Natural bridge, Yoho national park

Skew morphisms of cyclic groups and complete regular dessins

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Skew-morphisms

Skew-morphism A permutation ψ of elements of a group G is called a skew-morphism if

- $\psi(1_G) = 1_G$ and
- $\psi(a \cdot b) = \psi(a) \cdot (\psi(b))^{\pi(a)}$, where $\pi : G \to Z$ is a power function.

Jajcay and Siran $CM(G, P)$ is orientably regular iff P extends to a skew-morphism of G.

Structure $Aut(CM(G, P)) = G_L \cdot \langle \psi \rangle$

A fact Skew morphisms appear in constructions of groups acting on orientable surfaces, the main reason is that stabilisers of such actions are cyclic and sometimes the stabiliser has a complement

Skew morphisms and skew products of groups

Given a skew morphism σ of a group H we may form a product $G = H_L \cdot \langle\sigma\rangle$, where H_L is the left-regular representation.

Vice-versa, let $H_L \leq G \leq Sym(H)$. Suppose the stabiliser G_1 is cyclic. Then any generator σ of $G_1 = \langle\sigma\rangle$ is in $Skew(H)$.

Hence, $Skew(H)$ contains a complete information on the products $H \cdot C$, where C is cyclic.

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Group products and skew morphisms

Let G be a finite group having a factorisation $G = AB$ into subgroups A and B with B cyclic and $A \cap B = 1$; and let b be a generator of B. Then there exists a bijective mapping $f : A \to A$ well defined by the equality $baB = f(a)B$, moreover, f is a skew morphism of A.

Cyclic case: If both A and B are cyclic, we can transpose the role of A and B since $AB = BA$. Let a be a fixed generator of A. Then the equality $abA = f^*(b)A$ determines a skew morphism of A.

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Products of cyclic groups

Let $G = Z_n$ then $G_L = \langle \tau \rangle$, where $\tau(x) = x + 1$.

Let σ be a skew-morphism of Z_n. Then the characteristic identity for σ can be expressed as

$$\sigma \tau^y = \tau^{\sigma(y)} \sigma^{\pi(y)} \text{ for all } y \in Z_n.$$

So it defines a commuting rule in the product of cyclic groups $\langle \tau \rangle \langle \sigma \rangle$.

Classification of products of cyclic groups - an old project of Wieland, Huppert, Schur,.... For instance, if the product is a p-group, p is odd we know that it is a **metacycyclic group and a parametrized presentation is known**,

But we want more...

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Why skew-morphisms are difficult to understand?

- understanding $Aut(G)$ for a general group G is a hard problem, and automorphisms are the simplest skew-morphisms,
- $Skew(G)$ is not closed under composition,
- $Skew(G)$ allows understanding products $G_L \cdot \langle\psi\rangle$ and in general products of the form $G \cdot C$, where C is cyclic, a difficult problem in group theory even for G cyclic,

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Pure skew-morphisms in Z_{p^e}, p an odd prime

- Set $\sigma_{a,b}(x) = ax + b\frac{x(x-1)}{2}$, $a \equiv 1 \bmod p$ and $b \in \langle p^{e-1} \rangle$.
- The permutations $\sigma_{a,b}$ are pure skew-morphisms of Z_{p^e} if and only if $a \equiv 1 \bmod p$, $a \notin 1 + \langle p^{e-1} \rangle$ and $b \neq 0$.
- The total number of s-m of Z_{p^e}, $e \geq 2$ is $(p-1)(p^{2e-1} - p^{2e-2} + 2)/(p+1) > (p-1)p^{e-1}$

Regular dessins

- Dessin is a bipartite bicoloured map.
- **Regular dessin** is a dessin D such that $Aut(D)$ is regular on the edges.
- Regular dessins corresponds to the triples $(G; x, y)$, where $G = \langle x, y \rangle$
- **Complete regular dessin**: $G = \langle x \rangle \langle y \rangle$ and $\langle x \rangle \cap \langle y \rangle = 1$,
- If $|x| = m$ and $|y| = n$, then the underlying graph is $K_{m,n}$.

Skew morphisms and complete regular dessins

Let φ and φ' be skew-morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, and let π and π' be the associated power functions, respectively. If

(i) the orders of φ and φ' divide m and n respectively,

(ii) $\pi(x) \equiv -\varphi'^{-x}(-1) \pmod{|\varphi|}$ and $\pi'(y) \equiv -\varphi^{-y}(-1) \pmod{|\varphi'|}$.

then the skew-morphism pair (φ, φ') will be called *admissible*.

Theorem

Given pair of positive integers (m, n) the admissible pairs of skew morphisms are in one-to-one correspondence with the isomorphism classes of complete regular dessins with the underlying graph $K_{m,n}$.

Trivial admissible pairs

Since (id_m, id_n) is a (trivial) admissible pair of skew morphisms, for each (m, n) there exists at least one complete regular dessin with the underlying graph $K_{m,n}$.

When there are no others?

Remark: The admissible pair (id_n, id_n) determines the dessin that corresponds to the Fermat curve $x^n + y^n = 1$.

Uniqueness theorem

An integer is called *singular* if and only if $(n, \phi(n)) = 1$ A pair (m, n) of positive integers m and n will be called *singular* if $(m, \phi(n)) = (n, \phi(m)) = 1$.

Theorem. Let (m, n) be a pair of positive integers. Then there exists a unique complete regular dessin with the underlying graph $K_{m,n}$ if and only if (m, n) is a singular pair of integers.

Remark 1: There is a unique group of order n if and only if n is singular.

Remark 2 Erdos proved that the portion of integers $n \leq N$ that are singular is asymptotically $\frac{1}{e^{\gamma} \log \log \log N}$, where $\gamma = 0.577215664901532860606512090082402431042159335939 92...$ is the Euler constant

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Regular embeddings of $K_{n,n}$

- self-dual dessins with respect to swapping the colours,
- the respective admissible pairs are of the form (φ, φ)
- Classification done by Jones, Škoviera, Du, Kwak, N. in a serie of papers using group theoretical methods, for instance if $n = p^e$ there are p^{e-1} such dessins,
- **Corollary:** There is 1-1 correspondence between regular embeddings of $K_{n,n}$ and skew-morphisms, of cyclic group Z_n such that $ord(\sigma)|n$ and $\pi(x) = -\sigma^{-x}(-1)$.
- One implication observed by Kwak and Kwon.

Problems

1. Classify complete regular dessins.
2. What is the densite of singular pairs of integers?
3. What are the curves that correspond to complete regular dessins?
4. Characterise regular dessins defined by their automorphism groups. The corresponding curves will be nice, since such dessins are fixed by the action of the absolute Galois group.

End of Talk

Continuation: A Suplement

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Number of epimorphisms $\Pi_1(O) \to Z_\ell$

THEOREM: (Mednykh,N. 2006) Let $\Gamma = F[g; m_1, \ldots, m_r]$ be an $F-$group of signature $(g; m_1, \ldots, m_r)$ and $m = lcm\,(m_1, \ldots, m_r)$, $m|\ell$.
Then the number of order-preserving epimorphisms of the group Γ onto a cyclic group Z_ℓ is given by the formula

$$Epi_0(\Gamma, \mathsf{Z}_\ell) = m^{2g}\phi_{2g}(\ell/m)E(m_1, m_2, \ldots, m_r),$$

where

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m}\sum_{k=1}^{m} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r).$$

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ is a surface group of genus g we have

$$Epi_0(\Gamma, \mathsf{Z}_\ell) = \phi_{2g}(\ell).$$

Additional info

Jordan function: $\varphi_p(\ell) = \sum_{d|\,\ell} \mu(\frac{\ell}{d})\, d^p$

VonSerneck function:

$$\Phi(x,n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})}\,\mu\left(\frac{n}{(x,n)}\right) = \sum_{\substack{1\le k\le n\\(k,n)=1}} \exp(\frac{2\,ikx}{n}).$$

References:

1. A. Mednykh,R.N.,J. Combin. Theory B, 2006. Additive form of the orbicyclic function.
2. V. Liskovets, Integers 2010. Multiplicative form of the orbicyclic function
3. http://www.savbb.sk/ karabas/science.html
   Tables and enumeration of actions for small genera (up to 100).

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Keynote speakers: L. Babai, P. Cameron, E.van Dam, T. Ito, E. Lux (not confirmed), A. Munemasa, M. Muzychuk, D. Pasechnik, I. Ponomarenko, S. Shpectorov.

Further info: nedela savbb.sk


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