Fluctuations for point vortex models

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Regularity and blow-up of Navier-Stokes type PDEs using harmonic and stochastic analysis
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[F. Grotto (Scuola Normale Superiore), C. Geldhauser (Chebychev Institute)]

Consider the Euler equation

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

with either

- periodic boundary conditions on the 2D torus, or
- Dirichlet boundary conditions on a bounded regular domain.

Here

- u is the velocity, $u = \nabla^{\perp}\psi$,
- ψ is the stream function, with $-\Delta\psi = \omega$

The equation has a infinite number of conserved quantities. The most relevant are

- kinetic energy $\int |u|^2\, dx$,
- enstrophy $\int |\omega|^2\, dx$.

A (exact) measure valued solution is given by point vortices,

$$\omega_N = \sum_{j=1}^{N} \xi_j \delta_{X_j}$$

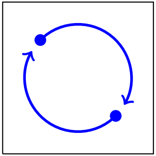
where the point vortex positions evolve according to

$$\dot{X}_k = \sum_{j \neq i} \xi_j (\nabla^\perp G)(X_k - X_j).$$

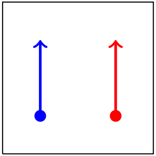
Here G is the Green function, $\nabla^\perp G$ is the Biot-Savart kernel, and the **self-interaction** has been neglected.
This is a Hamiltonian system with Hamiltonian

$$H_N(x_1, x_2, \ldots, x_N) = \frac{1}{2} \sum_{j \neq k} \xi_j \xi_k G(x_j - x_k)$$

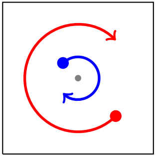
with invariant distribution $\mu_{\beta,N} = Z_{\beta,N}^{-1} \, \mathrm{e}^{-\beta H_N(x)} \, dx$.



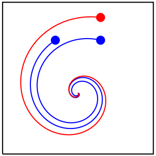
Positive vortices of equal intensity



non concordant vortices of equal intensity



Vortices of different intensity



Three vortices – collapse

Onsager's theory (in a nutshell)

Onsager's theory is a statistical theory of the formation of large scale vortex structures in 2D turbulence, where vorticity is replaced by a dilute gas of point vortices.

- Not relevant for homogeneous turbulence.
- Relevant at large scales (so viscosity≈ 0 and Euler).
- long time distribution of point vortices governed by equilibrium statistics

In $\mathbb{R}^2$,

$$H_N(x) = -\frac{1}{4\pi}\sum_{j\neq k}\xi_j\xi_k \log|x_j - x_k|$$

therefore,

- at $\beta > 0$: attractive for non concordant vortices,
- at $\beta < 0$: attractive for concordant vortices

The energy spectrum, averaged over $\mu_{\beta,N}$ has the leading order

$$E(k) \sim \frac{\xi^2 N}{|k|}$$

(due to self–interaction, yields infinite kinetic energy).
To obtain finite energy and a non–trivial limit one expects

$$\xi N \sim 1, \qquad \beta/N \sim 1,$$

and the corresponding Gibbs measure is

$$\frac{1}{Z_{\beta,N}} \, \mathrm{e}^{-\frac{\beta}{N}\sum_{j\neq k}\xi_j\xi_k G(x_j - x_k)} \, dx_1 \, dx_2 \, \ldots \, dx_N$$

of mean field type. The vorticity

$$\frac{1}{N}\sum_j \xi_j \delta_{X_j}.$$

obeys a LLN.

[Fröhlich, Ruelle] [Caglioti, Lions, Marchioro, Pulvirenti]

Consider the mean field model with random intensities

$$\mu_{\beta,N} = \frac{1}{Z_{\beta,N}} \mathrm{e}^{-\frac{\beta}{2N}\sum_{j\neq k}\xi_j\xi_k G(x_j-x_k)}\, d\ell^{\otimes N} d\nu^{\otimes N}$$

in the **neutral** case (only for Dirichlet boundary conditions),

$$\mathbb{E}_\nu[\xi] = 0.$$

with ν probability supported on a bounded interval K.
We have that

- The partition function $\log Z_{\beta,N} \approx N$,
- Finite dimensional distributions ρ_k^N are bounded in L^p for all p.
- Existence of limit points that are (by exchangeability) mixture of independent vortices: $\int \rho^{\otimes \mathbb{N}} \pi(d\rho)$.

[Joyce, Montgomery] [Bodineau, Guionnet] [Kiessling] [Neri]

Consider the free energy for probabilities on $\mathbb{T}_2 \times K$,

$$F(\rho) = \frac{\beta}{2} \int \int H(\xi_1, x_1, \xi_2, x_2) \rho(\xi_1, x_1) \rho(\xi_2, x_2) + \int \rho(\xi, x) \log \rho(\xi, x)$$

then

- π is supported over minimizers of F,
- ρ solves the **mean field equation**

$$\rho(\xi, x) = \frac{e^{-\beta \xi \psi(x)}}{\int \int e^{-\beta \xi \psi(x)}\, dx\, \nu(d\xi)},$$

  where ψ is the averaged (in ξ) stream function for ρ.
- $\beta > 0$ (or $\beta < 0$ small enough): F has a unique minimizer $\rightsquigarrow$ propagation of chaos.
- $\beta < 0$: in general non-unique minimizers.

Large deviations

A large deviation principle holds for the distribution of vortices,

$$\frac{1}{N}\sum_j \delta_{\xi_j, X_j}$$

with speed N and rate function

$$\mathcal{F}(\mu) = \mathcal{E}(\mu|\mathsf{Leb}^{\otimes N}\otimes\nu^{\otimes N}) + \frac{\beta}{2}\int\int \xi\xi' G(x,x')\mu(dx\, d\xi)\mu(dx'\, d\xi')$$

Central limit theorem

In the special case of a disk, Bernoulli $\pm\xi_0$ neutral intensities, and $\beta > 0$, the central limit theorem holds (for a restricted class of observables) with limit Gaussian measure

$$\frac{1}{Z}\,\mathrm{e}^{-\text{Enstrophy}/\xi_0^2 - \beta\text{Kinetic energy}}$$

[Bodineau, Guionnet]

Theorem

Assume

- *either periodic boundary conditions on the torus,*
- *or Dirichlet boundary conditions on a bounded regular domain,*

and

- $\beta > 0$,
- *Bernoulli* $\pm\xi_0$ *(neutral) intensities.*

If $\beta\xi_0^2$ *is small enough then the Central limit theorem holds with limit Gaussian measure*

$$\frac{1}{Z}\,\mathrm{e}^{-\textit{Enstrophy}/\xi_0^2-\beta\textit{Kinetic energy}}$$

[R., Grotto]

- why $\beta > 0$ and neutral case. Recall the mean field equation:

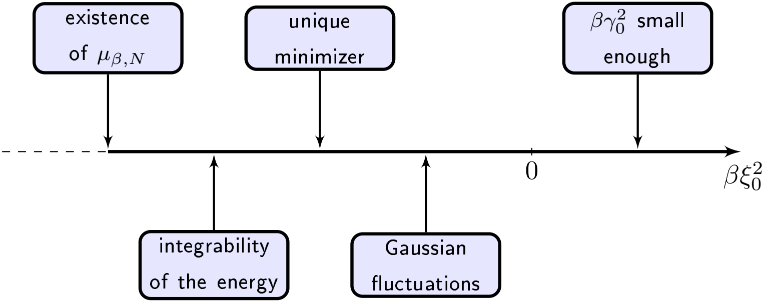
$$\rho(\xi, x) = \frac{\mathrm{e}^{-\beta\xi\psi(x)}}{\int\int \mathrm{e}^{-\beta\xi\psi(x)}\,dx\,\nu(d\xi)}.$$

- The proof is based on two main ideas,
  - **Gaussian integration**: the exponential in $\mu_{\beta,N}$ reformulated as a expectation wrt to a mean zero centred Gaussian random field with covariance βG,

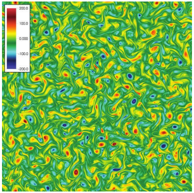
$$\mathrm{e}^{-\frac{\beta}{2N}\sum_{i\neq j}\xi_j\xi_k G(X_j,X_k)} = \mathrm{e}^{\frac{1}{2}\beta\xi_0^2 G(0,0)}\,\mathbb{E}_\phi\left[\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{N}}\sum_j \xi_j\phi(X_j)}\right]$$

  - **Spectral decomposition**: $G = G_R + G_S$.
- The condition on $\beta\xi_0^2$ due to a poor estimate of the partition function corresponding to G_S.

- Mean field limit for every β (Lions et al.)
- CLT (unconditional for the torus, in the neutral case in a bounded domain) for $\beta > 0$.
- Open problems:
  - non-neutral circulations (bounded domain)
  - $\beta < 0$?



- LLN gives (deterministic) stationary solutions,
- CLT gives (statistical) stationary solutions
- Connections with the Gaussian invariant measures of Albeverio-Cruzeiro.
- Connections with other turbulent regimes: re-interpret the vorticity as $\sum_i \frac{\xi_i}{\sqrt{N}} \delta_{X_i}$.
- CLT gives universality of fluctuations.
- Other models with similar features (point vortices, inverse cascade),
  - Euler equation,
  - surface quasi-geostrophic,
  - (a version of) plasma turbulence equation.



[Bernard, Boffetta, Celani, Falkovich]

We look at a slightly more general version of the model on the torus

$$\partial_t \theta + v \cdot \nabla \theta = 0,$$

with $v = \nabla^{\perp} \psi$, and

$$\psi(t, x) = \int G(x - y) \theta(y) \, dy,$$

with $G_{\mathbf{k}} = |\mathbf{k}|^{-m}$, thus $G(x - y) \sim |x - y|^{m-2}$ for $m < 2$.

- $m = 2$ Euler equation,
- $m = 1$ surface quasi-geostrophic,
- $m = -2$ plasma turbulence.

Two conserved quantities

- $\sum |\theta_{\mathbf{k}}|^2 \quad \rightsquigarrow \quad \int |\theta(x)|^2$
- $\sum |\mathbf{k}|^{-m} |\theta_{\mathbf{k}}|^2 \quad \rightsquigarrow \quad \int \theta(x) \psi(x)$

A similar vortex dynamics

$$\dot{X}_k = \sum_{j \neq k} \xi_j (\nabla^\perp G)(X_j - X_k)$$

Interaction too singular. Replace G by G_ϵ,

$$G_{\epsilon,\mathbf{k}} = \frac{\mathrm{e}^{-\epsilon|\mathbf{k}|^2}}{|\mathbf{k}|^m}$$

At finite ϵ one can obtain,

- existence of a limit distribution of a infinite number of vortices, as $N \to \infty$,
- propagation of chaos for $\beta > 0$ (or $\beta < 0$ and small),
- characterization of limit points as solutions of a mean field equation,
- and as minima of the free energy,

$$\frac{1}{2}\|(-\Delta)^{\frac{m}{4}}\, \mathrm{e}^{-\frac{1}{2}\epsilon(-\Delta)}\, \psi\|_{L^2}^2 + \frac{1}{\beta}\log\Big(\int \mathrm{e}^{-\beta\xi\psi(x)\, dx\, \nu(d\xi)}\Big)$$

Consider $\beta > 0$, and recall the empirical pseudo–vorticity

$$\bar{\theta} = \frac{1}{N} \sum_{j=1}^{N} \xi_j \delta_{X_j}.$$

Theorem

There is a choice $\epsilon = \epsilon(N)$ such that $\epsilon(N) \downarrow 0$ as $N \uparrow \infty$ and

- *Propagation of chaos holds and the law of (X, ξ) converges to $Leb_{\mathbb{T}_2} \otimes \nu$.*
- *(LLN) $\bar{\theta}$ converges in probability to 0,*
- *(CLT) $\sqrt{N}\bar{\theta}$ converges in law to a Gaussian distribution with covariance $(\gamma_0 I + \beta(-\Delta)^{-\frac{m}{2}})^{-1}$.*

Here $\gamma_0 = 1/\mathbb{E}_\nu[\xi^2]$, where the expectation is computed with respect to the prior ν.

[R., Geldhauser]

A few ideas

- It is sufficient to prove convergence on exponential functionals $\mathbb{E}_{\mu_{\beta,\epsilon,N}}[\mathrm{e}^{\mathrm{i}\langle\sqrt{N}\bar{\theta},f\rangle}]$
- The exponential in $\mu_{\beta,\epsilon,N}$ reformulated as a expectation wrt to a mean zero centred Gaussian random field with covariance βG_ϵ,

$$\mathrm{e}^{-\frac{\beta}{2N}\sum\xi_j\xi_k G_\epsilon(X_j,X_k)} = \mathrm{e}^{\frac{1}{2}\beta\xi_0^2 G_\epsilon(0,0)}\,\mathbb{E}_\phi\left[\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{N}}\sum_j \xi_j\phi(X_j)}\right]$$

- Taylor expansion of the exponential in terms of the small parameter $N^{-1/2}$ yields at leading order

$$\mathbb{E}_{\mu_{\beta,\epsilon,N}}[\mathrm{e}^{\mathrm{i}\langle\sqrt{N}\bar{\theta},f\rangle}] \sim \frac{1}{Z_{\beta,\epsilon,N}}\int \mathbb{E}_\phi\left[\mathrm{e}^{\frac{1}{2}\Gamma_N\|f+\phi\|_{L^2}^2}\right]d\nu^{\otimes N} + \mathtt{error}(\epsilon,N),$$

where $\Gamma_N = \frac{1}{N}\sum\xi_j^2$.

The $\epsilon = \epsilon(N)$ is chosen so to have $\mathtt{error}(\epsilon,N)\to 0$ as $N\uparrow\infty$. Here

$$\epsilon(N) \sim (\log N)^{\frac{2}{2-m}}.$$

[Benfatto, Picco, Pulvirenti]

Need to prove Γ–convergence of the free energy, in terms of the density of vortices μ,

$$\mathcal{F}(\mu) = \mathcal{E}(\mu|\mathsf{Leb}^{\otimes N}\otimes\nu^{\otimes N})+\frac{\beta}{2}\int\int \xi\xi' G_\epsilon(x,x')\mu(dx\,d\xi)\mu(dx'\,d\xi')$$

or in terms of the pseudo–stream function,

$$\frac{1}{2}\|(-\Delta)^{\frac{m}{4}}\,\mathrm{e}^{-\frac{1}{2}\epsilon(-\Delta)}\,\psi\|_{L^2}^2 + \frac{1}{\beta}\log\Big(\int \mathrm{e}^{-\beta\xi\psi(x)\,dx\,\nu(d\xi)}\Big)$$

Problem: Control of the energy and the entropy to ensure Γ–convergence (or at least lower semi–continuity of the candidate limit).
At $m = 0$ Moser–Trudinger inequality.

[Bellettini, Bertini, Mariani, Novaga]