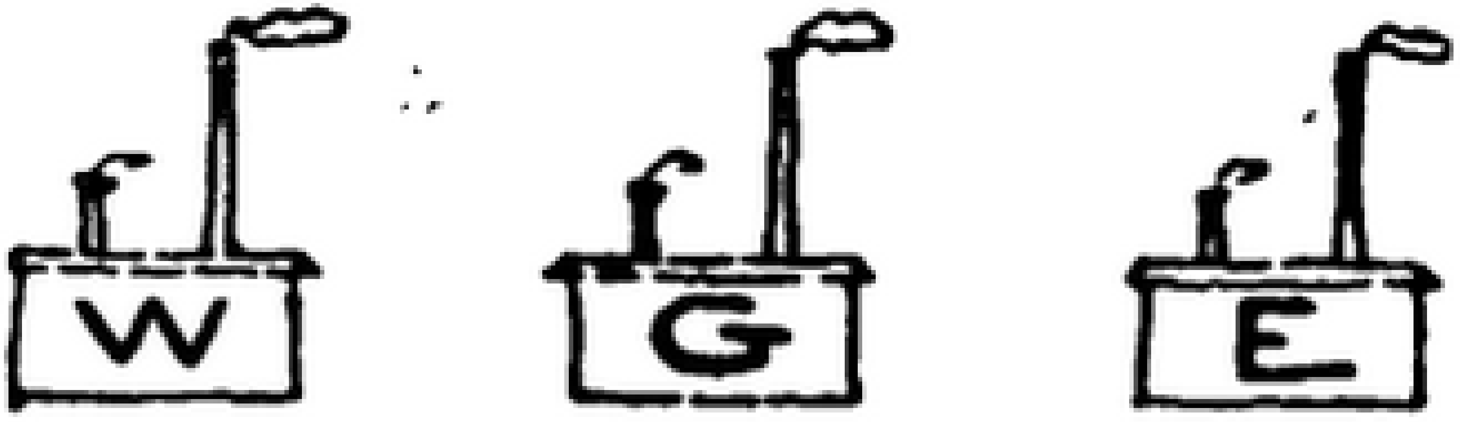


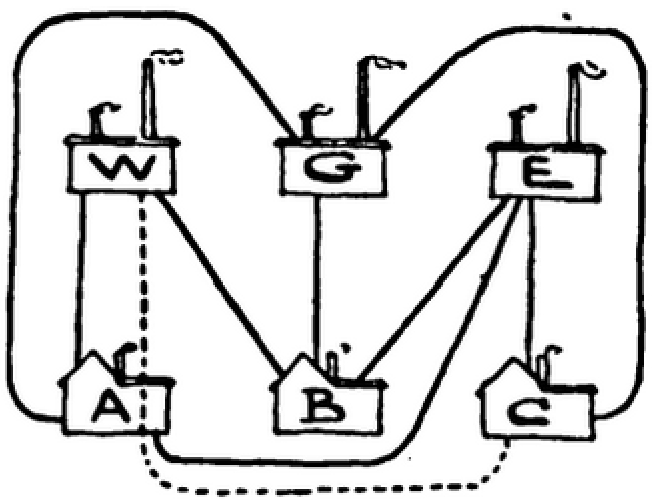
Crossing Numbers: Some Open Questions

Banff, 2018

Crossing Numbers ?

Solutions to Last Month's Puzzles.

146.—WATER, GAS, AND ELECTRICITY.



ACCORDING to the conditions, in the strict sense in which one at first understands them, there is no possible solution to this puzzle. In such a dilemma one always has to look for some verbal quibble or trick. If the owner of house A will allow the water company to run their pipe for house C through his property (and we are not bound to assume that he would object), then the difficulty is got over, as shown in our illustration. It will be seen that the dotted line from W to C passes through house A, but no pipe ever crosses another pipe.

Ernest H. Dudeney, *Strand Magazine* (46), 110, 1913 (original question)

Definition

The *(graph) crossing number,* $cr(G)$, of a graph G is the smallest number of crossings in any drawing of G.

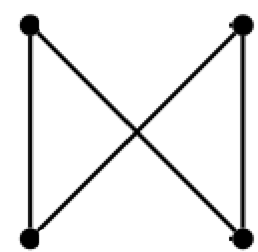
- drawn where? Plane, surface, book, …
- drawn how?
  - how is the graph represented?
  - how is the representation visualized?
- how are crossings counted?

Maximum Crossing Numbers

Definition

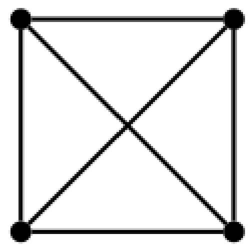
$max\text{-}\overline{cr}(G)$ = largest # crossings in
rectilinear (straight-line) drawing of G

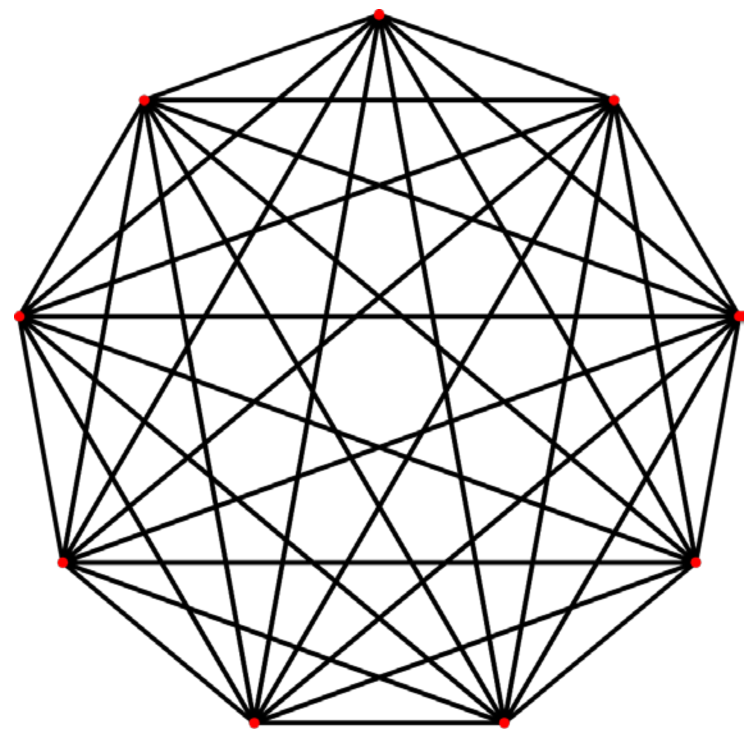
$max\text{-}cr(G)$ = largest # crossings in
good drawing of G



$max\text{-}cr(C_4) = max\text{-}\overline{cr}(C_4)$ =1

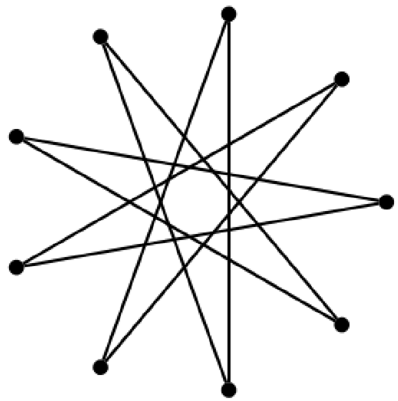
Maximum Crossing Numbers

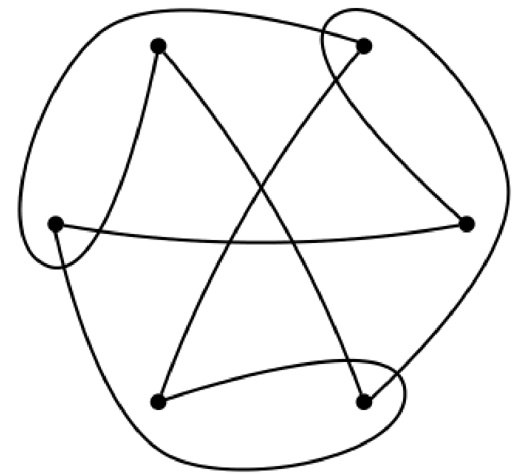


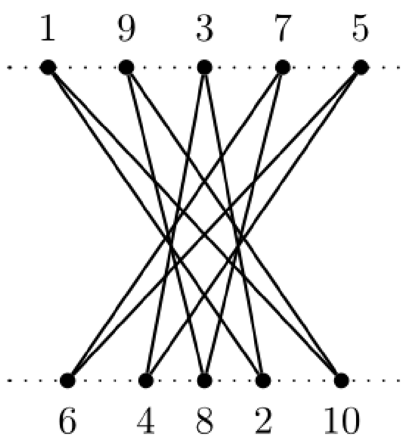


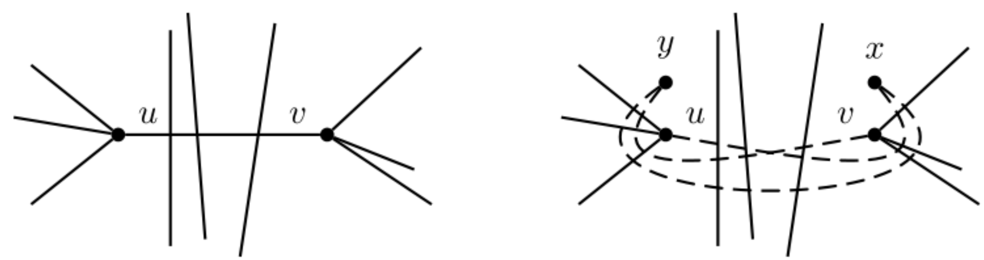
$$max\text{-}cr(K_n) = max\text{-}\overline{cr}(K_n) = \binom{n}{4}$$

Polygons and Cycles









$max\text{-}cr(C_n) = \frac{n(n-3)}{2}$ for $n \geq 4$

$$\text{max-}\,\overline{\text{cr}}(C_n) = \begin{cases} n(n-3)/2 & \textit{if } n \textit{ is odd,} \\ n(n-4)/2+1 & \textit{if } n \textit{ is even.} \end{cases}$$

Thrackle Bound

$$\vartheta(G) \coloneqq \frac{1}{2}\sum_{uv}(m - \deg(u) - \deg(v) + 1)$$

Lemma

$$max\text{-}cr(G) \leq \vartheta(G)$$

Theorem (Piazza, Ringeisen, Stueckle [PRS88])

$max\text{-}cr(F) = \vartheta(F)$ if F is a forest

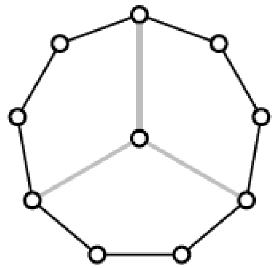
Theorem (Verbitsky [V08])

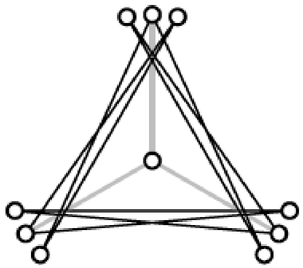
$$\frac{\vartheta(G)}{3} \leq max\text{-}\overline{cr}^{o}(G) \leq \vartheta(G)$$

Maximum Crossing Number

Question: $max\text{-}\overline{cr}^{o}(G) = max\text{-}\overline{cr}(G)$?

Answer: no. [CFKUVW17]





What if G is bipartite? A tree?

yes for spiders and trees of diameter at most 4 [BET18, FHKLSS18]

What if drawing is *separated* (all edges crossed by a line)?

Question: $\exists\mathbb{R}$-complete?

Known to be NP-hard [BJL16]

Subgraph Monotonicity

Question: If G is (induced) subgraph of H,

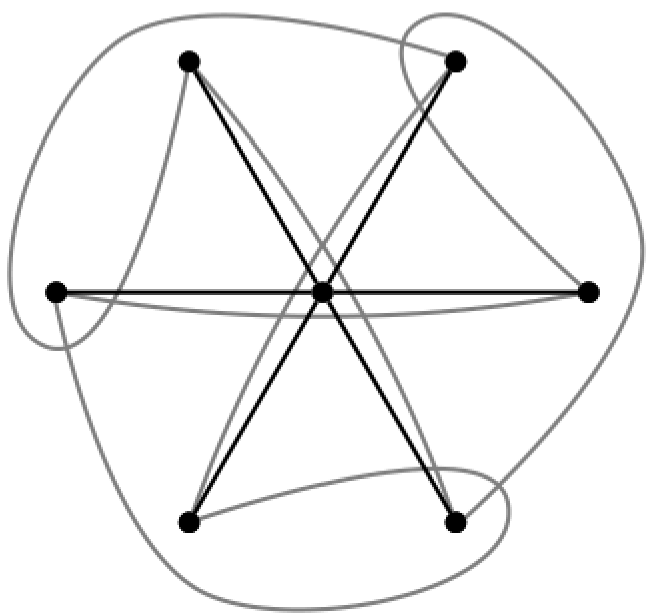
then $max\text{-}cr(G) \leq max\text{-}cr(H)$?

- True for $max\text{-}\overline{cr}(G)$ (even pseudolinear cr)
- Conditions on G which make this happen?

“Aura”/partial join Technique

Theorem

If C_n is induced subgraph of H,
then $max\text{-}cr(C_n) \leq max\text{-}cr(H)$.



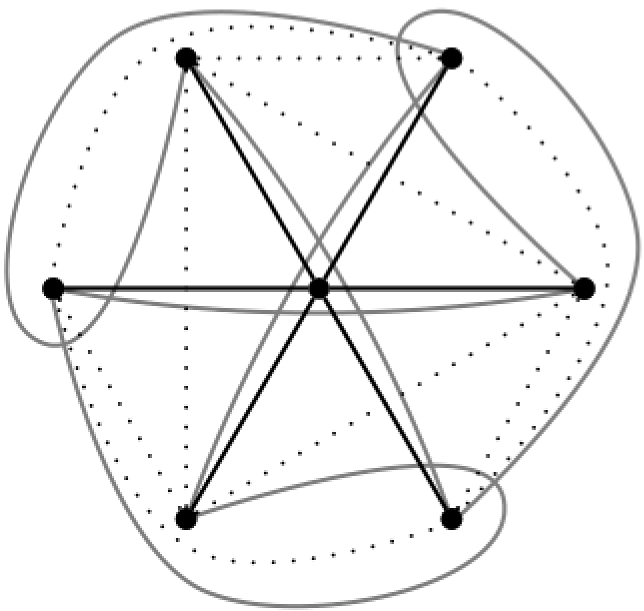
- True for odd n and C_4
- True for C_6 (pic on left)
- Replace edge with P_3
- Add spokes

"Aura"/partial join Technique

Theorem

If C_n is ~~induced~~ subgraph of H,

then $max\text{-}cr(C_n) \leq max\text{-}cr(H)$.



Question: Is monotonicity conjecture true for (induced) subgraphs?

Exercise: Show that $max\text{-}cr(G) \leq max\text{-}cr(H)$ for G apex, and induced subgraph of H.

Maximum Crossing Number

Question: Is there a formula for $max\text{-}\overline{cr}(T)$ if T is a tree?

Theorem (Piazza, Ringeisen, Stueckle [PRS88])
$max\text{-}cr(F) = \vartheta(F)$ if F is a forest

All caterpillars are thrackleable,
Subdivided $K_{1,3}$ is not.

yes for spiders and trees of diameter
at most 4 [BET18, FHKLSS18]

Conway's Thrackle Conjecture

If a graph can be drawn in the plane so that every two edges intersect once, then $|E| \leq |V|$.

Current best bound: $|E| \leq 1.3984|V|$ (Fulek, Pach [FP17])

Maximum Crossing Number Literature

[BJL16] Samuel Bald, Matthew P. Johnson, and Ou Liu. Approximating the maximum rectilinear crossing number. COCOON 2016, Springer, 2016.

[BET18] Patrick Bennett, Sean English, and Maria Talanda-Fisher. Weighted Turan Problems with Applications. ArXiv e-prints, abs/1809.05028, 2018. [CFKUVW17] Markus Chimani, Stefan Felsner, Stephen G. Kobourov, Torsten Ueckerdt, Pavel Valtr, and Alexander Wol�. On the maximum crossing number. CoRR, abs/1705.05176, 2017. arXiv:1705.05176

[FHKLSS18] Joshua Fallon, Kirsten Hogenson, Lauren Keough, Mario Lomelí, Marcus Schaefer, and Pablo Soberón. A Note on the Maximum Rectilinear Crossing Number of Spiders. ArXiv e-prints, abs/1808.00385, 2018.

[FP17] Radoslav Fulek, János Pach. Thrackles: An Improved Upper Bound. ArXiv e-prints, abs/2017arXiv170808037F, 2017.

[PRS88] B. L. Piazza, R. D. Ringeisen, and S. K. Stueckle. Properties of nonminimum crossings for some classes of graphs. In Graph theory, combinatorics, and applications. Vol. 2 (Kalamazoo, MI, 1988), Wiley-Intersci. Publ., pages 975�989. Wiley, New York, 1991

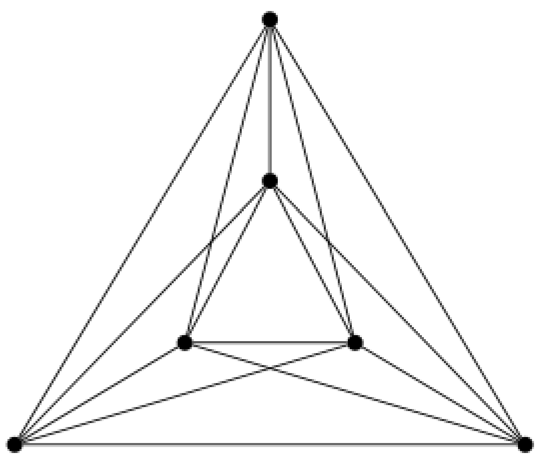
[V08] Oleg Verbitsky. On the obfuscation complexity of planar graphs. Theoret. Comput. Sci., 396(1-3):294�300, 2008

Local Crossing Number

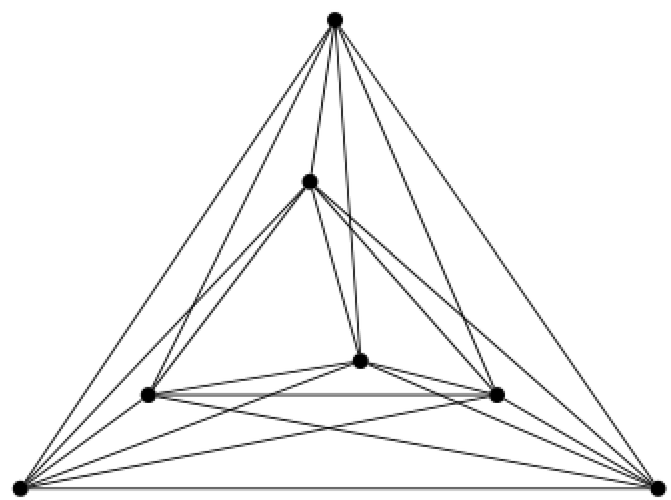
Definition

The *local crossing number,* $lcr(D)$, of a drawing D is the largest number of crossings along any edge in D.

The *local crossing number,* $lcr(G)$, of a graph G is the smallest $lcr(D)$ of any drawing D of G.



$lcr(K_6) = 1$



$lcr(K_7) = 2$

Simple Local Crossing Number

Definition

The *simple local crossing number,* $lcr^*(G)$, of a graph G is the smallest $lcr(D)$ of any **good** drawing D of G.

Theorem (Pach, Radoičič, Tardos, Tóth [PRTT06])

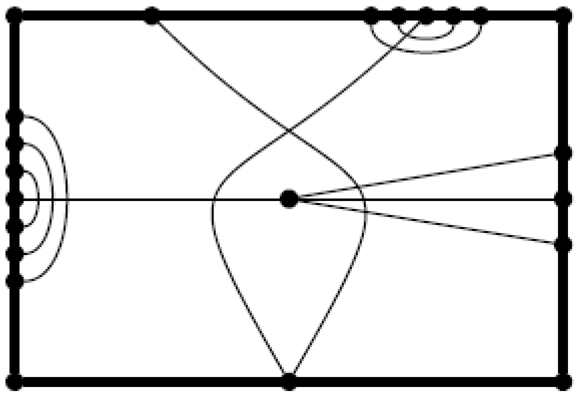
If $lcr(G) \leq 3$, then $lcr(G) = lcr^*(G)$.

Used for better lower bound in crossing lemma

Simple $=/\neq$ Non-Simple

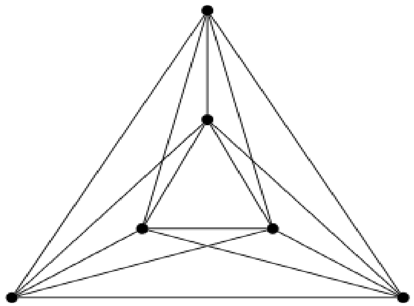
Theorem

There is G with $lcr(G) = 4$ and $lcr^*(G) = 5$.

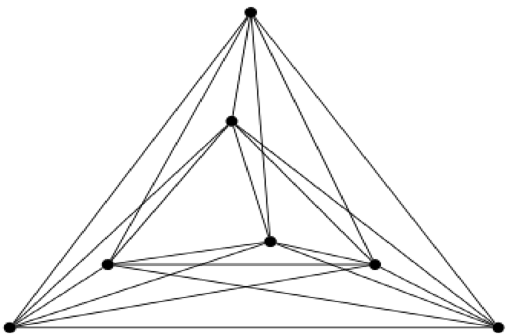


Question: can $lcr^*(G)$ be bounded in $lcr(G)$?

Local Crossing Number



$lcr(K_6) = 1$



$lcr(K_7) = 2$

Question: $lcr(K_n)$?
$lcr(K_{m,n})$?

$\overline{lcr}(K_n)$ is known for all n (Ábrego, Fernandez-Merchant [AF17])
$\overline{lcr}(K_{m,n})$ is known for m = 3, 4 [ADFLSS17].

Local Crossing Number Literature

[AF17] Bernardo M. Ábrego and Silvia Fernández-Merchant. The rectilinear local crossing number of Kn. J. Combin. Theory Ser. A, 151:131–145, 2017.

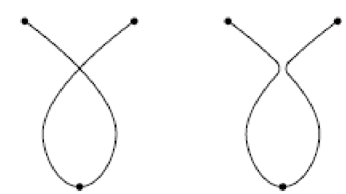
[ADFLSS17] Bernardo M. Ábrego, Kory Dondzila, Silvia Fernández-Merchant, Evgeniya Lagoda, Seyed Sajjadi, and Yakov Sapozhnikov. On the rectilinear local crossing number of K(m,n). Journal of Information Processing, 25:542–550, August 2017

[PRTT06] János Pach, Radoš Radoičić, Gábor Tardos, and Géza Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. Discrete Comput. Geom., 36(4):527–552, 2006

Independent Crossing Number

Definition

The *independent crossing number, $cr_{-}(G)$,* of a graph G is the smallest number of crossings between independent edges in any drawing of G.



Lemma $cr_{-}(G) = cr(G)$

Proof

- fix drawing D of G with smallest number of crossings
- D cannot have dependent crossings
- so $cr_{-}(D) = cr(D)$

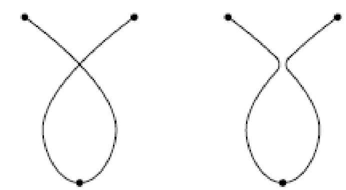
Independent Crossing Number

Definition

The *independent crossing number,* $cr\_(G)$*,* of a graph G i... est number of crossings between independ

Open

n any drawing of G.



Lemma $cr\_(G) = cr(G)$

Proof

- fix drawing D of G with smallest number of crossings
- D cannot have dependent crossings
- so $cr\_(D) = cr(D)$

Independent Crossing Number

Question: $cr(G) = cr_{-}(G)$?

- bound cr in terms of cr_{-} (best bound quadratic)
- bound $cr(D)$ of a $cr_{-}(G)$-minimal drawing
  (exponential bound possible)
- easier for $G = K_n$?

Independent Crossing Number

Conjecture:

If a graph can be drawn on a surface without independent crossings. Then graph can be embedded in surface.

Known for

plane (Hanani-Tutte theorem)
projective plane (HT for PP)
HT fails for orientable surfaces of genus ≥ 4.

Rectilinear Crossing number

Definition

The *rectilinear crossing number*, $\overline{cr}(G)$, of a graph G is the smallest number of crossings in any straight-line drawing of G.

Theorem (Fary, 48; Wagner, 36)

If *G* is planar, then *G* has a planar straight-line drawing.

Theorem (Bienstock, Dean, 93)

$$\overline{cr}(G) = cr(G) \text{ for } cr(G) \leq 3.$$

Independent Crossing Number Literature

[S10] Marcus Schaefer. Removing Incident Crosssings. Manuscript, 2010.
[

Rectilinear Crossing number

Definition

The *rectilinear crossing number,* $\overline{cr}(G)$*,* of a graph G is the smallest number of crossings in any straight-line drawing of G.

Theorem (Fary, 48; Wagner, 36)

If *G* is planar, then *G* has a planar straight-line drawing.

?

Theorem (Bienstock, Dean, 93)

$$\overline{cr}(G) = cr(G) \text{ for } cr(G) \leq 3.$$

Conjectures and Question

Conjecture (Harary, Kainen, Schwenk [HLS73])

$\overline{cr}(C_m \square C_n) = n(m-2)$ for $n \geq m \geq 3$.

Also open for $cr(G)$. For that case, partial results known.

Conjecture (Hernández-Vélez, Leaños, Salazar [HLS17])

$\overline{cr}(G)$ can be bounded in $cr(G)$ for 3-connected G.

Question: Can $\overline{cr}$ be bounded in $\widetilde{cr}(G)$?

Question: What's the complexity of $\overline{cr}(G) \leq 4$?

Grid Drawings

Question: Is there an f so that

if $\overline{cr}(G) \leq k$ and $n = |V(G)|$, then

G can be realized on a $f(k)n \times f(k)n$ grid?

Alternatively: on an $n^{f(k)} \times n^{f(k)}$ grid?

Question: If we restrict drawing D of G to a $t \times t$ grid, what is the best $\overline{cr}(D)$ we can guarantee?

If $t = \Omega(2^{2^n})$, then we can achieve $\overline{cr}(G)$.

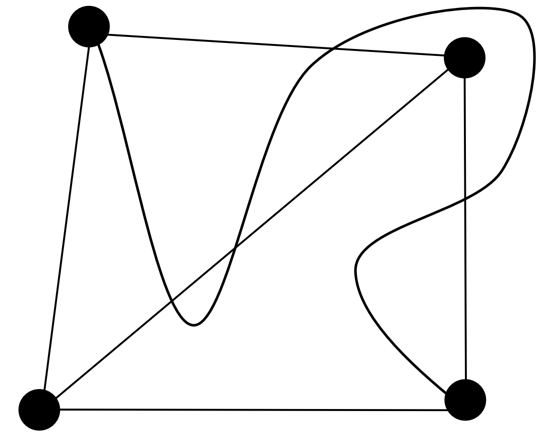
Rectilinear Crossing Number Literature

[BD93] Daniel Bienstock and Nathaniel Dean. Bounds for rectilinear crossing numbers. J. Graph Theory, 17(3):333–348, 1993.

[HKS73] Frank Harary, Paul C. Kainen, and Allen J. Schwenk. Toroidal graphs with arbitrarily high crossing numbers. Nanta Math., 6(1):58–67, 1973.

[HLS17] César Hernández-Vélez, Jesús Leaños, and Gelasio Salazar. On the pseudolinear crossing number. Journal of Graph Theory, 84(3):297–310, 2017.

Counting Crossings



Drawing D of K_4

$cr(D)$ = # of crossings in D

$pcr(D)$ = #r of pairs of edges crossing in D

$ocr(D)$ = # of pairs of edges crossing oddly in D

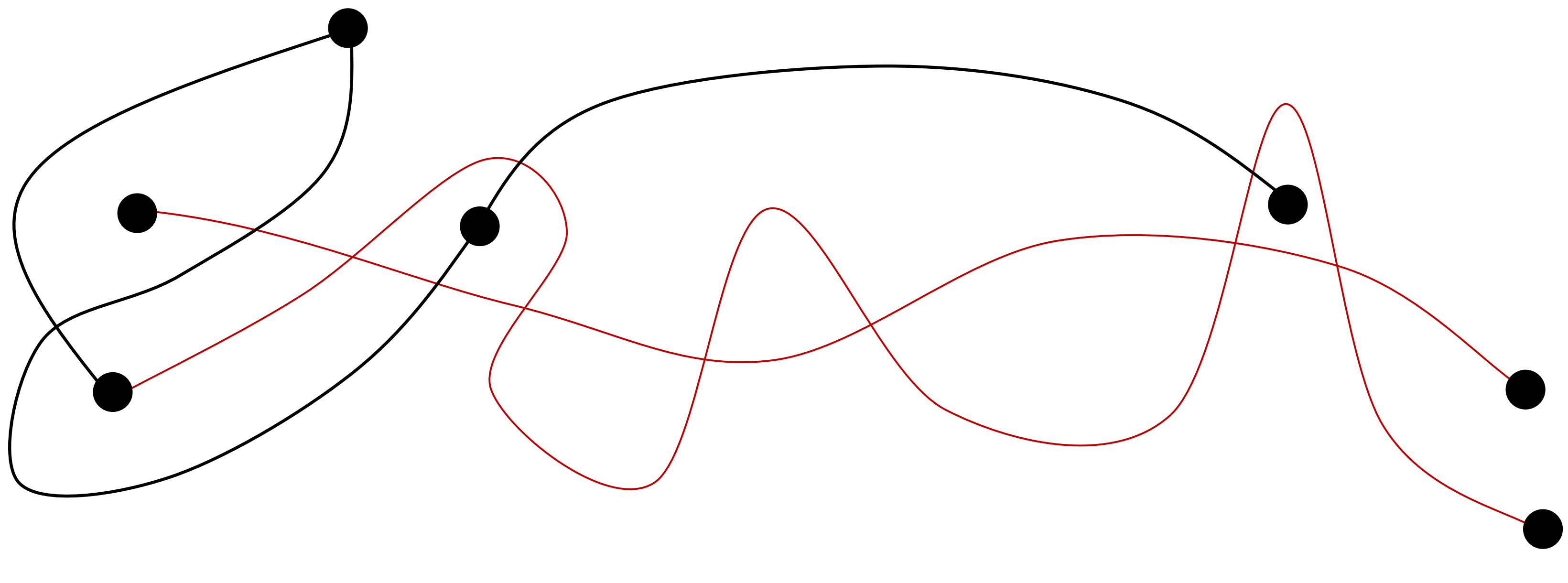
$ocr_{-}(D)$ = # of pairs of independent edges crossing oddly in D

$$ocr_{-}(G) \leq ocr(G) \leq pcr(G) \leq cr(G)$$

The Hanani-Tutte Theorem

Theorem (Hanani, 1934; Tutte, 1970)

$ocr_{-}(G) = 0$ implies $cr(G) = 0$



$$ocr_{-}(G) = ocr(G) = pcr(G) = cr(G)\,?$$

Answer: Not all. [FPSS12]

There is G so that $ocr_{-}(G) < ocr(G) < pcr(G)$

Conjecture $pcr(G) = cr(G)$

Bounds?

Theorem (Matousek [M14])

$$cr(G) \leq pcr(G)^{\frac{3}{2}} \log^2(pcr(G))$$

$\log^2(pcr(G))$ can be improved to $\log(pcr(G))$ using string graph separator by Lee.

Theorem (Pelsmajer, Schaefer, Štefankovič [PSS10])

$$cr(G) \leq \binom{2ocr_{-}(G)}{2}$$

Question: sub-quadratic bounds for cr in ocr_{-}?

Crossing Lemma for $pcr(G)$

remove each vertex of $G = (V, E)$ with prob. p i.a.r.

$$G' = (V', E')$$

$$cr(G') \geq |E'| - 3|V'|$$

$$\mathbb{E}\big(cr(G')\big) \geq \mathbb{E}(|E'|) - 3\mathbb{E}(|V'|)$$

$$p^4 cr(G) \geq p^2|E| - 3p|V|$$

can replace cr with pcr?

$$cr(G) \geq p^{-2}|E| - 3p^{-3}|V|$$

e.g. $p = 4|V|/|E|$

$$cr(G) \geq \frac{|E|^3}{64|V|^2}$$

Improved Crossing Lemma for $pcr_+(G)$

pcr_+: no adjacent crossings allowed

Theorem (Ackerman, Schaefer [AS14])

$$pcr_+(G) \geq \frac{1}{32.4} \frac{m^3}{n^2} \text{ for } m \geq 6.75\, n.$$

use $lpcr(G) \leq 2$, then $lcr(G) \leq lpcr(G)$

Questions:

$lpcr(G) \leq 3$, then $lcr(G) \leq lpcr(G)$

$lpcr_-(G) \leq 1$, then $lcr(G) \leq 1$?

Odd and Pair Crossing Number Literature

[AS14] Eyal Ackerman and Marcus Schaefer. A crossing lemma for the pair-crossing number. *Graph Drawing*, 222-233, 2014.

[FPSS12] Radoslav Fulek, Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič, Adjacent crossings do matter. J. Graph Algorithms Appl., 16(3):759–782, 2012.

[PSS10] Michael Pelsmajer, Marcus Schaefer, Daniel Štefankovič. Removing independently even crossings. SIAM Journal on Discrete Mathematics, 24(2):379–393, 2010

Thank You

Open Questions on Crossing Numbers

[A09] Dan Archdeacon. Open problems. In Topics in topological graph theory, volume 128 of Encyclopedia Math. Appl., Cambridge, 2009.

[BMP05] Peter Brass, William Moser, and János Pach. Research Problems in Discrete Geometry. Springer, New York, 2005.

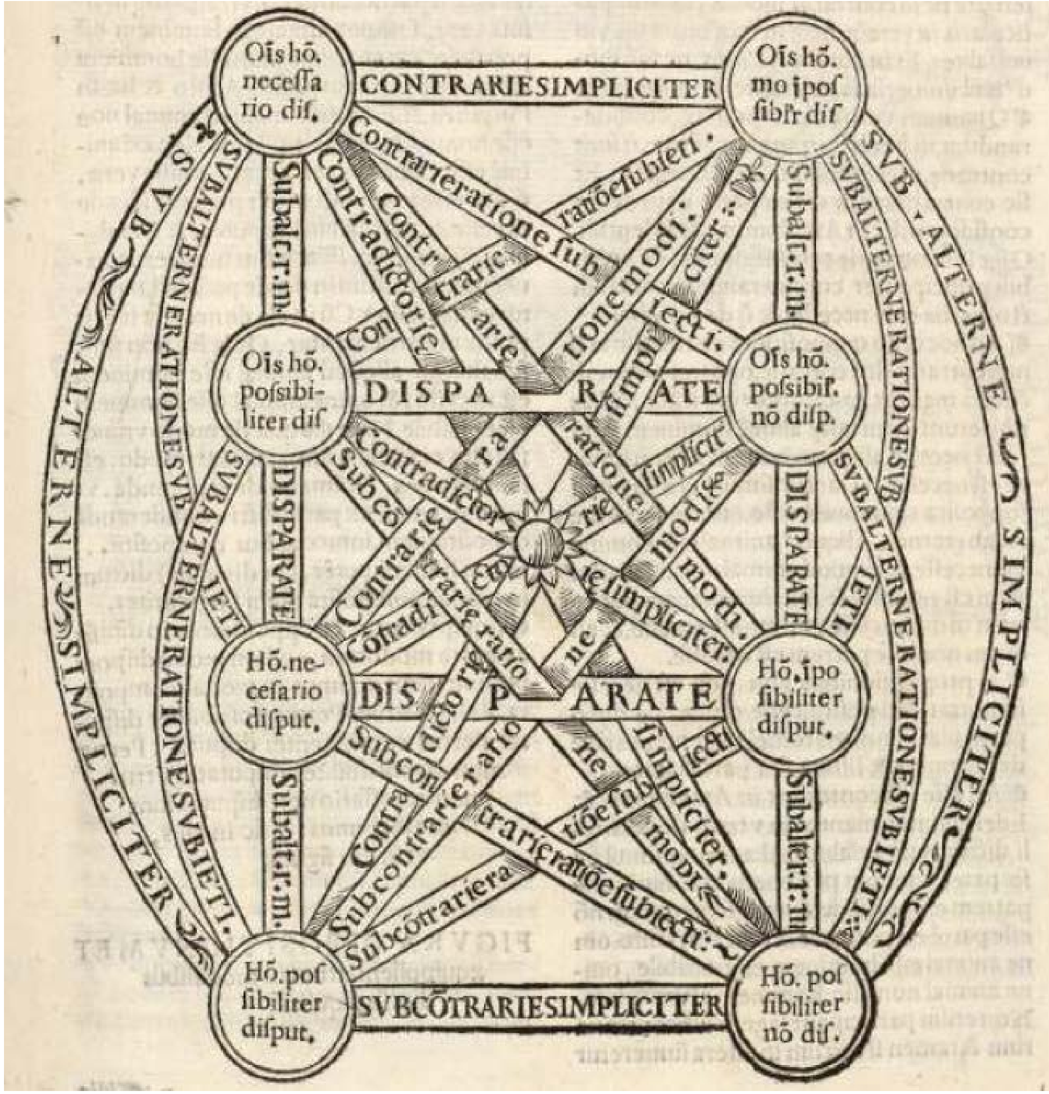
[PT00] János Pach and Géza Tóth. Thirteen problems on crossing numbers. Geombinatorics, 9(4):194�207, 2000.

[S17] Marcus Schaefer. The Graph Crossing Number and its Variants: A Survey, Electronic Journal of Combinatorics, 2017.

[S16] László A. Székely. Turán's brick factory problem: The status of the conjectures of Zarankiewicz and Hill. In Ralucca Gera, Stephen Hedetniemi, and Craig Larson, editors, Graph Theory: Favorite Conjectures and Open Problems, Springer, 2016.

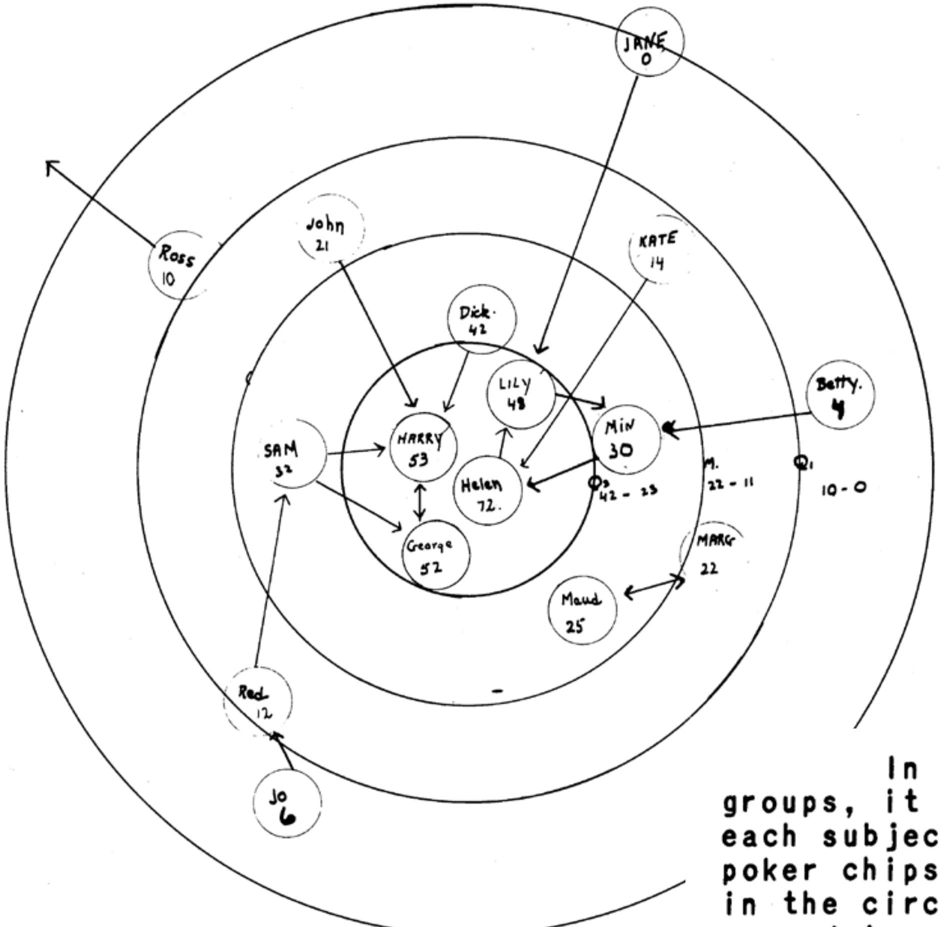
Also:

- http://www.openproblemgarden.org/category/topological_graph_theory
- http://www.cems.uvm.edu/TopologicalGraphTheoryProblems/



Question: What is the oldest reference to crossing numbers?

From de la Vera Cruz’ *Recognitio Summularum, 1554* (http://www.primeroslibros.org/browse.html)



In actual practice, especially with large groups, it has been convenient to use counters with each subject's name and score written on them. (White poker chips have served admirably.) These are moved in the circles to which their score belongs and arranged to get the best "fit" among the individuals, i.e., to have as few long lines and crossing lines as possible. In plotting the diagram when public school

From Mary L. Northway. *A Method for Depicting Social Relationships Obtained by Sociometric Testing*. Sociometry, Vol. 3, No. 2 (Apr., 1940), pp. 144-150