# SQG in Bounded Domains

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# SQG in $\mathbb{R}^2$ (or $\mathbb{T}^2$)

Nonlinear, nonlocal, scalar

$$\partial_t \theta + u \cdot \nabla \theta = 0$$

$\theta(x, t)$ is a real valued function of $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$

$$u = R^{\perp} \theta$$

$R$ is a vector of Riesz transforms

$$R_i f(x) = \partial_i (-\Delta)^{-\frac{1}{2}} f(x) = cPV \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x - y|^3} f(y)\, dy$$

$$R^{\perp} = (-R_2, R_1)$$

The velocity $u$ is divergence-free.
Held, Pierrhumbert, Garner, Swanson '95: SQG is an equation for frontogenesis in meteorology

- model for rapidly rotating, stratified fluids
- $\theta$ temperature (or surface buoyancy) in a 2D layer

# Analogies with the 3D incompressible Euler equations

- Conservation of kinetic energy, $||u||_{L^2}$.
- The integral curves of $\nabla^\perp\theta$ are carried by the flow.
- $\nabla^\perp\theta$ is like 3D vorticity $\omega$, Constantin–Majda–Tabak ('94): it satisfies the stretching equation

$$(\partial_t + u\cdot\nabla)(\nabla^\perp\theta) = (\nabla u)(\nabla^\perp\theta), \quad u = \nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta$$

  3D Euler: $(\partial_t + u\cdot\nabla)\,\omega = (\nabla u)\,\omega, \quad u = \nabla^\perp(-\Delta)^{-1}\omega$
- The Beal-Kato-Majda theorem holds: a smooth solution blows up at time $t = T$ if and only if $\int_0^T ||\nabla^\perp\theta(\cdot,t)||_\infty dt = \infty$.
- If the direction of level lines is locally nice, geometric depletion of nonlinearity.

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Major open problem: global existence of smooth solutions vs blow up.

# Dissipative SQG in $\mathbb{R}^2$

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda^s \theta = 0$$

$$u = \nabla^\perp \Lambda^{-1} \theta, \quad \Lambda = (-\Delta)^{\frac{1}{2}}$$

The fractional Laplacian has an explicit kernel in $\mathbb{R}^2$,

$$\Lambda^s f(x) = cPV \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^{2+s}} \, dy$$

for $0 < s < 2$.

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Scaling invariance: $\theta_\lambda(x,t) = \lambda^{s-1}\theta(\lambda x, \lambda^s t)$

- $s > 1$, subcritical SQG: global smooth solutions. Resnick '95, Constantin, Wu '99
- $s = 1$, critical SQG: global smooth solutions.
  - Small data in $L^\infty$: Cordoba–Constantin–Wu '01
  - Large data: Caffarelli–Vasseur '07, Kiselev–Nazarov–Volberg '07, Kiselev–Nazarov '09, Constantin–Vicol '12, Constantin–Tarfulea–Vicol '15
- $s < 1$, supercritical SQG: The problem of global existence of smooth solutions is open.

# Global regularity ideas in the whole space

- The stretching equation

$$(\partial_t + u\cdot\nabla + \Lambda)\,\nabla^\perp\theta = (\nabla u)\nabla^\perp\theta.$$

- Take the scalar product with $\nabla^\perp\theta$

$$\frac{1}{2}(\partial_t + u\cdot\nabla + \Lambda)q^2 + D(q) = Q$$

for $q^2 = |\nabla^\perp\theta|^2$, with

$$Q = (\nabla u)\nabla^\perp\theta\cdot\nabla^\perp\theta \le |\nabla u| q^2.$$

$|\nabla u| \sim q$, $Q$ is cubic.

- Nonlinear lower bounds

$$D(q) = q\Lambda q - \frac{1}{2}\Lambda\left(q^2\right) \ge c\left(\|\theta\|_{L^\infty}\right)^{-1} q^3$$

hold pointwise, for $q = \partial_i\theta$. (Useful when $\|\theta\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$.)

# Critical SQG in bounded domains

Let $\Omega \subset \mathbb{R}^2$ be open, bounded, smooth.

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda_D \theta = 0$$
$$u = R_D^{\perp} \theta, \; R_D = \nabla \Lambda_D^{-1}$$
$$\theta_{|t=0} = \theta_0$$

Main result: Global interior Lipschitz regularity

Additional challenges to the whole space case:

1. No explicit kernels. Need eigenfunction expansion and heat kernel.
2. No translation invariance. Need commutators of $\Lambda_D$ with finite difference operators, properly localized.

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Strategy of proof:
1. $L^\infty$ bounds (Convex damping inequality)

$$||\theta||_{L^\infty} \le ||\theta_0||_{L^\infty}.$$

2. Global interior Hölder estimates with exponent $\alpha$, where

$$\alpha ||\theta_0||_{L^\infty} \ll 1.$$

3. Global interior gradient bounds.

# The Dirichlet Fractional Laplacian

Recall the eigenfunction expansion for the Dirichlet Laplacian:

$$-\Delta w_j = \lambda_j w_j, \quad w_{j|\partial\Omega} = 0$$

We have

$$f = \sum f_j w_j, \quad f_j = \int_\Omega f w_j dx, \quad \Lambda_D f = \sum \lambda_j^{\frac{1}{2}} f_j w_j$$

We mainly use a formula based on the heat kernel:

$$((-\Delta)^{\frac{s}{2}} f)(x) = c_s \int_0^\infty [f(x) - e^{t\Delta} f(x)] t^{-1-\frac{s}{2}} dt$$

where $(e^{t\Delta} f)(x) = \int_\Omega H_D(t,x,y) f(y) dy$ is the heat operator.

$$\Lambda_D = (-\Delta)^{\frac{1}{2}}, \qquad \mathcal{D}(\Lambda_D) = H_0^1(\Omega)$$

Gaussian bounds for $H_D$ in $\Omega$. Denote

$$d(x) = dist(x, \partial\Omega).$$

We have

$$\frac{|\nabla_x H_D(t,x,y)|}{H_D(t,x,y)} \le C \begin{cases} \frac{1}{d(x)}, & \text{if } \sqrt{t} \ge d(x), \\ \frac{1}{\sqrt{t}}\left(1 + \frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \le d(x) \end{cases}$$

# The convex damping inequality

## Proposition (C, I '16)

*Let $\Omega$ be a bounded domain with smooth boundary. There exists a constant $c > 0$ depending only on $\Omega$ such that for any $\Phi$, a $C^2$ convex function satisfying $\Phi(0) = 0$, and any $f \in C_0^\infty(\Omega)$, the inequality*

$$\Phi'(f)\Lambda_D f - \Lambda_D(\Phi(f)) \geq \frac{c}{d(x)}(f\Phi'(f) - \Phi(f)) \geq 0$$

*holds pointwise in $\Omega$.*

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The proof follows from approximation, convexity, and the fact that $\Theta = e^{t\Delta}1$ obeys $0 \leq \Theta \leq 1$ and

$$\Lambda_D 1 = \int_0^\infty t^{-\frac{3}{2}}(1 - \Theta(x,t))dt \geq cd(x)^{-1}$$

# The Nonlinear Bound for derivatives

## Theorem (C, I '16)

*Let $f \in L^\infty(\Omega) \cap \mathcal{D}(\Lambda_D)$. Assume that $f = \partial\theta$ with $\theta \in L^\infty(\Omega)$ and $\partial$ a first order derivative. Then there exist constants $c$, $C$ depending on $\Omega$ such that*

$$f\Lambda_D f - \frac{1}{2}\Lambda_D f^2 \geq c(\|\theta\|_{L^\infty})^{-1}|f_d|^3 + \frac{c}{d(x)}f^2$$

*holds pointwise in $\Omega$, with*

$$|f_d(x)| = \begin{cases} |f(x)| & \text{if } |f(x)| \geq C\frac{\|\theta\|_{L^\infty(\Omega)}}{d(x)}, \\ 0 & \text{if } |f(x)| \leq C\frac{\|\theta\|_{L^\infty(\Omega)}}{d(x)}. \end{cases}$$

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Proof: uses precise bounds on the heat kernel and

$$D(f) = f\Lambda_D f - \tfrac{1}{2}\Lambda_D f^2$$
$$= \gamma_0 \int_0^\infty t^{-\frac{3}{2}} dt \int_\Omega H_D(x,y,t)(f(x) - f(y))^2 dy + \gamma_0 f^2(x)\Lambda_D 1$$

holds for all $x \in \Omega$.

# Global interior Hölder bounds for the critical SQG

$\Omega$ smooth bounded domain.

## Theorem (C, I '16)

*Let* $\theta(x,t)$ *be a smooth solution of*

$$\partial_t \theta + (R_D^{\perp}\theta)\cdot\nabla\theta + \Lambda_D\theta = 0$$

*on a time interval* $[0,T)$, *with* $T \le \infty$, *with initial data* $\theta(x,0) = \theta_0(x)$. *Then the solution is uniformly bounded,*

$$\sup_{0\le t<T} \|\theta(t)\|_{L^\infty(\Omega)} \le \|\theta_0\|_{L^\infty(\Omega)}.$$

*There exists* $\alpha$ *depending only on* $\|\theta_0\|_{L^\infty(\Omega)}$ *and* $\Omega$, *and a constant* $\Gamma$ *depending only on the domain* $\Omega$ *such that*

$$\sup_{0\le t<T} \|\theta(t)\|_{C^\alpha(\Omega)} \le \Gamma\|\theta_0\|_{C^\alpha(\Omega)},$$

*where the interior* $C^\alpha$ *norm is* $||f||_{C^\alpha(\Omega)} = ||f||_{L^\infty(\Omega)} + [f]_{C^\alpha(\Omega)}$ *with*

$$[f]_{C^\alpha(\Omega)} = \sup_{x\in\Omega} d(x)^\alpha \sup_{h\neq 0, |h|<d(x)} \frac{|f(x+h)-f(x)|}{|h|^\alpha}$$

# Global interior gradient bounds

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*on a time interval* $[0,T)$*, with* $T\le\infty$*, with initial data* $\theta(x,0)=\theta_0(x)$*. There exists a constant* $\Gamma_1$ *depending only on* $\Omega$ *such that*

$$\sup_{x\in\Omega, 0\le t<T} d(x)|\nabla_x\theta(x,t)| \le \Gamma_1\left[\sup_{x\in\Omega} d(x)|\nabla_x\theta_0(x)| + \left(1+\|\theta_0\|_{L^\infty(\Omega)}\right)^4\right]$$

*holds.*

# Commutator estimates, $\Omega \subset \mathbb{R}^2$

## Theorem (C, I '16)

*Let* $a \in W^{2,p}(\Omega)$ *with* $p > 2$. *There exists a constant* $C$, *such that*

$$\|[a, \Lambda_D]f\|_{\frac{1}{2},D} \leq C\|a\|_{W^{2,p}(\Omega)}\|f\|_{\frac{1}{2},D}$$

*holds for any* $f \in \mathcal{D}\left(\Lambda_D^{\frac{1}{2}}\right)$.

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## Theorem (C, I '16)

*Let* $a \in \left(W^{2,p}(\Omega)\right)^2$ *with* $p > 2$. *Assume that* $a_{|\,\partial\Omega} \cdot n = 0$. *There exists a constant* $C$ *such that*

$$\|[a \cdot \nabla, \Lambda_D]f\|_{\frac{1}{2},D} \leq C\|a\|_{W^{2,p}(\Omega)}\|f\|_{\frac{3}{2},D}$$

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*holds for any $f \in \mathcal{D}\left(\Lambda_D^{\frac{3}{2}}\right)$.*

The proofs are based on harmonic extension, cancellation, and elliptic regularity.

# Linear drift-diffusion equation with nonlocal diffusion

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary.

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda_D \theta = 0$$

$$\theta(x, 0) = \theta_0$$

with the constraint

$$\theta_{|\,\partial\Omega} = 0$$

Assumptions for $u = u(x, t)$:

- $\nabla \cdot u = 0,$
- $u \in L^2(0, T; (W^{2,p}(\Omega))^2), \; p > 2$
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## Theorem (C, I '16)

*The equation with $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ has unique solutions*

$$\theta \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0, T; H^{2.5}(\Omega)).$$

*If $\theta_0 \in L^p(\Omega)$, $1 \le p \le \infty$, then*

$$\sup_{0 \le t \le T} \|\theta(\cdot, t)\|_{L^p(\Omega)} \le \|\theta_0\|_{L^p(\Omega)}.$$

# Critical SQG in bounded domains

Local existence of smooth solutions: proof using methods above for linear drift-diffusion equations.

Global weak solutions:

## Theorem (C, I '16)

*Let $\theta_0 \in L^2(\Omega)$ and let $T > 0$. There exists a weak solution*

$$\theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{D}(\Lambda_D^{\frac{1}{2}}))$$

*satisfying $lim_{t\to 0}\theta(t) = \theta_0$ weakly in $L^2(\Omega)$.*

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- $\theta$ obeys the energy inequality

$$\frac{1}{2}\|\theta(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega |\Lambda_D^{\frac{1}{2}}\theta|^2 dx d\tau \le \frac{1}{2}\|\theta_0\|_{L^2(\Omega)}^2$$

  for a.e. $t > 0$.
- the dissipation $\Lambda_D\theta$ can be replaced by $\Lambda_D^s\theta$ for $s \in (0, 2]$.

# Inviscid SQG in bounded domains

Constantin, Nguyen '17: Let $\theta_0 \in L^2(\Omega)$. There exists a weak solution $\theta \in L^\infty([0,\infty); L^2(\Omega))$; that is, for any $T \geq 0$ and $\phi \in C_0^\infty((0,T) \times \Omega)$

$$\int_0^T \int_\Omega \theta(x,t)\partial_t\phi(x,t)dxdt + \int_0^T \int_\Omega \theta(x,t)u(x,t) \cdot \nabla\phi(x,t)dxdt = 0.$$

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Moreover, $\theta$ obeys the energy inequality

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Constantin, I., Nguyen '18: Weak solutions of $\partial_t \theta + u \cdot \nabla \theta + \nu \Lambda_D^s \theta = 0$ converge to weak solutions of $\partial_t \theta + u \cdot \nabla \theta = 0$ as $\nu \to 0$.

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Commutator structure:

$$\int_\Omega \Lambda \psi \nabla^\perp \psi \cdot \nabla \phi dx = \frac{1}{2} \int_\Omega [\Lambda, \nabla^\perp]\psi \cdot \nabla \phi \psi dx - \frac{1}{2} \int_\Omega \nabla^\perp \psi \cdot [\Lambda, \nabla \phi]\psi dx$$

for $\psi = \Lambda^{-1}\theta \in H_0^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$.

# Elements of the proof for the Hölder bound

- Gaussian bounds for the heat kernel; cancelation due to translation invariance effective for small time
- Good cutoff $\chi$ and bound for the commutator $[\delta_h, \Lambda_D]$ away from boundary (the most expensive term, fighting boundary repulsion)
- Nonlinear maximum principle (lower bound for $\Lambda_D$) giving smoothing and a strong boundary repulsion damping effect
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Equation for the finite difference $\delta_h\theta(x) = \theta(x+h) - \theta(x)$:

$$(\partial_t + u\cdot\nabla + \delta_h u\cdot\nabla_h)(\delta_h\theta) + \Lambda_D(\delta_h\theta) + [\delta_h, \Lambda_D]\theta = 0.$$

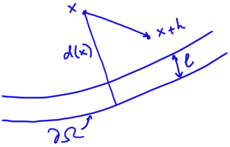
# Good cutoff

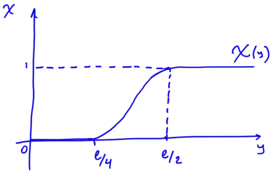
## Lemma

*Let $\Omega$ be a bounded domain with $C^2$ boundary. For $\ell > 0$ small enough (depending on $\Omega$) there exist cutoff functions $\chi$ with the properties:*

- $0 \leq \chi \leq 1$
- $\chi(y) = 0$ *if* $d(y) \leq \frac{\ell}{4}$
- $\chi(y) = 1$ *for* $d(y) \geq \frac{\ell}{2}$
- $|\nabla^k \chi| \leq C\ell^{-k}$ *with* $C$ *independent of* $\ell$
- $\int_\Omega \frac{(1-\chi(y))}{|x-y|^{2+j}} dy \leq \frac{C}{d(x)^j}$
- $\int_\Omega \frac{|\nabla \chi(y)|}{|x-y|^2} \leq \frac{C}{d(x)}$

  *hold for* $j > 0$ *and* $d(x) \geq \ell$.




Useful because of the Gaussian bounds on the heat kernel. Makes work in $\Omega$ look like work in half-space without changing coordinates.

# Translation invariance effect

Using the definition of $\Lambda_D$ and integration by parts

$$[\nabla, \Lambda_D]f(x) = c_s \int_0^\infty t^{-\frac{3}{2}} \int_\Omega (\nabla_x + \nabla_y) H_D(x,y,t) f(y) dydt.$$

Important additional bounds we need are

$$|(\nabla_x + \nabla_y) H_D(x,y,t)| \le C t^{-\frac{1}{2}-\frac{d}{2}} e^{-\frac{d(x)^2}{Ct}}$$

and

$$I_1(x,t) = \int_\Omega |(\nabla_x + \nabla_y) H_D(x,y,t)|\, dy \le C t^{-\frac{1}{2}} e^{-\frac{d(x)^2}{\tilde{K}t}}$$

valid for $t \le cd(x)^2$. Nonsingular at $x = y$.

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valid for $t \leq c d(x)^2$. Nonsingular at $x = y$. These imply that

$$\int_0^t s^{-\frac{3}{2}} I_1(x, s) ds \leq d(x)^{-2}$$

for small time.

# Commutator

Let $\chi$ be a good cutoff with scale $\ell > 0$. Denote

$$\delta_h\theta(x) = \theta(x+h) - \theta(x).$$

## Lemma

*There exists a constant* $\Gamma_0$ *such that the commutator*

$$C_h(\theta) = \delta_h\Lambda_D\theta - \Lambda_D(\chi\delta_h\theta)$$

*obeys*

$$|C_h(\theta)(x)| \le \Gamma_0 \frac{|h|}{d(x)^2} \|\theta\|_{L^\infty(\Omega)}$$

*for* $d(x) \ge \ell$, $|h| \le \frac{\ell}{16}$ *and* $\theta \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

# The nonlinear bound for finite differences

**Theorem**

*Let $\chi \in C_0^\infty(\Omega)$ be a good cutoff with scale $\ell > 0$ and let*

$$f(x) = \chi(x)(\delta_h \theta(x)) = \chi(x)(\theta(x+h) - \theta(x)).$$

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*Then*

$$D(f) = (f \Lambda_D f)(x) - \frac{1}{2}(\Lambda_D f^2)(x) \geq \gamma_1 |h|^{-1} \frac{|f_d(x)|^3}{\|\theta\|_{L^\infty}} + \gamma_1 \frac{f^2(x)}{d(x)}$$

*holds pointwise in* $\Omega$ *when* $|h| \leq \frac{\ell}{16}$ *and* $d(x) \geq \ell$ *with*

$$|f_d(x)| = \begin{cases} |f(x)|, & \text{if } |f(x)| \geq M\|\theta\|_{L^\infty(\Omega)} \frac{|h|}{d(x)}, \\ 0, & \text{if } |f(x)| \leq M\|\theta\|_{L^\infty(\Omega)} \frac{|h|}{d(x)}. \end{cases}$$

# Finite difference of Riesz transform

## Lemma

*Let* $u$ *be given by*

$$u = \nabla^{\perp}\Lambda_D^{-1}\theta$$

*and let* $\chi$ *be a good cutoff with a length scale* $\ell$. *Then*

$$|\delta_h u(x)| \le C\left(\sqrt{\rho D(f)(x)} + \|\theta\|_{L^\infty}\left(\frac{|h|}{d(x)} + \frac{|h|}{\rho}\right) + |\delta_h \theta(x)|\right)$$

*holds for* $d(x) \ge \ell$, $\rho \le cd(x)$, $f = \chi\delta_h\theta$ *and with* $C$ *a constant depending on* $\Omega$.

# Hölder bound, idea of proof:

Let $\chi$ be a good cutoff with a scale $\ell > 0$, and $|h| \leq \frac{\ell}{16}$. The equation for $\delta_h\theta$ implies:

$$\frac{1}{2}L_\chi\left(\delta_h\theta\right)^2 + D(f) + (\delta_h\theta)C_h(\theta) = 0$$

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for $f = \chi \delta_h \theta$. Multiply by $|h|^{-2\alpha}$ where $\alpha > 0$ will be chosen small enough:

$$\frac{1}{2} L_\chi \left( \frac{\delta_h \theta(x)^2}{|h|^{2\alpha}} \right) + |h|^{-2\alpha} D(f) - 2\alpha \frac{|\delta_h u|}{|h|} \left( \frac{\delta_h \theta(x)^2}{|h|^{2\alpha}} \right) \leq |C_h(\theta)| |\delta_h \theta| |h|^{-2\alpha}.$$

# $\epsilon$-approximation of critical SQG

Let $\epsilon > 0$ and consider the $\epsilon$-approximation of SQG

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon + \Lambda_D \theta^\epsilon = 0$$

where

$$u^\epsilon = \nabla^\perp \psi^\epsilon = \nabla^\perp \int_\epsilon^\infty t^{-\frac{1}{2}} e^{t\Delta} \theta^\epsilon dt$$

with initial data $\theta^\epsilon(0) = \theta_0$.

## Theorem

*For each $\epsilon > 0$, the $\epsilon$-approximation has unique, global, smooth solutions up to the boundary. The solutions obey bounds*

$$d(x)|\nabla \theta^\epsilon(x, t)| \le C$$

*with $C$ depending on $\Omega$ and $\|\theta_0\|_{W^{1,\infty}(\Omega)}$ but not on $t$ nor on $\epsilon$.*

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For the proof: note that

$$\|\Lambda_D^M \psi^\epsilon\|_{L^2(\Omega)} \leq C_{M,\epsilon} \|\theta_0\|_{L^2(\Omega)}$$

for any $M > 0$, and therefore $u^\epsilon$ is smooth.

# Convergence to critical SQG

## Theorem

*Let $\theta_0 \in L^\infty(\Omega)$ and let $T > 0$. Any sequence of solutions of $\epsilon$-approximations of SQG with $\epsilon \to 0$ contains a subsequence $\theta_n$ converging strorngly in $L^2([0,T], L^2(\Omega))$ to a weak solution $\theta \in L^\infty([0,T], L^\infty(\Omega)) \cap L^2([0,T], \mathcal{D}(\Lambda_D^{\frac{1}{2}}))$ of critical SQG.*
*If $\theta_0 \in W^{1,\infty}(\Omega)$, then $\theta$ obeys*

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For the proof we use that $\theta_n$ are uniformly bounded in $L^\infty([0,T], L^\infty(\Omega))$ hence $u_n\theta_n$ are bounded in $L^\infty([0,T], L^2(\Omega))$, and $\partial_t\theta_n$ are bounded in $L^\infty([0,T], H^{-1}(\Omega))$. We then use an Aubin-Lions lemma with based on $L^2$ in time, and with spaces $\mathcal{D}(\Lambda_D^{\frac{1}{2}}) \subset\subset L^2(\Omega) \subset H^{-1}(\Omega)$.

# Electroconvection

Electric field determined by charge density:

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with $\sigma$ electric conductivity. Conducting fluid confined to domain $\Omega$:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = qE^{||}, \quad \nabla \cdot u = 0.$$

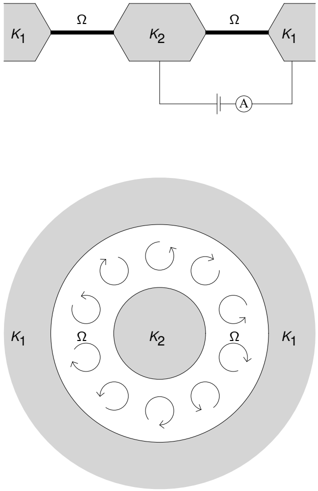


Figure: Schematic of the experiment of Morris et al. Side view and top view.

# The electrical potential

Smectic conducting fluid, driven by electric current. 2DNS in fluid region $\Omega \subset \mathbb{R}^2 \times \{0\}$.

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defined in $Q = \Omega \times \mathbb{R}$ with inhomogeneous boundary conditions for the electric potential $\Phi$.

$$-\Delta_3 \Phi = 2q\delta_\Omega, \quad \Phi_{\partial Q} = V, \quad 0.$$

Solution is

$$\Phi(x,z) = \Phi_0(x) + \left\{ \begin{array}{ll} e^{-z\Lambda_D}\Lambda_D^{-1} q, & z \ge 0, \\ e^{z\Lambda_D}\Lambda_D^{-1} q, & z < 0 \end{array} \right.$$

Parallel component of $E$

$$E^{||} = (-\partial_1\Phi, -\partial_2\Phi, 0)_{|\,\Omega}$$

Fractional Laplacian emerges:

$$\nabla \cdot E^{||} = \Lambda_D q$$

# Global Regularity in Bounded Domains

## Theorem (Constantin, Elgindi, Ignatova, Vicol ('17))

*Let $\Omega \subset \mathbb{R}^2$ open, bounded, with smooth boundary. Let $u_0 \in [H_0^1(\Omega) \cap H^2(\Omega)]^2$ be divergence-free. Let $q_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. Then the electroconvection system*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - q \nabla(\Phi_0 + \Lambda_D^{-1} q), \\ \nabla \cdot u = 0, \\ \partial_t q + u \cdot \nabla q + \sigma \Lambda_D q = 0 \end{cases}$$

*with homogeneous Dirichlet boundary conditions for both u and q has global unique strong solutions,*

$$u \in L^\infty(0, T; [H_0^1(\Omega) \cap H^2(\Omega)]^2) \cap L^2(0, T; H^{\frac{5}{2}}(\Omega)^2),$$

$$q \in L^\infty(0, T; W_0^{1,4}(\Omega) \cap H^2(\Omega)) \cap L^2(0, T; H^{\frac{5}{2}}(\Omega)).$$

# Strategy of Proof

1. Good approximation:

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coupled with Galerkin for NSE:

$$\partial_t u_m + Au_m + \mathbb{P}_m B(u_m, u_m) = -\mathbb{P}_m(qR_D q)$$

From the $q$ equation we get a priori bounds for $q \in L^\infty(0, T; L^p(\Omega))$, independent of $u_m$, using the convex damping inequality in bounded domains.

2. We use NSE energy bounds to deduce $u_m \in L^\infty(0, T; H_0^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2)$ are controlled uniformly. ($R_D = \nabla\Lambda^{-1}$ are bounded in $L^p(\Omega)$ spaces.)
3. We obtain higher regularity for $q$.
4. Then we obtain higher uniform regularity for $u_m$.
5. Pass to the limit $m \to \infty$.

# Conclusion and Outlook

- Nonlinear lower bounds for $\Lambda_D$ can be used to prove global interior regularity for SQG and electroconvection.
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- Construction of global unique weak solution with uniform interior smoothness is in progress.
- Electroconvection: different configurations.