Homomorphisms between mapping class groups of surfaces

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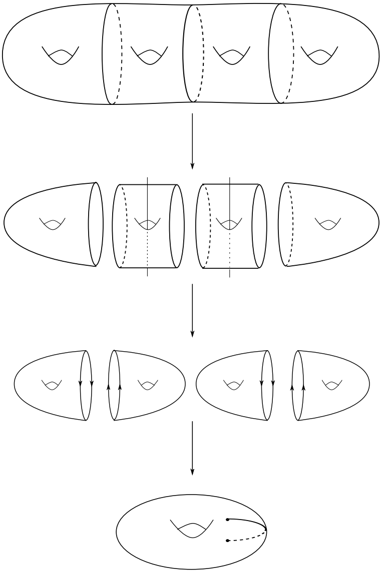
Mapping class group

S – closed surface possibly with punctures or marked points
MCG(S) – isotopy classes of orientation preserving diffeomorphisms of S that preserve the set of marked points

Motivating Question



$p : \widetilde{S} \to S$ – possibly branched covering space of surfaces.

Is there a natural relationship between $\mathrm{MCG}(\widetilde{S})$ and $\mathrm{MCG}(S)$?

Birman–Hilden Property

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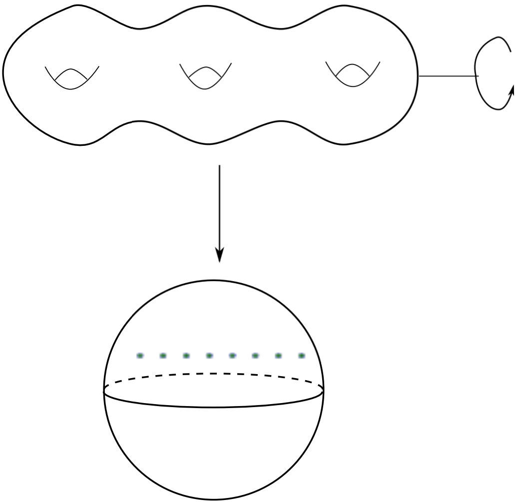
$p : \widetilde{S} \to S$ has Birman–Hilden property if for all $f \in \mathrm{SMCG}(\widetilde{S})$, the projections of any two representatives of f are isotopic in S.
That is, isotopy $\Rightarrow$ fiber-preserving isotopy.

Example



Does the covering have Birman–Hilden property?

The Birman–Hilden Theorem

Theorem (Birman, Hilden)

Let $\widetilde{S}$ be a hyperbolic surface. Let G be a finite group of diffeomorphisms of $\widetilde{S}$. Any finite connected (possibly branched) $p : \widetilde{S} \to \widetilde{S}/G$ has the Birman–Hilden Property.

Known Answers

	regular	irregular
unbranched	Yes-Birman–Hilden	
branched	Yes-Birman–Hilden	

Curve Lifting Property

$\widetilde{S} \to S$ branched covering space

Curve lifting property –The preimage of every essential, simple closed curve in S is an essential, simple closed multicurve in $\widetilde{S}$.

Proposition

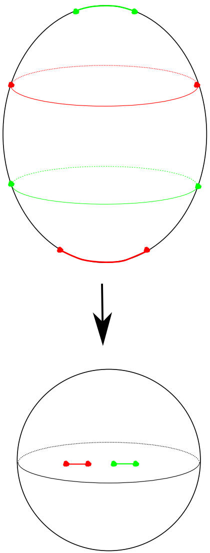
Let $p : \widetilde{S} \to S$ be a cover that has the Birman–Hilden Property. Then p has the curve lifting property.

Curve Lifting Property

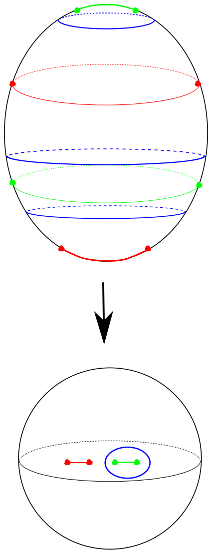
Theorem

There is an algorithm to check the curve lifting property.

Example without the Birman–Hilden Property



Example without the Birman–Hilden Property



Simple Covers

A simple n-fold cover – each branch point has $n-1$ preimages

Theorem (Berstein–Edmonds, W)

Let $p : \widetilde{S} \to S$ be an n-fold simple connected cover that has at least two branch points and $n \geq 3$. Then p does not have the Birman–Hilden property.

Known Answers

	regular	irregular
unbranched	Yes-Birman–Hilden	
branched	Yes-Birman–Hilden	No/??

Unbranched covers

Proposition (Aramayona–Leininger–Souto)

Let $\widetilde{S}$ *be a hyperbolic surface, and* $p : \widetilde{S} \to S$ *an unbranched cover. Then* p *has the Birman–Hilden Property.*

Known Answers

	regular	irregular
unbranched	Yes-Birman–Hilden	Yes, Aramayona–Leininger–Souto
branched	Yes-Birman–Hilden	No/??

Known Answers

	regular	irregular
unbranched	Yes-Birman–Hilden	Yes, Aramayona–Leininger–Souto
branched	Yes-Birman–Hilden	Sometimes Yes (ALS)/No

Sufficient Condition

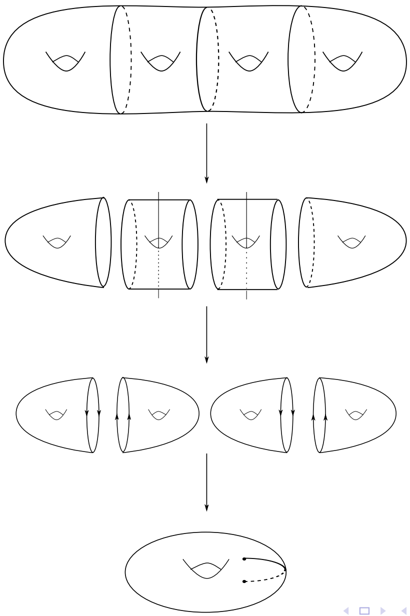
Theorem (W)

Let $p : \widetilde{S} \to S$ be a cover such that no branch points have unramified preimages. Then p has the Birman–Hilden Property.

Corollary (Birman–Hilden, Aramayona–Leininger–Souto)

Regular and unbranched covers have the Birman–Hilden property.

A new example



Isotopy projection property

A cover $p : \widetilde{S} \to S$ is said to have the isotopy projection property if all simple closed curves $\alpha, \beta \subset S$ are isotopic whenever $p^{-1}(\alpha)$ and $p^{-1}(\beta)$ are isotopic.

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Proposition

The isotopy projection property is equivalent to the Birman–Hilden property.

Future Work

Problem:

Checking the isotopy projection property requires checking infinite pairs of curves.

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Checking the isotopy projection property requires checking infinite pairs of curves.

Goal:

Find a property that is equivalent to the Birman–Hilden property and can be checked algorithmically.