BV quantization in perturbative AQFT: gauge theories and effective quantum gravity

Kasia Rejzner

University of York

Banff, 02.08.2018

Outline of the talk

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT,

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT,
- Basic ingredients:

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT,
- Basic ingredients:
  - Free theory obtained by the formal deformation quantization of the Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, ...]).

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT,
- Basic ingredients:
  - Free theory obtained by the formal deformation quantization of the Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, ...]).
  - Interaction introduced in the causal approach to renormalization due to Epstein and Glaser ([Epstein-Glaser 73]),

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT,
- Basic ingredients:
  - Free theory obtained by the formal deformation quantization of the Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, ...]).
  - Interaction introduced in the causal approach to renormalization due to Epstein and Glaser ([Epstein-Glaser 73]),
  - Generalization to curved spacetime in the framework of general local covariance ([Brunetti-Fredenhagen-Verch 03, Brunetti-Dütsch-Fredenhagen 09]).

Perturbative algebraic quantum field theory

- Algebraic approach ([Haag 59, Haag-Kastler 64]): allows to separate the dynamics from the specification of the state.
- We can follow the spirit of AQFT also in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT,
- Basic ingredients:
  - Free theory obtained by the formal deformation quantization of the Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, ...]).
  - Interaction introduced in the causal approach to renormalization due to Epstein and Glaser ([Epstein-Glaser 73]),
  - Generalization to curved spacetime in the framework of general local covariance ([Brunetti-Fredenhagen-Verch 03, Brunetti-Dütsch-Fredenhagen 09]).
  - Generalization to gauge theories using homological algebra ([Hollands 07, Fredenhagen-KR 11]).

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).
- Allows to quantize observables that are non-local, which is particularly important for QG.

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).
- Allows to quantize observables that are non-local, which is particularly important for QG.
- Uncovers underlying geometrical structures and leads to interesting mathematics (homological algebra, homotopy).

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).
- Allows to quantize observables that are non-local, which is particularly important for QG.
- Uncovers underlying geometrical structures and leads to interesting mathematics (homological algebra, homotopy).
- Justifies constructions, which otherwise seem ad hoc.

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).
- Allows to quantize observables that are non-local, which is particularly important for QG.
- Uncovers underlying geometrical structures and leads to interesting mathematics (homological algebra, homotopy).
- Justifies constructions, which otherwise seem ad hoc.
- Delivers an abstract definition of the quantum BRST differential, without the need for constructing the charge in a given representation.

References:

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).
- Allows to quantize observables that are non-local, which is particularly important for QG.
- Uncovers underlying geometrical structures and leads to interesting mathematics (homological algebra, homotopy).
- Justifies constructions, which otherwise seem ad hoc.
- Delivers an abstract definition of the quantum BRST differential, without the need for constructing the charge in a given representation.

References:

- K. Fredenhagen, KR *Batalin-Vilkovisky formalism in the functional approach to classical field theory* (CMP 2012),

Motivation

Why use pAQFT for quantization of theories with local symmetries?

- Applies to a very general class of models, including string quantization and effective quantum gravity (QG).
- Allows to quantize observables that are non-local, which is particularly important for QG.
- Uncovers underlying geometrical structures and leads to interesting mathematics (homological algebra, homotopy).
- Justifies constructions, which otherwise seem ad hoc.
- Delivers an abstract definition of the quantum BRST differential, without the need for constructing the charge in a given representation.

References:

- K. Fredenhagen, KR *Batalin-Vilkovisky formalism in the functional approach to classical field theory* (CMP 2012),
- K. Fredenhagen, KR *Batalin-Vilkovisky formalism in perturbative algebraic quantum field theory* (CMP 2013).

Physical input

- A globally hyperbolic spacetime (M, g).

Physical input

- A globally hyperbolic spacetime (M, g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).

Physical input

- A globally hyperbolic spacetime (M, g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.

Physical input

- A globally hyperbolic spacetime (M,g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
    - For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M,\mathbb{R})$.

Physical input

- A globally hyperbolic spacetime (M, g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
  - For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$.
  - For Yang-Mills with trivial bundle: $\mathcal{E}(M) \equiv \Omega^1(M, \mathfrak{k})$, where $\mathfrak{k}$ is a Lie algebra of a compact Lie group.

Physical input

- A globally hyperbolic spacetime (M, g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
  - For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$.
  - For Yang-Mills with trivial bundle: $\mathcal{E}(M) \equiv \Omega^1(M, \mathfrak{k})$, where $\mathfrak{k}$ is a Lie algebra of a compact Lie group.
  - For effective QG: $\mathcal{E}(M) = \Gamma((T^*M)^{\otimes 2})$.

Physical input

- A globally hyperbolic spacetime (M, g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
  - For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$.
  - For Yang-Mills with trivial bundle: $\mathcal{E}(M) \equiv \Omega^1(M, \mathfrak{k})$, where $\mathfrak{k}$ is a Lie algebra of a compact Lie group.
  - For effective QG: $\mathcal{E}(M) = \Gamma((T^*M)^{\otimes 2})$.
- We use notation $\varphi \in \mathcal{E}(M)$, also if it has several components.

Physical input

- A globally hyperbolic spacetime (M, g).
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
  - For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$.
  - For Yang-Mills with trivial bundle: $\mathcal{E}(M) \equiv \Omega^1(M, \mathfrak{k})$, where $\mathfrak{k}$ is a Lie algebra of a compact Lie group.
  - For effective QG: $\mathcal{E}(M) = \Gamma((T^*M)^{\otimes 2})$.
- We use notation $\varphi \in \mathcal{E}(M)$, also if it has several components.
- Dynamics: we use a modification of the Lagrangian formalism (fully covariant).

Classical observables

- Classical observables are smooth functionals on $\mathcal{E}(M)$, i.e. elements of $\mathcal{C}^\infty(\mathcal{E}(M), \mathbb{C})$.

Classical observables

- Classical observables are smooth functionals on $\mathcal{E}(M)$, i.e. elements of $\mathcal{C}^\infty(\mathcal{E}(M),\mathbb{C})$.
- For simplicity of notation (and because of functoriality), we drop M, if no confusion arises, i.e. write $\mathcal{E}$, $\mathcal{C}^\infty(\mathcal{E},\mathbb{C})$, etc.

Classical observables

- Classical observables are smooth functionals on $\mathcal{E}(M)$, i.e. elements of $\mathcal{C}^\infty(\mathcal{E}(M),\mathbb{C})$.
- For simplicity of notation (and because of functoriality), we drop M, if no confusion arises, i.e. write $\mathcal{E}$, $\mathcal{C}^\infty(\mathcal{E},\mathbb{C})$, etc.
- Localization of functionals governed by their spacetime support:

$$\begin{aligned}\operatorname{supp} F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi, \psi \in \mathcal{E},\\ \operatorname{supp} \psi \subset U \text{ such that } F(\varphi+\psi) \neq F(\varphi)\}\ .\end{aligned}$$

Classical observables

- Classical observables are smooth functionals on $\mathcal{E}(M)$, i.e. elements of $\mathcal{C}^\infty(\mathcal{E}(M),\mathbb{C})$.
- For simplicity of notation (and because of functoriality), we drop M, if no confusion arises, i.e. write $\mathcal{E}$, $\mathcal{C}^\infty(\mathcal{E},\mathbb{C})$, etc.
- Localization of functionals governed by their spacetime support:

$$\begin{aligned}\operatorname{supp} F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi,\psi \in \mathcal{E},\\ \operatorname{supp}\psi \subset U \text{ such that } F(\varphi+\psi) \neq F(\varphi)\}\ .\end{aligned}$$

- F is local, $F \in \mathcal{F}_{\text{loc}}$ if it is of the form: $F(\varphi) = \int_M f(j_x(\varphi))\, d\mu_g(x)$, where f is a function on the jet bundle over M and $j_x(\varphi)$ is the jet of φ at the point x. $\mathcal{F}$ is the space of multilocal functionals (products of local).

Classical observables

- Classical observables are smooth functionals on $\mathcal{E}(M)$, i.e. elements of $\mathcal{C}^\infty(\mathcal{E}(M),\mathbb{C})$.
- For simplicity of notation (and because of functoriality), we drop M, if no confusion arises, i.e. write $\mathcal{E}$, $\mathcal{C}^\infty(\mathcal{E},\mathbb{C})$, etc.
- Localization of functionals governed by their spacetime support:

$$\operatorname{supp} F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi,\psi \in \mathcal{E},\\ \operatorname{supp}\psi \subset U \text{ such that } F(\varphi+\psi) \neq F(\varphi)\}\ .$$

- F is local, $F \in \mathcal{F}_{\mathrm{loc}}$ if it is of the form: $F(\varphi) = \int_M f(j_x(\varphi))\, d\mu_g(x)$, where f is a function on the jet bundle over M and $j_x(\varphi)$ is the jet of φ at the point x. $\mathcal{F}$ is the space of multilocal functionals (products of local).
- A functional is regular, $F \in \mathcal{F}_{\mathrm{reg}}$ if $F^{(n)}(\varphi)$ is as smooth section (in general it would be distributional).

Dynamics

- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S : \mathcal{D} \to \mathcal{F}_{\mathrm{loc}}$, where $\mathcal{D}(M) = \mathcal{C}_0^\infty(M, \mathbb{R})$. Examples:

Dynamics

- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S : \mathcal{D} \to \mathcal{F}_{\mathrm{loc}}$, where $\mathcal{D}(M) = \mathcal{C}_0^\infty(M, \mathbb{R})$. Examples:
  - $S(f)[\varphi] = \displaystyle\int_M \Big(\tfrac{1}{2}\varphi^2 + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\Big) f d\mu_g,$

Dynamics

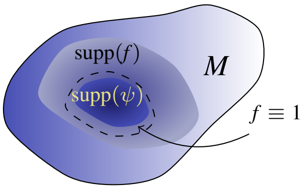
- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S:\mathcal{D}\to\mathcal{F}_{\mathrm{loc}}$, where $\mathcal{D}(M)=\mathcal{C}_0^\infty(M,\mathbb{R})$. Examples:
  - $S(f)[\varphi]=\int_M\left(\frac{1}{2}\varphi^2+\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\right)fd\mu_g$,
  - $S(f)[A]=-\frac{1}{2}\int_M f\,\mathrm{tr}(F\wedge *F)$, F being field strength for A,

Dynamics

- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S:\mathcal{D}\to\mathcal{F}_{\mathrm{loc}}$, where $\mathcal{D}(M)=\mathcal{C}_0^\infty(M,\mathbb{R})$. Examples:
  - $S(f)[\varphi]=\int_M\Big(\frac{1}{2}\varphi^2+\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\Big)fd\mu_g$,
  - $S(f)[A]=-\frac{1}{2}\int_M f\,\mathrm{tr}(F\wedge *F)$, F being field strength for A,
  - $S(f)[g]\doteq\int R[g]f\,d\mu_g$

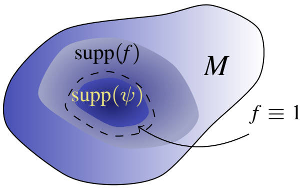
Dynamics

- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S:\mathcal{D}\to\mathcal{F}_{\mathrm{loc}}$, where $\mathcal{D}(M)=\mathcal{C}_0^\infty(M,\mathbb{R})$. Examples:
  - $S(f)[\varphi]=\int_M\left(\frac{1}{2}\varphi^2+\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\right)fd\mu_g$,
  - $S(f)[A]=-\frac{1}{2}\int_M f\,\mathrm{tr}(F\wedge *F)$, F being field strength for A,
  - $S(f)[g]\doteq\int R[g]f\,d\mu_g$
- The Euler-Lagrange derivative of S is denoted by dS and defined by $\langle dS(\varphi),\psi\rangle=\left\langle S^{(1)}(f)[\varphi],\psi\right\rangle$, where $f\equiv 1$ on supp$h$.



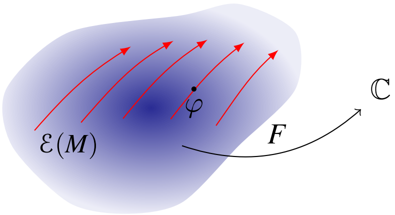
Dynamics

- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S : \mathcal{D} \to \mathcal{F}_{\mathrm{loc}}$, where $\mathcal{D}(M) = \mathcal{C}_0^\infty(M, \mathbb{R})$. Examples:
  - $S(f)[\varphi] = \int_M \left(\frac{1}{2}\varphi^2 + \frac{1}{2}\nabla_\mu \varphi \nabla^\mu \varphi\right) f d\mu_g,$
  - $S(f)[A] = -\frac{1}{2}\int_M f \,\mathrm{tr}(F \wedge *F)$, F being field strength for A,
  - $S(f)[g] \doteq \int R[g] f \, d\mu_g$
- The Euler-Lagrange derivative of S is denoted by dS and defined by $\langle dS(\varphi), \psi\rangle = \left\langle S^{(1)}(f)[\varphi], \psi \right\rangle$, where $f \equiv 1$ on supp$h$.
- The field equation is: $dS(\varphi) = 0$, so geometrically, the solution space is the zero locus of the 1-form dS.



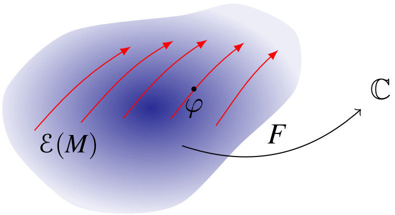
Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.



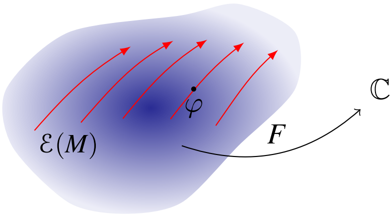
Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.
- We consider vector fields that are local, compactly supported and sufficiently regular and use notation $\mathcal{V}$.



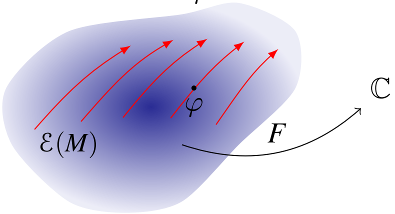
Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.
- We consider vector fields that are local, compactly supported and sufficiently regular and use notation $\mathcal{V}$.
- They act on $\mathcal{F}$ as derivations: $\partial_X F(\varphi) := \langle F^{(1)}(\varphi), X(\varphi)\rangle$



Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.
- We consider vector fields that are local, compactly supported and sufficiently regular and use notation $\mathcal{V}$.
- They act on $\mathcal{F}$ as derivations: $\partial_X F(\varphi) := \langle F^{(1)}(\varphi), X(\varphi)\rangle$
- A symmetry of S is a direction in $\mathcal{E}$ in which the action is constant, i.e. it is a vector field $X \in \mathcal{V}$ such that $\forall \varphi \in \mathcal{E}$: $0 = \langle dS(\varphi), X(\varphi)\rangle =: \delta_S(X)(\varphi)$.



Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.

Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.
- The space of on-shell functionals $\mathcal{F}_S$ is the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$.

Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.
- The space of on-shell functionals $\mathcal{F}_S$ is the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$.
- δ_S is called the Koszul differential. Symmetries constitute its kernel.

Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.
- The space of on-shell functionals $\mathcal{F}_S$ is the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$.
- δ_S is called the Koszul differential. Symmetries constitute its kernel.
- We obtain a sequence: $0 \to \mathrm{Sym} \hookrightarrow \mathcal{V} \xrightarrow{\delta_S} \mathcal{F} \to 0$.

Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.
- The space of on-shell functionals $\mathcal{F}_S$ is the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$.
- δ_S is called the Koszul differential. Symmetries constitute its kernel.
- We obtain a sequence: $0 \to \mathrm{Sym} \hookrightarrow \mathcal{V} \xrightarrow{\delta_S} \mathcal{F} \to 0$.
- For the beginning we consider the case where there are no non-trivial (not vanishing on $\mathcal{E}_S$) local symmetries,

Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.
- The space of on-shell functionals $\mathcal{F}_S$ is the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$.
- δ_S is called the Koszul differential. Symmetries constitute its kernel.
- We obtain a sequence: $0 \to \mathrm{Sym} \hookrightarrow \mathcal{V} \xrightarrow{\delta_S} \mathcal{F} \to 0$.
- For the beginning we consider the case where there are no non-trivial (not vanishing on $\mathcal{E}_S$) local symmetries,
- Let $\mathcal{KT} \doteq \left(\bigwedge \mathcal{V}, \delta_S\right)$. Then $\mathcal{F}_S = H_0(\mathcal{KT})$ and higher homologies vanish.

Equations of motion and symmetries

- Space of solutions: $\mathcal{E}_S \subset \mathcal{E}$. Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$. Assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.
- The space of on-shell functionals $\mathcal{F}_S$ is the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$.
- δ_S is called the Koszul differential. Symmetries constitute its kernel.
- We obtain a sequence: $0 \to \mathrm{Sym} \hookrightarrow \mathcal{V} \xrightarrow{\delta_S} \mathcal{F} \to 0$.
- For the beginning we consider the case where there are no non-trivial (not vanishing on $\mathcal{E}_S$) local symmetries,
- Let $\mathcal{KT} \doteq \left(\bigwedge \mathcal{V}, \delta_S\right)$. Then $\mathcal{F}_S = H_0(\mathcal{KT})$ and higher homologies vanish.
- This is called the Koszul complex.

Antifields and antibracket

- Vector fields $\mathcal{V}$ can be written formally as: $X = \int dx\, X(x) \frac{\delta}{\delta\varphi(x)}$.

Antifields and antibracket

- Vector fields $\mathcal{V}$ can be written formally as: $X = \int dx\, X(x) \frac{\delta}{\delta\varphi(x)}$.
- The action on functionals $F \in \mathcal{F}$ can be written as:

$$X(F)(\varphi) = \int dx\, X(\varphi)(x) \frac{\delta F}{\delta\varphi(x)}(\varphi)\,.$$

Antifields and antibracket

- Vector fields $\mathcal{V}$ can be written formally as: $X = \int dx\, X(x) \frac{\delta}{\delta\varphi(x)}$.
- The action on functionals $F \in \mathcal{F}$ can be written as:

$$X(F)(\varphi) = \int dx\, X(\varphi)(x) \frac{\delta F}{\delta\varphi(x)}(\varphi)\,.$$

- We can think of derivatives $\frac{\delta}{\delta\varphi(x)}$ as "generators" of $\mathcal{V}$.

Antifields and antibracket

- Vector fields $\mathcal{V}$ can be written formally as: $X = \int dx\, X(x) \frac{\delta}{\delta\varphi(x)}$.
- The action on functionals $F \in \mathcal{F}$ can be written as:

$$X(F)(\varphi) = \int dx\, X(\varphi)(x) \frac{\delta F}{\delta\varphi(x)}(\varphi)\,.$$

- We can think of derivatives $\frac{\delta}{\delta\varphi(x)}$ as "generators" of $\mathcal{V}$.
- In literature those objects are called *antifields* and are denoted by $\varphi^\ddagger(x)$, i.e.: $\varphi^\ddagger(x) \doteq \frac{\delta}{\delta\varphi(x)}$. The grading of Koszul complex is called antifield number #af.

Antifields and antibracket

- Vector fields $\mathcal{V}$ can be written formally as: $X = \int dx\, X(x) \frac{\delta}{\delta\varphi(x)}$.
- The action on functionals $F \in \mathcal{F}$ can be written as:

$$X(F)(\varphi) = \int dx\, X(\varphi)(x) \frac{\delta F}{\delta\varphi(x)}(\varphi)\,.$$

- We can think of derivatives $\frac{\delta}{\delta\varphi(x)}$ as "generators" of $\mathcal{V}$.
- In literature those objects are called *antifields* and are denoted by $\varphi^\ddagger(x)$, i.e.: $\varphi^\ddagger(x) \doteq \frac{\delta}{\delta\varphi(x)}$. The grading of Koszul complex is called antifield number #af.
- There is a graded bracket (called antibracket) identified with the Schouten bracket $\{.,.\}$ on multivector fields.

Antifields and antibracket

- Vector fields $\mathcal{V}$ can be written formally as: $X = \int dx\, X(x) \frac{\delta}{\delta\varphi(x)}$.
- The action on functionals $F \in \mathcal{F}$ can be written as:

$$X(F)(\varphi) = \int dx\, X(\varphi)(x) \frac{\delta F}{\delta\varphi(x)}(\varphi)\,.$$

- We can think of derivatives $\frac{\delta}{\delta\varphi(x)}$ as "generators" of $\mathcal{V}$.
- In literature those objects are called *antifields* and are denoted by $\varphi^\ddagger(x)$, i.e.: $\varphi^\ddagger(x) \doteq \frac{\delta}{\delta\varphi(x)}$. The grading of Koszul complex is called antifield number $\#$af.
- There is a graded bracket (called antibracket) identified with the Schouten bracket $\{.,.\}$ on multivector fields.
- Derivation δ_S is not inner with respect to $\{.,.\}$, but locally it can be written as $\delta_S X = \{X, S(f)\}$ for $f \equiv 1$ on supp$X$, $X \in \mathcal{V}$.

Invariants

- The space of symmetries is a Lie subalgebra of $\mathcal{V}$ and has a natural action on $\mathcal{F}$. Assume that this space is of the form $\mathcal{F}\widehat{\otimes}\mathfrak{s}$, for some Lie algebra $\mathfrak{s}$.

Invariants

- The space of symmetries is a Lie subalgebra of $\mathcal{V}$ and has a natural action on $\mathcal{F}$. Assume that this space is of the form $\mathcal{F}\widehat{\otimes}\mathfrak{s}$, for some Lie algebra $\mathfrak{s}$.
- In YM theories, we have $\mathfrak{s}(M) = \mathcal{C}^\infty(M, \mathfrak{k})$, while in gravity by $\mathfrak{s}(M) = \Gamma(TM)$.

Invariants

- The space of symmetries is a Lie subalgebra of $\mathcal{V}$ and has a natural action on $\mathcal{F}$. Assume that this space is of the form $\mathcal{F}\widehat{\otimes}\mathfrak{s}$, for some Lie algebra $\mathfrak{s}$.
- In YM theories, we have $\mathfrak{s}(M) = \mathcal{C}^\infty(M, \mathfrak{k})$, while in gravity by $\mathfrak{s}(M) = \Gamma(TM)$.
- In physics we are interested in the space of on-shell functionals, invariant under the action of symmetries. We denote this space by $\mathcal{F}_S^{\text{inv}}$ and call it gauge invariant on-shell functionals.

Invariants

- The space of symmetries is a Lie subalgebra of $\mathcal{V}$ and has a natural action on $\mathcal{F}$. Assume that this space is of the form $\mathcal{F}\widehat{\otimes}\mathfrak{s}$, for some Lie algebra $\mathfrak{s}$.
- In YM theories, we have $\mathfrak{s}(M) = \mathcal{C}^\infty(M, \mathfrak{k})$, while in gravity by $\mathfrak{s}(M) = \Gamma(TM)$.
- In physics we are interested in the space of on-shell functionals, invariant under the action of symmetries. We denote this space by $\mathcal{F}_S^{\mathrm{inv}}$ and call it gauge invariant on-shell functionals.
- $\mathcal{F}_S^{\mathrm{inv}}$ is characterized with the Chevalley-Eilenberg complex

$$\mathcal{CE} \doteq \left(\bigwedge \mathfrak{s}^* \widehat{\otimes} \mathcal{F}, \gamma\right).$$

Invariants

- The space of symmetries is a Lie subalgebra of $\mathcal{V}$ and has a natural action on $\mathcal{F}$. Assume that this space is of the form $\mathcal{F}\widehat{\otimes}\mathfrak{s}$, for some Lie algebra $\mathfrak{s}$.
- In YM theories, we have $\mathfrak{s}(M)=\mathcal{C}^\infty(M,\mathfrak{k})$, while in gravity by $\mathfrak{s}(M)=\Gamma(TM)$.
- In physics we are interested in the space of on-shell functionals, invariant under the action of symmetries. We denote this space by $\mathcal{F}_S^{\mathrm{inv}}$ and call it gauge invariant on-shell functionals.
- $\mathcal{F}_S^{\mathrm{inv}}$ is characterized with the Chevalley-Eilenberg complex

$$\mathcal{CE}\doteq\left(\bigwedge\mathfrak{s}^*\widehat{\otimes}\mathcal{F},\gamma\right).$$

- In degree 0, γ acts as: $(\gamma F)(\xi)\doteq\partial_\xi F$, $\xi\in\mathfrak{s}$, $F\in\mathcal{F}$.

Invariants

- The space of symmetries is a Lie subalgebra of $\mathcal{V}$ and has a natural action on $\mathcal{F}$. Assume that this space is of the form $\mathcal{F}\widehat{\otimes}\mathfrak{s}$, for some Lie algebra $\mathfrak{s}$.
- In YM theories, we have $\mathfrak{s}(M)=\mathcal{C}^\infty(M,\mathfrak{k})$, while in gravity by $\mathfrak{s}(M)=\Gamma(TM)$.
- In physics we are interested in the space of on-shell functionals, invariant under the action of symmetries. We denote this space by $\mathcal{F}^{\text{inv}}_S$ and call it gauge invariant on-shell functionals.
- $\mathcal{F}^{\text{inv}}_S$ is characterized with the Chevalley-Eilenberg complex

$$\mathcal{CE}\doteq\left(\bigwedge\mathfrak{s}^*\widehat{\otimes}\mathcal{F},\gamma\right).$$

- In degree 0, γ acts as: $(\gamma F)(\xi)\doteq\partial_\xi F$, $\xi\in\mathfrak{s}$, $F\in\mathcal{F}$.
- If $F\in\mathcal{F}^{\text{inv}}$ then $\gamma F\equiv 0$, so the $H^0(\gamma)$ characterizes the gauge invariant functionals.

BV complex

- Now we combine gauge invariant and on-shell, to be able to characterize the space $\mathcal{F}_S^{\mathrm{inv}}$.

BV complex

- Now we combine gauge invariant and on-shell, to be able to characterize the space $\mathcal{F}_S^{\mathrm{inv}}$.
- Observation: $\mathcal{CE}$ is a graded manifold $\mathcal{E} \oplus \mathfrak{s}[1]$, so instead of vector fields on $\mathcal{E}$, we should consider the vector fields on the extended configuration space $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$.

BV complex

- Now we combine gauge invariant and on-shell, to be able to characterize the space $\mathcal{F}_S^{\mathrm{inv}}$.
- Observation: $\mathcal{CE}$ is a graded manifold $\mathcal{E} \oplus \mathfrak{s}[1]$, so instead of vector fields on $\mathcal{E}$, we should consider the vector fields on the extended configuration space $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$.
- This way we obtain the BV complex: $\mathcal{BV}(M)$. Its underlying algebra is the algebra of multivector fileds on $\overline{\mathcal{E}}$.

BV complex

- Now we combine gauge invariant and on-shell, to be able to characterize the space $\mathcal{F}_S^{\mathrm{inv}}$.
- Observation: $\mathcal{CE}$ is a graded manifold $\mathcal{E} \oplus \mathfrak{s}[1]$, so instead of vector fields on $\mathcal{E}$, we should consider the vector fields on the extended configuration space $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$.
- This way we obtain the BV complex: $\mathcal{BV}(M)$. Its underlying algebra is the algebra of multivector fileds on $\overline{\mathcal{E}}$.
- $\mathcal{BV}$ is equipped with the BV differential, which in simple cases is just $s = \delta + \gamma$ (in general, more work needed).

BV complex

- Now we combine gauge invariant and on-shell, to be able to characterize the space $\mathcal{F}_S^{\text{inv}}$.
- Observation: $\mathcal{CE}$ is a graded manifold $\mathcal{E} \oplus \mathfrak{s}[1]$, so instead of vector fields on $\mathcal{E}$, we should consider the vector fields on the extended configuration space $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$.
- This way we obtain the BV complex: $\mathcal{BV}(M)$. Its underlying algebra is the algebra of multivector fileds on $\overline{\mathcal{E}}$.
- $\mathcal{BV}$ is equipped with the BV differential, which in simple cases is just $s = \delta + \gamma$ (in general, more work needed).
- We have $H^0(s) = H^0(H_0(\delta), \gamma) = \mathcal{F}_S^{\text{inv}}$, which is the reason to work with $\mathcal{BV}$ as it contains the same information as $\mathcal{F}_S^{\text{inv}}$, but has a simpler algebraic structure (quotients and spaces of orbits are resolved).

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a graded bracket (the Schouten bracket again).

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a graded bracket (the Schouten bracket again).
- Derivation δ_S is not inner with respect to $\{.,.\}$, but locally it can be written as:

$$\delta_S X = \{X, S(f)\}\,, \quad f \equiv 1 \text{ on supp} X\,,\ X \in \mathcal{V}$$

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a graded bracket (the Schouten bracket again).
- Derivation δ_S is not inner with respect to $\{.,.\}$, but locally it can be written as:

$$\delta_S X = \{X, S(f)\}\,, \quad f \equiv 1 \text{ on } \mathrm{supp} X\,,\ X \in \mathcal{V}$$

- Similarly $sX = \{X, S^{\mathrm{ext}}(f)\}$, where S^{ext} is the extended action, which contains ghosts (odd generators of $\mathcal{CE}$), antifields and often non-minimal sector needed for implementing the gauge fixing (see the talk of Hollands).

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a graded bracket (the Schouten bracket again).
- Derivation δ_S is not inner with respect to $\{.,.\}$, but locally it can be written as:

$$\delta_S X = \{X, S(f)\}\,, \quad f \equiv 1 \text{ on supp} X\,,\ X \in \mathcal{V}$$

- Similarly $sX = \{X, S^{\text{ext}}(f)\}$, where S^{ext} is the extended action, which contains ghosts (odd generators of $\mathcal{CE}$), antifields and often non-minimal sector needed for implementing the gauge fixing (see the talk of Hollands).
- The BV differential s has to be nilpotent, i.e.: $s^2 = 0$, which leads to the classical master equation (CME):

$$\{S^{\text{ext}}(f), S^{\text{ext}}(f)\} = 0\,,$$

modulo terms that vanish in the limit of constant f.

Poisson structure and the $\star$-product

- Firstly, linearize S^{ext} around a fixed configuration φ_0, and write $S^{\text{ext}} = S_0 + V$, where S_0 might contain both fields and antifields.

Poisson structure and the $\star$-product

- Firstly, linearize S^{ext} around a fixed configuration φ_0, and write $S^{\text{ext}} = S_0 + V$, where S_0 might contain both fields and antifields.
- The Poisson bracket of the free theory is

$$\{F, G\} \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

where $\Delta = \Delta^{\text{R}} - \Delta^{\text{A}}$ is the Pauli-Jordan function for the $\#\text{af} = 0$ part of S_0.

Poisson structure and the $\star$-product

- Firstly, linearize S^{ext} around a fixed configuration φ_0, and write $S^{\text{ext}} = S_0 + V$, where S_0 might contain both fields and antifields.
- The Poisson bracket of the free theory is

$$\{F, G\} \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

where $\Delta = \Delta^{\text{R}} - \Delta^{\text{A}}$ is the Pauli-Jordan function for the $\#\text{af} = 0$ part of S_0.
- Define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle ,$$

where W is the 2-point function of a Hadamard state and it differs from $\frac{i}{2}\Delta$ by a symmetric bidistribution: $W = \frac{i}{2}\Delta + H$.

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,
- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle ,$$

where $\Delta^{\mathrm{F}} = \dfrac{i}{2}(\Delta^{\mathrm{A}} + \Delta^{\mathrm{R}}) + H$ and $H = W - \dfrac{i}{2}\Delta$.

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,
- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle ,$$

where $\Delta^{\mathrm{F}} = \frac{i}{2}(\Delta^{\mathrm{A}} + \Delta^{\mathrm{R}}) + H$ and $H = W - \frac{i}{2}\Delta$.

- Formally it corresponds to the operator of convolution with the oscillating Gaussian measure "with covariance $i\hbar\Delta^{\mathrm{F}}$",

$$\mathcal{T}F(\varphi) \overset{\text{formal}}{=} \int F(\varphi - \phi)\, d\mu_{i\hbar\Delta_F}(\phi) .$$

Time-ordered product

- Let $\mathcal{F}_{\text{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,
- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\text{F}})^{\otimes n} \right\rangle ,$$

where $\Delta^{\text{F}} = \frac{i}{2}(\Delta^{\text{A}} + \Delta^{\text{R}}) + H$ and $H = W - \frac{i}{2}\Delta$.

- Formally it corresponds to the operator of convolution with the oscillating Gaussian measure "with covariance $i\hbar\Delta^{\text{F}}$",

$$\mathcal{T}F(\varphi) \overset{\text{formal}}{=} \int F(\varphi - \phi)\, d\mu_{i\hbar\Delta_F}(\phi) .$$

- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\text{reg}}[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G \,,$$

  if the support of F is later than the support of G.

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that
$$F \cdot_{\mathcal{T}} G = F \star G\,,$$
if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}$.

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

  if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

  if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar,\lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star -1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

if the support of F is later than the support of G.

- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star -1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

- We define the interacting star product as:

$$F \star_{int} G \doteq R_V^{-1}\left(R_V(F) \star R_V(G)\right)\,,$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

if the support of F is later than the support of G.

- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star -1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

- We define the interacting star product as:

$$F \star_{int} G \doteq R_V^{-1}\left(R_V(F) \star R_V(G)\right)\,,$$

- Renormalization problem: extend $\cdot_{\mathcal{T}}$ to V local and non-linear.

QME on regular functionals

- The quantum master equation is the condition that the S-matrix is invariant under the quantum Koszul operator:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star = 0\,,$$

where $\{.,.\}_\star$ is the antibracket where the pointwise product is replaced by $\star$.

QME on regular functionals

- The quantum master equation is the condition that the S-matrix is invariant under the quantum Koszul operator:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star = 0\,,$$

where $\{.,.\}_\star$ is the antibracket where the pointwise product is replaced by $\star$.

- The left-hand side can be rewritten as:

$$e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\frac{1}{2}\{S+V, S+V\}_{\mathcal{T}} - i\hbar \bigtriangleup (S+V)\right) = \{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star\,.$$

QME on regular functionals

- The quantum master equation is the condition that the S-matrix is invariant under the quantum Koszul operator:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star = 0\,,$$

where $\{.,.\}_\star$ is the antibracket where the pointwise product is replaced by $\star$.

- The left-hand side can be rewritten as:

$$e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\frac{1}{2}\{S+V, S+V\}_{\mathcal{T}} - i\hbar \bigtriangleup (S+V)\right) = \{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star\,.$$

- We obtain the standard form of the QME:

$$\frac{1}{2}\{S+V, S+V\}_{\mathcal{T}} = i\hbar \bigtriangleup_{S+V}\ .$$

QME on regular functionals

- The quantum master equation is the condition that the S-matrix is invariant under the quantum Koszul operator:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star = 0\,,$$

where $\{.,.\}_\star$ is the antibracket where the pointwise product is replaced by $\star$.
- The left-hand side can be rewritten as:

$$e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\frac{1}{2}\{S+V, S+V\}_{\mathcal{T}} - i\hbar \bigtriangleup (S+V)\right) = \{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star\,.$$

- We obtain the standard form of the QME:

$$\frac{1}{2}\{S+V, S+V\}_{\mathcal{T}} = i\hbar \bigtriangleup_{S+V}\,.$$

- This should be understood as a condition on V, which guarantees that the S-matrix on-shell doesn't depend on the gauge fixing.

Quantum BV operator I

- The linearized BV operator is defined by

$$\hat{s}_0 X = \{X, S_0\}_\star \,.$$

Under appropriate conditions on the 2-point function W, $\hat{s}_0 = s_0$.

Quantum BV operator I

- The linearized BV operator is defined by

$$\hat{s}_0 X = \{X, S_0\}_\star \, .$$

Under appropriate conditions on the 2-point function W, $\hat{s}_0 = s_0$.

- The quantum BV operator $\hat{s}$ is defined on regular functionals by:

$$R_V \circ \hat{s} = \hat{s}_0 \circ R_V \, ,$$

the twist of the free quantum BV operator by the (non-local!) map that intertwines the free and the interacting theory.

Quantum BV operator I

- The linearized BV operator is defined by

$$\hat{s}_0 X = \{X, S_0\}_\star \, .$$

  Under appropriate conditions on the 2-point function W, $\hat{s}_0 = s_0$.
- The quantum BV operator $\hat{s}$ is defined on regular functionals by:

$$R_V \circ \hat{s} = \hat{s}_0 \circ R_V \, ,$$

  the twist of the free quantum BV operator by the (non-local!) map that intertwines the free and the interacting theory.
- The 0th cohomology of $\hat{s}$ characterizes quantum gauge invariant observables.

Quantum BV operator II

- Assuming QME, $\hat{s}X = e_{\mathcal{T}}^{-iV/\hbar} \cdot_{\mathcal{T}} \left(\{e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} X, S_0\}_{\star}\right)$.

Quantum BV operator II

- Assuming QME, $\hat{s}X = e_{\mathcal{T}}^{-iV/\hbar} \cdot_{\mathcal{T}} \left(\{ e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} X, S_0 \}_\star \right)$.
- $\hat{s}$ on regular functionals can also be written as:

$$\hat{s} = \{., S + V\}_{\mathcal{T}} - i\hbar\triangle\,,$$

where $\triangle$ is the BV Laplacian, which on regular functionals is

$$\triangle X = (-1)^{(1+|X|)} \int dx \frac{\delta^2 X}{\delta\varphi^\ddagger(x)\delta\varphi(x)}\,.$$

Quantum BV operator II

- Assuming QME, $\hat{s}X = e_{\mathcal{T}}^{-iV/\hbar} \cdot_{\mathcal{T}} \left(\{ e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} X, S_0 \}_\star \right)$.
- $\hat{s}$ on regular functionals can also be written as:

$$\hat{s} = \{.,S+V\}_{\mathcal{T}} - i\hbar\triangle\,,$$

where $\triangle$ is the BV Laplacian, which on regular functionals is

$$\triangle X = (-1)^{(1+|X|)} \int dx \frac{\delta^2 X}{\delta\varphi^\ddagger(x)\delta\varphi(x)}\,.$$

- In our framework this is a mathematically rigorous result, no path integral needed (in contrast to other approaches).

Towards renormalization

To extend QME and $\hat{s}$ to local observables, we need to replace $\cdot_{\mathcal{T}}$ with the renormalized time-ordered product.

Theorem (K. Fredenhagen, K.R. 2011)

The renormalized time-ordered product $\cdot_{\mathcal{T}_{\mathrm{r}}}$ is an associative product on $\mathcal{T}_{\mathrm{r}}(\mathcal{F})$ given by

$$F \cdot_{\mathcal{T}_{\mathrm{r}}} G \doteq \mathcal{T}_{\mathrm{r}}(\mathcal{T}_{\mathrm{r}}^{-1} F \cdot \mathcal{T}_{\mathrm{r}}^{-1} G)\,,$$

where $\mathcal{T}_{\mathrm{r}} : \mathcal{F}[[\hbar]] \to \mathcal{T}_{\mathrm{r}}(\mathcal{F})[[\hbar]]$ is defined as

$$\mathcal{T}_{\mathrm{r}} = (\oplus_n \mathcal{T}_{\mathrm{r}}^n) \circ \beta\,,$$

where $\beta : \mathcal{T}_{\mathrm{r}} : \mathcal{F} \to S^{\bullet}\mathcal{F}_{\mathrm{loc}}^{(0)}$ is the inverse of multiplication m.

Renormalized QME and the quantum BV operator

- Since $\cdot_{\mathcal{T}_r}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$\{e_{\mathcal{T}_r}^{iV/\hbar}, S_0\}_\star = 0$$

$$\hat{s}(X) \doteq e_{\mathcal{T}_r}^{-iV/\hbar} \cdot_{\mathcal{T}_r} \left(\{e_{\mathcal{T}_r}^{iV/\hbar} \cdot_{\mathcal{T}_r} X, S_0\}_\star \right) ,$$

Renormalized QME and the quantum BV operator

- Since $\cdot_{\mathcal{T}_r}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$\{e^{iV/\hbar}_{\mathcal{T}_r}, S_0\}_\star = 0$$
$$\hat{s}(X) \doteq e^{-iV/\hbar}_{\mathcal{T}_r} \cdot_{\mathcal{T}_r} \left(\{e^{iV/\hbar}_{\mathcal{T}_r} \cdot_{\mathcal{T}_r} X, S_0\}_\star\right) ,$$

- These formulas get even simpler if we use the anomalous Master Ward Identity ([Brenecke-Dütsch 08, Hollands 07]).

Renormalized QME and the quantum BV operator

- Using the MWI we obtain following formulas:

$$0 = \frac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}_r} - \triangle_V \,,$$
$$\hat{s}X = \{X, V + S_0\} - \triangle_V(X) \,,$$

where $\triangle_V$ is identified with the anomaly term and $\triangle_V(X) \doteq \frac{d}{d\lambda} \triangle_{V+\lambda X} \Big|_{\lambda=0}$.

Renormalized QME and the quantum BV operator

- Using the MWI we obtain following formulas:

$$0 = \frac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}_r} - \triangle_V \,,$$
$$\hat{s}X = \{X, V + S_0\} - \triangle_V(X) \,,$$

  where $\triangle_V$ is identified with the anomaly term and $\triangle_V(X) \doteq \frac{d}{d\lambda} \triangle_{V+\lambda X} \big|_{\lambda=0}$.
- Hence, by using the renormalized time ordered product $\cdot_{\mathcal{T}_r}$, we obtained in place of $\triangle(X)$, the interaction-dependent operator $\triangle_V(X)$ (the anomaly). It is of order $\mathcal{O}(\hbar)$ and local.

Renormalized QME and the quantum BV operator

- Using the MWI we obtain following formulas:

$$0 = \frac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}_r} - \triangle_V ,$$
$$\hat{s}X = \{X, V + S_0\} - \triangle_V(X) ,$$

where $\triangle_V$ is identified with the anomaly term and $\triangle_V(X) \doteq \frac{d}{d\lambda} \triangle_{V+\lambda X} \big|_{\lambda=0}$.
- Hence, by using the renormalized time ordered product $\cdot_{\mathcal{T}_r}$, we obtained in place of $\triangle(X)$, the interaction-dependent operator $\triangle_V(X)$ (the anomaly). It is of order $\mathcal{O}(\hbar)$ and local.
- In the renormalized theory, $\triangle_V$ is well-defined on local vector fields, in contrast to $\triangle$.

Conclusions

- We combined geometrical structures underlying the BV formalism with pAQFT, to develop a general framework to quantize theories with local symmetries.

Conclusions

- We combined geometrical structures underlying the BV formalism with pAQFT, to develop a general framework to quantize theories with local symmetries.
- Our approach avoids using path integrals and ill-defined quantities in intermediate steps.

Conclusions

- We combined geometrical structures underlying the BV formalism with pAQFT, to develop a general framework to quantize theories with local symmetries.
- Our approach avoids using path integrals and ill-defined quantities in intermediate steps.
- We showed that for regular objects our definitions agree with the standard ones.

Conclusions

- We combined geometrical structures underlying the BV formalism with pAQFT, to develop a general framework to quantize theories with local symmetries.
- Our approach avoids using path integrals and ill-defined quantities in intermediate steps.
- We showed that for regular objects our definitions agree with the standard ones.
- We proved the associativity of the renormalized time-ordered product and this allowed us to use $\mathcal{T}_r$ instead of $\mathcal{T}$ in algebraic formulas for the QME and $\hat{s}$ (which we postulated).

Conclusions

- We combined geometrical structures underlying the BV formalism with pAQFT, to develop a general framework to quantize theories with local symmetries.
- Our approach avoids using path integrals and ill-defined quantities in intermediate steps.
- We showed that for regular objects our definitions agree with the standard ones.
- We proved the associativity of the renormalized time-ordered product and this allowed us to use $\mathcal{T}_r$ instead of $\mathcal{T}$ in algebraic formulas for the QME and $\hat{s}$ (which we postulated).
- The renormalized QME and the quantum BV operator are defined in a natural way and don't suffer from divergent terms,

Conclusions

- We combined geometrical structures underlying the BV formalism with pAQFT, to develop a general framework to quantize theories with local symmetries.
- Our approach avoids using path integrals and ill-defined quantities in intermediate steps.
- We showed that for regular objects our definitions agree with the standard ones.
- We proved the associativity of the renormalized time-ordered product and this allowed us to use $\mathcal{T}_{\mathrm{r}}$ instead of $\mathcal{T}$ in algebraic formulas for the QME and $\hat{s}$ (which we postulated).
- The renormalized QME and the quantum BV operator are defined in a natural way and don't suffer from divergent terms,
- Example applications: Yang-Mills theories, bosonic string, perturbative quantum gravity.

Thank you for your attention!