Why in some cases the asymptotic linearized problem yields optimal results for a nonlinear version of the “carré du champ"

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Symmetry, symmetry breaking and nonlinear flows

In this talk I will discuss symmetry and symmetry breaking issues for the positive solutions of equations like

$$-\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w\right)=|x|^{-\gamma}\left(w^{2p-1}-w^p\right)\quad\text{in}\quad\mathbb{R}^d\setminus\{0\}\,,$$

Alternatively, we could consider the equation

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Alternatively, we could consider the equation

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We will see that this discussion about symmetry is strongly linked to the use of nonlinear flows and entropies.

- Elliptic approach (DEL, Invent 2016; DELM, CRAS, 2017).

- Parabolic approach (DEL, J. Ell. Parab. Eqs 2016).

- Linearization around symmetric solutions and optimality (DEL, J. Ell. Parab. Eqs 2016).

Caffarelli-Kohn-Nirenberg (CKN) critical and subcritical inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|w|^{2p}}{|x|^\gamma}\,dx\right)^{\frac{1}{2p}} \le \mathcal{C}_{\beta,\gamma,p}\left(\int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^\beta}\,dx\right)^{\frac{\theta}{2}}\left(\int_{\mathbb{R}^d} \frac{|w|^{p+1}}{|x|^\gamma}\,dx\right)^{\frac{1-\theta}{p+1}}$$

$$\gamma-2<\beta<\frac{d-2}{d}\,\gamma\,,\quad \gamma\in(-\infty,d)\,,\quad p\in(1,p_\star]\,,$$

$$p_\star:=\frac{d-\gamma}{d-\beta-2}\quad\text{and}\quad \vartheta=\frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}\,.$$

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When $p=p_\star=q/2$, $\theta=1$, the above inequality becomes

$$\left(\int_{\mathbb{R}^d}\frac{|w|^q}{|x|^{b\,q}}\,dx\right)^{2/q}\le\mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla w|^2}{|x|^{2\,a}}\,dx$$

with $a\le b\le a+1$ if $d\ge3\,,\qquad a<b\le a+1$ if $d=2\,,\qquad a\neq\frac{d-2}{2}$

$$q=\frac{2\,d}{d-2+2\,(b-a)}$$

The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|w|^q}{|x|^{b\,q}}\,dx\right)^{2/q} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2\,a}}\,dx$$

$\mathsf{C}_{a,b}$ = best constant for general functions w
$\mathsf{C}^*_{a,b}$ = best constant for radially symmetric functions w

$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

$$w^*_{a,b}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

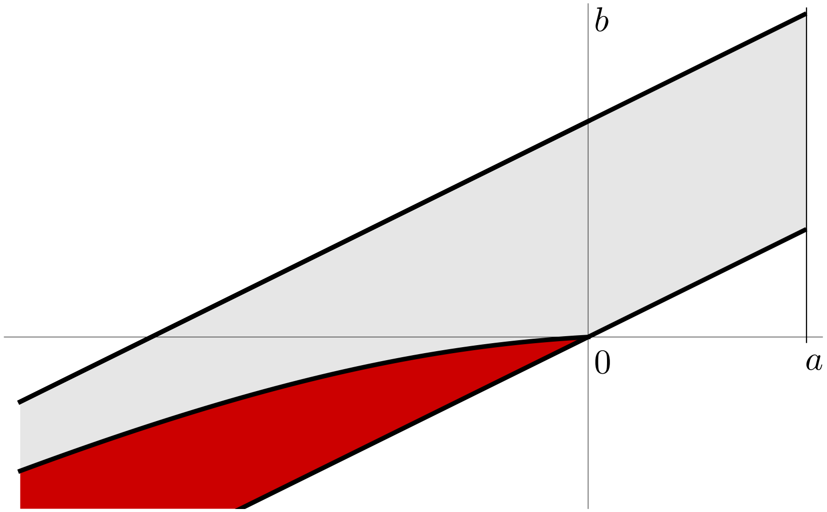
Question: is optimality (equality) achieved ? do we have $w_{a,b} = w^*_{a,b}$?

Linear instability of radial minimizers: the Felli-Schneider curve

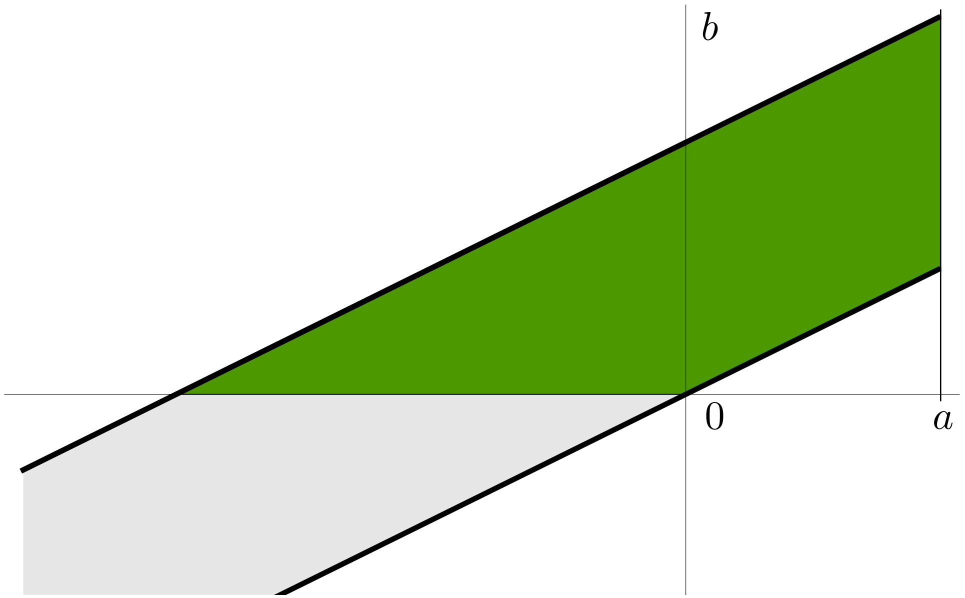
Looking for the set of pairs (a, b) such that the functional

$$\mathsf{C}^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|w|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $w = w^*_{a,b}$ (Catrina, Wang; Felli, Schneider).



The symmetry region:



Chou, Chu; Horiuchi ($a > 0$)

Betta, Brock, Mercaldo, Posteraro ($a < 0, b > 0$)

Perturbation results: C-S Lin, Z-Q Wang; Smets, Willem ; Dolbeault, E., Tarantello (2007 & 2009)

The conjecture

The minimizers are radially symmetric outside of the Felli-Schneider zone.

That is,

The symmetric minimizers are global minimizers whenever they are stable / whenever they are local minimizers.

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ANSWER: YES (Dolbeault, E., Loss, Inv. 2016).

Define

$$\alpha = \sqrt{\frac{d-1}{n-1}} =: \alpha_{\,\mathrm{FS}} \qquad \text{(Stability zone defined by } \alpha \le \alpha_{\mathrm{FS}}\text{)}$$

THEOREM [2016].- If $\alpha \le \alpha_{\,\mathrm{FS}}$ and $d \ge 2$, optimality is achieved by radial functions.

Idea of the proof : With the change of variables : $r \mapsto r^\alpha$, $w(r,\omega) = v(r^\alpha,\omega)$, and with

$$n = \frac{d - b\,p}{\alpha} = \frac{d - 2\,a - 2}{\alpha} + 2\,, \qquad \mathsf{D}v = \left(\alpha\,\frac{\partial v}{\partial r}, \frac{1}{r}\,\nabla_\omega v\right)$$

$p = \frac{2n}{n-2}$ and the CKN the inequality becomes

$$\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^d} |v|^p\,d\mu_n\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d} |\mathsf{D}v|^2\,d\mu_n\,, \quad d\mu := r^{n-1}\,dr\,d\omega \text{ “} = dx\,(\mathbb{R}^n)\text{”}$$

Resolution of the conjecture: A Sobolev type inequality

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$$n=\frac{d-b\,p}{\alpha}=\frac{d-2\,a-2}{\alpha}+2\,,\qquad \mathsf{D}v=\left(\alpha\,\frac{\partial v}{\partial r},\frac1r\,\nabla_\omega v\right)$$

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The parameters α and n vary in the ranges $0<\alpha<\infty$ and $d<n<\infty$ and the *Felli-Schneider curve* in the (α,n) variables is given by $\alpha=\alpha_{\rm FS}$.

Notations

If ∇_ω denotes the gradient with respect to the angular variables $\omega \in S^{d-1}$ and Δ_ω is the Laplace-Beltrami operator on S^{d-1}, we define

$$\mathsf{D}v = \left(\alpha \, \frac{\partial v}{\partial r}, \frac{1}{r} \, \nabla_\omega v \right) \; ,$$

we define the self-adjoint operator L by

$$\mathrm{L}v := -\,\mathsf{D}^* \, \mathsf{D}\, v = \alpha^2 \, v'' + \alpha^2 \, \frac{n-1}{r} \, v' + \frac{\Delta_\omega \, v}{r^2}$$

The fundamental property of L is the fact that

$$\int_{\mathbb{R}^d} v_1 \, \mathrm{L}v_2 \, d\mu_n = -\int_{\mathbb{R}^d} \mathsf{D}v_1 \cdot \mathsf{D}v_2 \, d\mu_n \quad \forall \, v_1 , \, v_2 \in \mathcal{D}(\mathbb{R}^d)$$

$\triangleright$ Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$

Fisher information decay and a fast diffusion equation

Let $u = |v|^p$, $\quad p = \frac{2\,n}{n-2}$.

Up to multiplicative constants, $\int_{\mathbb{R}^d} |v|^p \, d\mu_n = \int_{\mathbb{R}^d} u \, d\mu_n$, and $\int_{\mathbb{R}^d} |\mathsf{D}v|^2 \, d\mu_n = \mathcal{I}[u]$, with

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u \, |\mathsf{Dp}|^2 \, d\mu_n \,, \quad \mathsf{p} = \frac{m}{1-m} \, u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}$$

Here $\mathcal{I}$ is the *Fisher information* and p is the *pressure function.*

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Next, define the fast diffusion equation (flow)

$$\frac{\partial u}{\partial t} = \mathrm{L} u^m \,, \quad m = 1 - \frac{1}{n}$$

▷ STRATEGY: Assume that $\alpha \le \alpha_{\mathrm{FS}}$,

1) prove that for all $t \ge 0$, $\frac{d}{dt} \int_{\mathbb{R}^d} u(t) \, d\mu_n = 0$ and $\frac{d}{dt} \mathcal{I}[u(t,\cdot)] \le 0$,

2) prove that $\frac{d}{dt} \mathcal{I}[u(t,\cdot)] = 0$ means, in particular, that u is radially symmetric.

Mass conservation and Fisher information decay along the fast diffusion flow

Easy to see: the mass $\int_{\mathbb{R}^d} u\, d\mu_n$ is conserved along the flow.

$$\text{With} \quad \mathsf{p} = \frac{m}{1-m}\, u^{m-1}\,, \qquad \mathcal{I}[u] := \int_{\mathbb{R}^d} u\, |\mathsf{D}\mathsf{p}|^2\, d\mu_n\,,$$

Some calculations: Let $u_0 \geq 0$. Up to estimates near the origin and near infinity,

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\,2\,(n-1)^{n-1} \int_{\mathbb{R}^d} \mathcal{K}[\mathsf{p}]\,\mathsf{p}^{1-n}\, d\mu_n\,,$$

with $d\mu_n = r^{n-1}\, dx$ and $\zeta_\star > 0$,

$$\mathcal{K}[\mathsf{p}] := \alpha^4 \left(1-\frac{1}{n}\right)\left[\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_\omega\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla_\omega \mathsf{p}' - \frac{\nabla_\omega \mathsf{p}}{r}\right|^2$$
$$+\,\frac{1}{r^4}\left((n-2)\,(\alpha_{\mathrm{FS}}^2 - \alpha^2)\int_{S^d} |\nabla_\omega \mathsf{p}|^2\,\mathsf{p}^{1-n}\,d\omega + \zeta_\star\,(n-d)\int_{S^d} |\nabla_\omega \mathsf{p}|^4\,\mathsf{p}^{1-n}\,d\omega\right).$$

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So, if $\alpha \leq \alpha_{\mathrm{FS}}$, $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \leq 0$, and $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = 0$ implies that p is radially symmetric.

Elliptic proof for rigidity (uniqueness of positive solutions) if $\alpha \le \alpha_{\rm FS}$

If $\alpha \le \alpha_{\rm FS}$ and if p_0 is a critical point of the E-L equations for CKN, written in the good variables, then

$$\frac{\partial}{\partial t}\,\mathcal{I}[u(t)]_{|t=0} = \mathcal{I}'[\mathsf{u}(t)]\cdot\frac{\partial}{\partial t}\,u(t)_{|t=0} = \mathcal{I}'[\mathsf{u}_0]\cdot \mathrm{L}u_0^m = 0 = -C\int_{\mathbb{R}^d}\mathcal{K}[\mathsf{p}_0]\,\mathsf{p}_0^{1-n}\,d\mu_n\;,\quad \mathsf{p}_0 = \frac{m\,u_0^{m-1}}{1-m}$$

$$0 \;=\; \mathcal{K}[\mathsf{p}_0] \;\ge \int_{\mathbb{R}^d}\alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{p}_0'' - \frac{\mathsf{p}_0'}{r} - \frac{\Delta_\omega\,\mathsf{p}_0}{\alpha^2\,(n-1)\,r^2}\right]^2\mathsf{p}_0^{1-n}\,d\mu_n$$

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where $\zeta_\star > 0$ and $n > d$.

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where $\zeta_\star > 0$ and $n > d$.

So, $\nabla_\omega\mathsf{p}_0 \equiv 0$, that is, p_0 does not depend on ω, which means radial symmetry.

Moreover, $\mathsf{p}_0''-\frac{\mathsf{p}_0'}{r}-\frac{\Delta_\omega\,\mathsf{p}_0}{\alpha^2\,(n-1)\,r^2}\equiv 0$, which implies that for some $a, b > 0$, $\mathsf{p}_0 = a + b\,r^2$.

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In the case of subcritical CKN inequalities, the method has to be modified, and Renyi entropies shown to be concave:

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} u^m\,d\mu_n\;,\;\mathcal{E}' = (1-m)\mathcal{I}\,,\;\mathcal{R}[u] := \left(\int_{\mathbb{R}^d} u^m\,d\mu_n\right)^\sigma\;,\;\sigma = \frac{1}{d(1-m)}-1$$

Disadvantages of this approach

– Painful estimates of boundary terms in integrations by parts.

– No way to obtain improved inequalities from the remainder terms.

– No clear understanging of why a local stability result for the symmetric solutions yields a global result (non existence of other positive solutions apart from the symmetric ones, when these are stable).

Alternative parabolic approach in self-similar variables

With $\mu = 2 + n\,(m-1)$ and $\kappa = \left(\frac{2\,m}{1-m}\right)^{1/\mu}$, let

$$u(t,x) = \frac{1}{\kappa^n\,R^n}\,g\Big(\tau,\frac{x}{\kappa\,R}\Big) \quad \text{where} \quad \begin{cases} \frac{dR}{dt} = R^{1-\mu}\,, \quad R(0) = R_0 = \kappa^{-1}\,, \\ \tau(t) = \frac{1}{2}\,\log\left(\frac{R(t)}{R_0}\right)\,. \end{cases} \tag{0}$$

In self-similar variables the function g solves

$$\frac{\partial g}{\partial \tau} = \mathsf{D}^*\,(g\,z) \tag{0}$$

where, with the notation $\mathcal{B}_\alpha(x) := \left(1 + \frac{|x|^2}{\alpha^2}\right)^{\frac{1}{m-1}}$,

$$z(\tau,x) := \mathsf{D} g^{m-1} - \frac{2}{\alpha}\,x = \mathsf{D}\left(g^{m-1} - \frac{|x|^2}{\alpha^2}\right) = \mathsf{Dq}\,, \quad \mathsf{q} := g^{m-1} - \mathcal{B}_\alpha^{m-1}\,.$$

The exponent m is now in the range $m_1 \le m < 1$ with $m_1 = 1 - 1/n$.

Bakry-Emery type calculation

For any $R >$, let us consider the solution of the no-flux boundary problem

$$\frac{\partial g}{\partial \tau} = \mathsf{D}^* (g\, z) \ \text{ in } \ B_R\,; \quad z \cdot \omega = 0 \quad \text{on} \quad \partial B_R\,.$$

and suppose that g is smooth at the origin. Then, by defining $\mathsf{p} := g^{m-1}$, computing in B_R and taking the limit $R \to +\infty$, we get

$$\frac{d}{d\tau} \int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n + 4 \int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n$$

$$\le -2\,\frac{1-m}{m} \int_{\mathbb{R}^d} \left(\alpha^4 \left(1-\tfrac{1}{n}\right) \left[\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_\omega\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + \frac{2\,\alpha^2}{r^2} \left|\nabla_\omega \mathsf{p}' - \frac{\nabla_\omega \mathsf{p}}{r}\right|^2 \right) g^m\,d\mu_n$$

$$-\ 2\,\frac{1-m}{m}\,(m-m_1) \int_{\mathbb{R}^d} \left(\mathsf{L}_\alpha g^{m-1} - 2\,n\right)^2\,g^m\,d\mu_n$$

$$-\ 2\,\frac{1-m}{m} \int_{\mathbb{R}^d} \frac{\mathsf{Q}[\mathsf{p}]}{r^4}\,g^m\,d\mu_n - 2\,\frac{1-m}{m}\,(n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right) \int_{\mathbb{R}^d} \frac{|\nabla_\omega \mathsf{p}|^2}{r^4}\,g^m\,d\mu_n$$

$$\le 0 \quad \text{when} \quad \alpha \le \alpha_{\rm FS}\,.$$

Improved inequalities

For all $x \in \mathbb{R}^d$, $t > 0$ let

$$v_\star(t,x) := \frac{1}{\kappa^n\,(\mu\,t)^{n/\mu}}\,\mathcal{B}_\star\left(\frac{x}{\kappa^{\mu/\mu_\star}\,(\mu\,t)^{1/\mu_\star}}\right) \quad \text{where} \quad \mathcal{B}_\star(x) := \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(1-m)}$$

$$\mathsf{G}[v] := \left(\int_{\mathbb{R}^d} v^m\,|x|^{-\gamma}\,dx\right)^{\sigma-1}\int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,|x|^{-\beta}\,dx$$

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THEOREM.- Define $\mathsf{h}(t) := \left(1+\frac{2\,m}{1-m}\,\mu\,t\right)^{1/\mu} \quad \forall\, t \ge 0\,,$ with $\mu = 2\,\frac{2+\beta-d+m\,(d-\gamma)}{2+\beta-\gamma}\,,$

$$\mathsf{G}[v(t,\cdot)] - \mathsf{G}[v_\star] \ge \mathcal{C}\int_t^\infty \mathsf{h}(s)^{3\,\mu-2} \int_{\mathbb{R}^d} v^m(s,x)\,\frac{|\nabla_\omega v^{m-1}(s,x)|^2}{|x|^4}\,|x|^{\gamma-2\beta}\,dx\,ds \quad \forall\, t \ge 0\,.$$

if $\alpha \le \alpha_{\rm FS}$, if v is smooth at the origin, if $||v_0||_{1,\gamma} = ||\mathcal{B}_\star||_{1,\gamma}$, and if

$$\left(C_1+|x|^{2+\beta-\gamma}\right)^{-1/(1-m)} \le v_0(x) \le \left(C_2+|x|^{2+\beta-\gamma}\right)^{-1/(1-m)} \quad \forall\, x \in \mathbb{R}^d\,,$$

(REMARK.- Different, and better, remainder terms in the critical and in the subcritical case).

Linearization and optimality

Let us linearize the equation $\frac{\partial g}{\partial \tau} = \mathsf{D}^* \, (g \, z)$ around a Barenblatt profile $\mathcal{B}_\alpha$, by taking a solution g_ε s.t. $\int_{\mathbb{R}^d} g_\varepsilon = M_\star = \int_{\mathbb{R}^d} \mathcal{B}_\alpha$, $g_\varepsilon = \mathcal{B}_\alpha \left(1 + \varepsilon \, f \, \mathcal{B}_\alpha^{1-m} \right)$. Taking ε to 0 we find

$$\frac{\partial f}{\partial t} = \mathcal{L}_\alpha \, f \quad \text{where} \quad \mathcal{L}_\alpha \, f := (m-1) \, \mathcal{B}_\alpha^{m-2} \, \mathsf{D}^* \, \left(\mathcal{B}_\alpha \, \mathsf{D} f \right) \, .$$

We define the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1 \, f_2 \, \mathcal{B}_\alpha^{2-m} \, d\mu_n \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D} f_1 \cdot \mathsf{D} f_2 \, \mathcal{B}_\alpha \, d\mu_n$$

and its corresponding Hilbert spaces, X and Y ($Y \subset X$). We see that

$$\frac{1}{2} \, \frac{d}{dt} \langle f, f \rangle = - \, \langle\!\langle f, f \rangle\!\rangle \, ; \quad \frac{1}{2} \, \frac{d}{dt} \langle\!\langle f, f \rangle\!\rangle = - \, \langle\!\langle f, \mathcal{L}_\alpha \, f \rangle\!\rangle$$

and if λ_1 is the smallest positive eigenvalue of $\mathcal{L}_\alpha$, with $\mathcal{L}_\alpha \, f_1 = \lambda_1 \, f_1$, then it has been proved by Bonforte-Dolbeault-Muratori-Nazaret that $f_1 \in Y \subset X$. Moreover $\lambda_1 \geq 4$ iff $\alpha \leq \alpha_{\mathrm{FS}}$.
A simple expansion of a square tells us that λ_1 is also optimal in the inequality

$$- \, \langle\!\langle g, \mathcal{L}_\alpha \, g \rangle\!\rangle \geq \lambda_1 \, \langle\!\langle g, g \rangle\!\rangle \, , \quad \forall g, \ \text{s.t.} \ \langle\!\langle g, 1 \rangle\!\rangle = 0 \quad \text{(Hardy-Poincaré type inequality)} \, .$$

Define $\mathcal{I}[g] := \int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n\ ;\quad \frac{d}{d\tau}\mathcal{I}[g] = -\mathcal{K}[g]$

Since for $\alpha \le \alpha_{\rm FS}$, $-\mathcal{K}[g] + 4\,\mathcal{I}[g] = \frac{d}{d\tau}\mathcal{I}[g] + 4\,\mathcal{I}[g] = \frac{d}{d\tau}\int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n + 4\int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n \le 0$

the functional $g \mapsto \frac{\mathcal{K}[g]}{\mathcal{I}[g]} - 4$ is nonnegative and if $\alpha \le \alpha_{\rm FS}$, its minimizer is $g = \mathcal{B}_\alpha$.

Moreover, with $g_\varepsilon = \mathcal{B}_\alpha\left(1 + \varepsilon\,f\,\mathcal{B}_\alpha^{1-m}\right)$,

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[g]}{\mathcal{I}[g]} \le \liminf_{\varepsilon\to 0}\,\inf_f \frac{\mathcal{K}[g_\varepsilon]}{\mathcal{I}[g_\varepsilon]} = \inf_f \frac{\langle\langle f, \mathcal{L}_\alpha\,f\rangle\rangle}{\langle\langle f, f\rangle\rangle} = \frac{\langle\langle f_1, \mathcal{L}_\alpha\,f_1\rangle\rangle}{\langle\langle f_1, f_1\rangle\rangle} = \lambda_1\ .$$

Link between the nonlinear problem and the asymptotic linearized problem

$$\text{Define}\quad \mathcal{I}[g] := \int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n\ ;\qquad \frac{d}{d\tau}\mathcal{I}[g] = -\mathcal{K}[g]$$

Since for $\alpha \le \alpha_{\rm FS}$, $\quad -\mathcal{K}[g] + 4\,\mathcal{I}[g] = \frac{d}{d\tau}\mathcal{I}[g] + 4\,\mathcal{I}[g] = \frac{d}{d\tau}\int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n + 4\int_{\mathbb{R}^d} g\,|z|^2\,d\mu_n \le 0$

the functional $g \mapsto \frac{\mathcal{K}[g]}{\mathcal{I}[g]} - 4$ is nonnegative and if $\alpha \le \alpha_{\rm FS}$, its minimizer is $g = \mathcal{B}_\alpha$.
Moreover, with $g_\varepsilon = \mathcal{B}_\alpha\left(1 + \varepsilon\,f\,\mathcal{B}_\alpha^{1-m}\right)$,

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[g]}{\mathcal{I}[g]} \le \liminf_{\varepsilon\to 0}\,\inf_f \frac{\mathcal{K}[g_\varepsilon]}{\mathcal{I}[g_\varepsilon]} = \inf_f \frac{\langle\!\langle f, \mathcal{L}_\alpha\, f\rangle\!\rangle}{\langle\!\langle f, f\rangle\!\rangle} = \frac{\langle\!\langle f_1, \mathcal{L}_\alpha\, f_1\rangle\!\rangle}{\langle\!\langle f_1, f_1\rangle\!\rangle} = \lambda_1\,.$$

Summarizing, the infimum of $\mathcal{K}/\mathcal{I}$ is achieved in the asymptotic regime as $g \to \mathcal{B}_\alpha$ and determined by the spectral gap of $\mathcal{L}_\alpha$ when $\lambda_1 = 4$. And $\mathcal{K}/\mathcal{I} > 4$ if $\lambda_1 > 4$, that is, when $\alpha < \alpha_{\rm FS}$.
Finally,

$$\text{If}\quad \alpha > \alpha_{\rm FS},\quad \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \lambda_1 < 4$$

$$\text{If}\quad \alpha \le \alpha_{\rm FS},\quad 4 \le \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \lambda_1\,.$$

Thank you for your attention!