1/(r+1)}\big)\big(s - \gamma\,\rho(x)^{1/(r+1)}\big).$$

Idea of the proof (2).

Then

$$\mathcal{G}[f] := \int_{\mathbb{T}^d} G(x,f) = \mathcal{F}[f] - \frac{1}{\gamma^r}\int_{\mathbb{T}^d} \rho^{1/(r+1)}.$$

Thus our PDE is also the GF of $\mathcal{G}$.

We prove a Gronwall inequality on $\mathcal{G}$:

$$\frac{d}{dt}\mathcal{G}[f] \le -c_0\mathcal{G}[f] \quad \Rightarrow \quad \mathcal{G}[f(t)] \le e^{-c_0 t}\mathcal{G}[f(0)]$$

Also, using $0 < b_0 \le f(t) \le B_0$, we show that

$$\mathcal{G}[f(t)] \approx \|f(t) - \gamma\,\rho^{1/(r+1)}\|_{L^2(\mathbb{T}^d)} \qquad \forall\, t \ge 0.$$

$\square$

The general case

We now want to study the PDE (existence, uniqueness, convergence) under weaker regularity assumptions on $f(0)$ and ρ.

For this, we want to use the GF interpretation of our equation.

Remark: all the following discussion extends also to convex smooth bounded domains with Neumann boundary conditions.

Existence via JKO scheme

Given $f(0)$ and $\tau > 0$, we define

$$f_0^\tau := f(0)$$

and

$$f_{k+1}^\tau := \operatorname{argmin}\left\{ g \mapsto \frac{W_2(g, f_k^\tau)^2}{2\tau} + \mathcal{F}[g] \right\}.$$

Remark: we can rewrite the functional as

$$\mathcal{F}[f] = \int_{\mathbb{T}^d} U\left(\frac{f}{m}\right) m,$$

where

$$U(s) = s^{-r}, \qquad m(x) = \rho(x)^{1/(r+1)}.$$

Discrete maximum principle

Proposition (I. - Patacchini - Santambrogio, in progress)

Assume $c_0 m \leq f_k^\tau \leq C_0 m$. *Then* $c_0 m \leq f_{k+1}^\tau \leq C_0 m$.

Idea of the proof (1).

Step 1: find the Euler-Lagrange equations

$$U'\Big(\frac{f_{k+1}^\tau}{m}\Big) + \frac{\varphi}{\tau} = \mathrm{const},$$

where $T(x) := x - \nabla\varphi(x)$ is the optimal transport map from f_{k+1}^τ to f_k^τ.

Step 2: let $\bar{x}$ be a maximum point of f_{k+1}^τ / m.

Since U' is monotone increasing, $\bar{x}$ is a maximum point of $U'\big(\frac{f_{k+1}^\tau}{m}\big)$.
Thus, by Step 1, it is a minimum point of φ/τ, therefore

$$\nabla\varphi(\bar{x}) = 0, \quad D^2\varphi(\bar{x}) \geq 0.$$

Idea of the proof (2).

Step 3: By the transport condition $T_{\#} f_{k+1}^{\tau} = f_k^{\tau}$ we have

$$\det(\nabla T) = \frac{f_{k+1}^{\tau}}{f_k^{\tau} \circ T}.$$

Since $T(x) = x - \nabla \varphi(x)$, Step 2 and Brenier's Theorem give

$$T(\bar{x}) = \bar{x} - \nabla \varphi(\bar{x}) = \bar{x}, \quad 0 \leq \nabla T(\bar{x}) = \mathrm{Id} - D^2 \varphi(\bar{x}) \leq \mathrm{Id},$$

thus

$$1 = \det(\mathrm{Id}) \geq \det(\nabla T)(\bar{x}) = \frac{f_{k+1}^{\tau}(\bar{x})}{f_k^{\tau}(\bar{x})}.$$

Since by assumption $f_k^{\tau}(\bar{x}) \leq C_0 m(\bar{x})$, we get

$$\max_{\mathbb{T}^d} \frac{f_{k+1}^{\tau}}{m} = \frac{f_{k+1}^{\tau}(\bar{x})}{m(\bar{x})} \leq C_0 \frac{f_{k+1}^{\tau}(\bar{x})}{f_k^{\tau}(\bar{x})} \leq C_0,$$

as desired. □

By the discrete maximum principle, if $c_0 m \leq f(0) \leq C_0 m$ then $c_0 m \leq f_k^\tau \leq C_0 m$ for all k. This means that our functional is not degenerate on these solutions, and we can let $\tau \to 0$ to obtain a continuous gradient flow.

By approximation, under suitable regularity assumptions on m, we obtain existence of solutions also for more general initial data. More precisely, we can prove:

Theorem (I. - Patacchini - Santambrogio, in progress)

Assume $0 < \lambda \leq \rho \leq 1/\lambda$ and $D^2 \log m \leq \Lambda \mathrm{Id}$. Then, for any $f(0) \in L^{r+3}$ with $\mathcal{F}[f(0)] < \infty$ there exists a distributional solution of the PDE obtained as the limit of the JKO scheme.

Also, this solution satisfies

$$\frac{f}{m} \in L^2([0,\infty), H^1(\mathbb{T}^d)), \qquad \left(\frac{f}{m}\right)^{-r} \in L^2([0,\infty), H^1(\mathbb{T}^d)).$$

Uniqueness and L^1 contractivity

The uniqueness of solutions is nontrivial due to the degeneracy of the equation when f is close to 0. To prove uniqueness, we need a "good" notion of solution.

Definition

Given $f(0) \in L^{r+3}(\mathbb{T})$ with $\mathcal{F}[f(0)] < \infty$, we say that f is a weak solution of the PDE if f solves the PDE in the sense of distribution and

$$\frac{f}{m} \in L^2([0,\infty), H^1(\mathbb{T}^d)), \qquad \left(\frac{f}{m}\right)^{-r} \in L^2([0,\infty), H^1(\mathbb{T}^d)).$$

As shown before, weak solutions exist as limit of the JKO scheme.

We can prove the following uniqueness/stability result.

Theorem (I. - Patacchini - Santambrogio, in progress)

Assume $0 < \lambda \leq \rho \leq 1/\lambda$. *Let* f_1, f_2 *be weak solutions to the PDE starting from* $f_1(0), f_2(0) \in L^{r+3}(\mathbb{T}^d)$ *and* $\mathcal{F}[f_1(0)], \mathcal{F}[f_2(0)] < \infty$. *Then both*

$$t \mapsto \int_{\mathbb{T}^d} (f_1(t,x) - f_2(t,x))_+ \, dx$$

and

$$t \mapsto \int_{\mathbb{T}^d} (f_1(t,x) - f_2(t,x))_- \, dx$$

are decreasing in time.
In particular, weak solutions are unique and they are obtained as limit of the JKO scheme.

Regularization and convergence

By Moser's iteration techniques, we can prove that weak solutions become immediately bounded and positive if $f(0) \in L^q$ with $q > (r+1)d/2$.

Theorem (I. - Patacchini - Santambrogio, in progress)

Assume $0 < \lambda \leq \rho \leq 1/\lambda$, let f be a weak solutions to the PDE with $f(0) \in L^{r+3}(\mathbb{T})$ and $\mathcal{F}[f(0)] < \infty$. Assume in addition that $f(0) \in L^q(\mathbb{T}^d)$ for some $q > (r+1)d/2$. Then, for any $t_0 > 0$,

$$0 < c(t_0) \leq f(t) \leq C(t_0) \qquad \forall\, t \geq t_0.$$

In particular, after waiting a positive time $t_0 > 0$, we can apply the argument from I.-DCDS'17 to obtain exponential convergence to equilibrium.

Thanks for your attention