Electromagnetic Waves in Honeycomb Lattices

Photonic analogs of graphene and topological insulator

Yi Zhu

Zhou Pei-Yuan Center for Applied Mathematics
Tsinghua University
yizhu@tsinghua.edu.cn

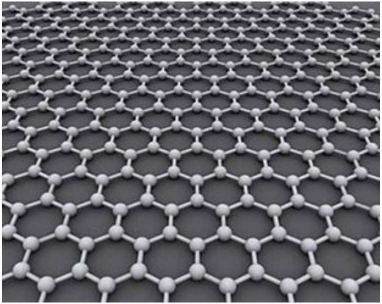
Joint work with:

Michael I. Weinstein	Columbia University
James Lee-Thorp	Courant Institute

Outline

Graphene

- A single layer carbon atoms in a two-dimensional **honeycomb** structure.



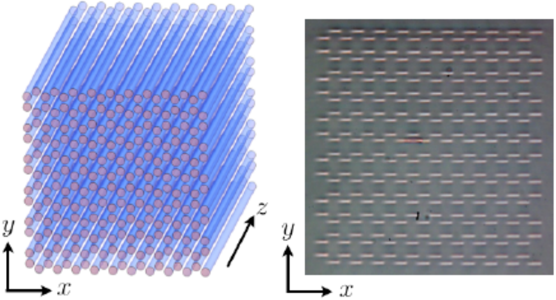
- The simplest one-electron model

$$i\partial_t \psi(\mathbf{x}, t) = \mathcal{H}\psi(\mathbf{x}, t) \equiv [-\Delta + V(\mathbf{x})]\psi(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^2, t > 0,$$

with $V(\mathbf{x})$ being a honeycomb lattice potential.

Artificial Graphene

Analogs in different physical systems, e.g., photonic graphene



(a) (b)

Refs: Segev, Rechtsman,Szameit, Khanikaev, Alu et al.

Maxwell's equation

Electromagnetic waves in a linear loss-free media

$$i\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

where the Maxwell operator is

$$\mathcal{M} \equiv \mathbf{R}^{-1} \begin{pmatrix} 0 & i\nabla\times \\ -i\nabla\times & 0 \end{pmatrix}$$

with a general 6×6 positive definite Hermitian material weight matrix

$$\mathbf{R} = \begin{pmatrix} \epsilon & \xi \\ \xi^\dagger & \mu \end{pmatrix}.$$

Paraxial VS In-plane propagations

In photonic crystals, $\mathbf{R}(\mathbf{x}) = \mathbf{R}(x, y)$ is invariant along the longitudinal direction. Simplified equations in two important propagations

1. **Paraxial propagation:** Fields propagate almost along the longitudinal direction (z direction). Paraxial wave equation (Schrödinger equation) is obtained.

$$i\partial_Z U + (\partial_{xx} + \partial_{yy})U + V(x_1, x_2)U = 0.$$

   See e.g., Segev, Rechtsman, Christodoulides, Chen, Yang, Ablowitz, Z. et al.

2. **In-plane propagation:** Fields propagate in the transverse plane:

$$(\mathbf{E}(x, y, z, t),\ \mathbf{H}(x, y, z, t))) = (\mathbf{E}(x, y, t),\ \mathbf{H}(x, y, t)).$$

   See e.g., Raghu, Haldane, Soljacic, Lu, Khanikaev, Alu et al.

Two examples

We will apply our analysis to the in-plane propagations of the following two physical systems.

1. magneto-optic media: Raghu and Haldane 2008

$$\mathbf{R} = \begin{pmatrix} \boldsymbol{\epsilon} & 0 \\ 0 & \boldsymbol{\mu} \end{pmatrix}, \ \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon & -i\gamma & 0 \\ i\gamma & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \boldsymbol{\mu} = I$$

2. bianisotropic media with dual symmetry ($\epsilon \approx \mu$): Khanikaev et al. 2013

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \xi = \begin{pmatrix} 0 & -i\chi & 0 \\ i\chi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Heterogeneous wave equation

In many in-plane propagations, Maxwell's equation is reduced to the 2-D heterogeneous wave equation

$$\partial_{tt}\psi(\mathbf{x}, t) - \nabla \cdot A(\mathbf{x})\nabla\psi(\mathbf{x}, t) = 0, \mathbf{x} \in \mathbb{R}^2$$

where the material weight $A(\mathbf{x})$ is a 2×2 positive definite Hermitian matrix.

- magneto-optic media: consider the TE mode, $\psi = H_3$ and

$$A(\mathbf{x}) = \epsilon^{-1} + \gamma\epsilon^{-2}\sigma_2.$$

- bianisotropic media: $\psi = H_3 \pm E_3$ and

$$A(\mathbf{x}) = \epsilon^{-1} \pm \chi\epsilon^{-2}\sigma_2.$$

Motivations

- The starting point of our mathematical analysis

$$\partial_{tt}\psi(\mathbf{x}, t) = -\mathcal{L}^A\psi(\mathbf{x}, t) \equiv \nabla \cdot A(\mathbf{x})\nabla\psi(\mathbf{x}, t).$$

- Fefferman, Weinstein, LeeThorp (2012-2016) developed a series of rigorous mathematical analysis on the Schrödinger equation with a honeycomb lattice including: existence of Dirac points, stability and instability of Dirac points, Dirac dynamics, strong binding limit, topological edge states, etc.
- **Question:** for the above wave equation (reduced version of Maxwell's equation), could we do similar analysis?
- **Remark:** $A(\mathbf{x})$ is a 2×2 Hermitian matrix and has more freedoms to manipulate compared to the potential $V(\mathbf{x})$ in the Schrödinger case.

General Material Weight Assumptions

In this talk, the 2×2 complex-valued matrix function $A(\mathbf{x})$ satisfies

1. $A(\mathbf{x})$ is smooth and Hermitian, *i.e.*, $A(\mathbf{x})^\dagger \equiv \overline{A(\mathbf{x})^T} = A(\mathbf{x})$ for all $\mathbf{x}$.
2. $A(\mathbf{x})$ is elliptic, *i.e.* there exist constants $c_\pm > 0$, such that for all $\mathbf{x} \in \mathbb{R}^2$ and all $\xi \in \mathbb{C}^2$: $c_-|\xi|^2 \;\le\; \langle \xi, A(\mathbf{x})\xi \rangle_{\mathbb{C}^2} \;\le c_+|\xi|^2$.

Remark: the smoothness assumption can be removed by using some technique treatments of the analysis.

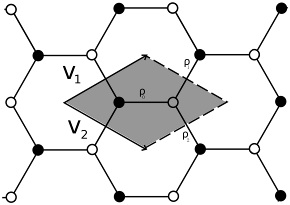
Hexagonal lattice

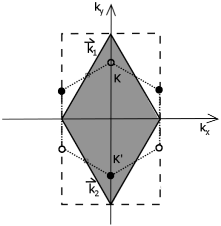
A hexagonal lattice is generated by

$$\mathbf{v}_1 = l\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \quad \mathbf{v}_2 = l\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)^T,$$

then

$$\mathbf{k}_1 = q\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T, \quad \mathbf{k}_2 = q\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^T, \quad q \equiv \frac{4\pi}{\sqrt{3}}.$$





Floquet-Bloch theory

To under stand the spectrum of the operator $\mathcal{L}^A$, we solve the Floquet-Bloch eigenvalue problem:

$$\mathcal{L}^A \Phi(\mathbf{x}) = E\Phi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$
$$\Phi(\mathbf{x}+\mathbf{v}) = e^{i\mathbf{k}\cdot\mathbf{v}}\Phi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad \mathbf{v} \in \Lambda_h,$$

for each $\mathbf{k}$ in the Brillouin Zone $\mathcal{B}$.
Alternatively, define $\Phi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}), \mathbf{k} \in \mathcal{B}$. Then $\phi(\mathbf{x})$ satisfies

$$\mathcal{L}^A(\mathbf{k})\phi(\mathbf{x}) = E(\mathbf{k})\phi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2$$
$$\phi(\mathbf{x}+\mathbf{v}) = \phi(\mathbf{x}), \mathbf{v} \in \Lambda$$

with $\quad \mathcal{L}^A(\mathbf{k}) = -(\nabla + i\mathbf{k}) \cdot A(\mathbf{x})(\nabla + i\mathbf{k}).$

Floquet-Bloch theory

Standard theory on the elliptic operator with periodic coefficients yields

- $\mathcal{L}^A(\mathbf{k})$ has discrete spectrum: $E_1(\mathbf{k}) \le E_2(\mathbf{k}) \le E_3(\mathbf{k}) \le \cdots$.
- The energy functions $\mathbf{k} \mapsto E_b(\mathbf{k})$, called band dispersion function, are Lipschitz-continuous. As $\mathbf{k}$ varies over $\mathcal{B}$, each function $E_b(\mathbf{k})$ sweeps out a closed real interval. The union over $b \ge 1$ of these closed intervals is exactly the $L^2(\mathbb{R}^2)$-spectrum of operator $\mathcal{L}^A$.
- The set $\{\Phi_b(\mathbf{x};\mathbf{k})\}_{b\ge 1,\mathbf{k}\in\mathcal{B}}$ is a complete (orthonormal) set in $L^2(\mathbb{R}^2)$:

$$f(\mathbf{x}) = \sum_{b\ge 1} \int_{\mathcal{B}} \langle \Phi_b(\cdot,\mathbf{k}), f(\cdot)\rangle_{L^2(\mathbf{R}^2)} \Phi_b(\mathbf{x};\mathbf{k})\, d\mathbf{k}$$

where the sum converges in the L^2 norm.

$\mathcal{C}$, $\mathcal{P}$ and $\mathcal{R}$ operators

Let $g(\mathbf{x})$ denote a function defined on $\mathbb{R}^2$. We define

- Complex conjugate operator

$$(\mathcal{C}g)(\mathbf{x}) \equiv \overline{g(\mathbf{x})}.$$

- Parity inversion operator

$$(\mathcal{P}g)(\mathbf{x}) \equiv g(-\mathbf{x}).$$

- 120-degree-rotation operator

$$(\mathcal{R}g)(\mathbf{x}) \equiv g(R^*\mathbf{x})$$

$$R = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Honeycomb structured media

A honeycomb-structured media is defined, if in addition to the general material weight assumption, $A(\mathbf{x})$ satisfies

1. $A(\mathbf{x}+\mathbf{v}) = A(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{v} \in \Lambda_h$.
2. $[\mathcal{PC}, \mathcal{L}^A] = 0 \Leftrightarrow \overline{A(-\mathbf{x})} = A(\mathbf{x})$.
3. $[\mathcal{R}, \mathcal{L}^A] = 0 \Leftrightarrow A(R^*\mathbf{x}) = R^* A(\mathbf{x}) R$.

A special case (**isotropic**):

$$A(\mathbf{x}) = a(\mathbf{x}) I_{2\times 2} + b(\mathbf{x})\sigma_2,$$

where $a(R^*\mathbf{x}) = a(\mathbf{x})$, $a(-\mathbf{x}) = a(\mathbf{x})$, and $b(R^*\mathbf{x}) = b(\mathbf{x})$, $b(-\mathbf{x}) = -b(\mathbf{x})$.

Fourier characterizations

The Fourier series of honeycomb-structured $A(\mathbf{x})$:

$$A(\mathbf{x}) = a_{\mathbf{0}} I + \sum_{\mathbf{m}\in\tilde{\mathcal{S}}\setminus\{\mathbf{0}\}} A_{\mathbf{m}}\, e^{i\mathbf{m}\vec{\mathbf{k}}\cdot\mathbf{x}} + R^* A_{\mathbf{m}} R\, e^{i(\tilde{R}\mathbf{m})\vec{\mathbf{k}}\cdot\mathbf{x}} + R A_{\mathbf{m}} R^*\, e^{i(\tilde{R}^2\mathbf{m})\vec{\mathbf{k}}\cdot\mathbf{x}}.$$

where $A_{\mathbf{m}}$ is **real** and satisfies $A_{-\mathbf{m}} = A_{\mathbf{m}}^T$.

Specially

- $A_{\mathbf{m}}$ is symmetric iff $A(\mathbf{x})$ is real;
- the isotropic case:

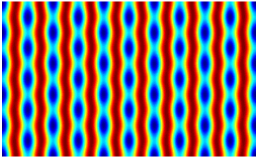
$$A_{\mathbf{m}} = \begin{pmatrix} a_{\mathbf{m}} & -i\ b_{\mathbf{m}} \\ i\ b_{\mathbf{m}} & a_{\mathbf{m}} \end{pmatrix}$$

  where $b_{\mathbf{m}}$ is purely imaginary.

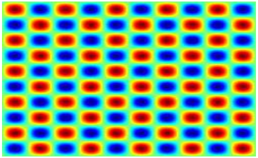
Anisotropic honeycomb material weight

Let $A_{0,1} = C$ real. Then $A_{1,0} = R^* C R$, $A_{-1,-1} = RCR^*$, $A_{0,-1} = C^T$, $A_{-1,0} = R^* C^T R$ and $A_{1,1} = RC^T R^*$. Let higher Fourier components be zero. Similar to the three wave interaction. $C = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$.

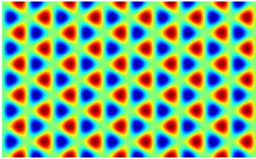
$A^{(1,1)}(x)$



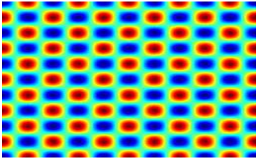
$\mathrm{Re}(A^{(1,2)})(x)$



$\mathrm{Im}(A^{(1,2)})(x)$



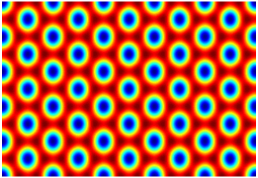
$A^{(2,2)}(x)$



Isotropic honeycomb material weight

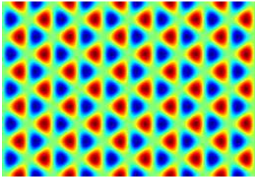
Taking $C = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

$A^{(1,1)}(x)$

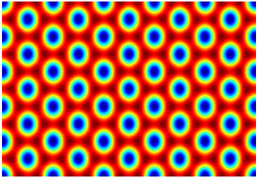


$Re(A^{(1,2)})(x)$

$Im(A^{(1,2)})(x)$



$A^{(2,2)}(x)$



Dirac points

Definition: A Dirac point is a quasi-momentum/energy pair $(\mathbf{K}_*, E_D)$ such that for $\mathbf{k}$ near $\mathbf{K}_*$:

$$E_\pm(\mathbf{k}) - E_D \approx \pm v_F |\mathbf{k} - \mathbf{K}_*|.$$

Theorem1: Conditions ensure the Dirac point $(\mathbf{K}_*, E_D)$.

1. Multiplicity 2: existence of two $\mathbf{K}_*$-quasi periodic Bloch modes $\Phi_j(\mathbf{x}), j = 1, 2$ such that

$$\mathcal{L}^A \Phi_j(\mathbf{x}) = E_D \Phi_j(\mathbf{x}), \quad \Phi_j(R^* \mathbf{x}) = \tau^j \Phi_j(\mathbf{x}), \quad j = 1, 2.$$

   where $\tau = e^{2\pi i/3}$.
2. Non-degeneracy: $|\langle \Phi_1, (1, -i) \cdot \mathcal{A} \Phi_2 \rangle| > 0$, where $\mathcal{A} = \frac{1}{i} A(\mathbf{x}) \nabla + \frac{1}{i} \nabla \cdot A(\mathbf{x})$ is like the $\hat{\mathbf{p}}$ operator in the Schrödinger case.

Existence of Dirac points of honeycomb structured media

Let $\mathcal{L}^A = -\nabla \cdot (I + \delta A_0(\mathbf{x}))\nabla$, where $I + \delta A_0(\mathbf{x})$ is a honeycomb structured medium. Fix $\mathbf{K}_\star = \mathbf{K}, \mathbf{K}'$ and assume

$$K_*^T A_{0,-1} R K_* \neq 0.$$

Theorem: If δ ensures positivity of $I + \delta A_0(\mathbf{x})$ and is not in a discrete set $\tilde{\mathcal{C}}$ where existence conditions fail, then

1. $\mathcal{L}^A$ has Dirac points in its band structure.
2. If $A(\mathbf{x})$ is further assumed to be real, then $E_D^{\mathbf{K}} = E_D^{\mathbf{K}'}$.

With some considerable modifications, the proof can be done by using the Fefferman & Weinstein's strategies developed for the Schrödinger operator (JAMS 2012).

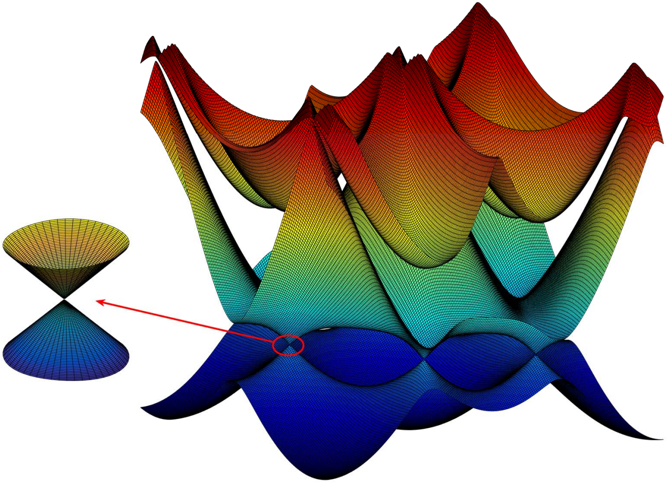
Stability and instability under perturbations

Consider $\mathcal{L}^{\delta} = \mathcal{L}^{A} + \delta\mathcal{L}^{B} \equiv -\nabla \cdot A\nabla - \delta\nabla \cdot B\nabla$. $A(\mathbf{x})$ is honeycomb lattice and the perturbation $B(\mathbf{x})$ is Λ_h periodic.

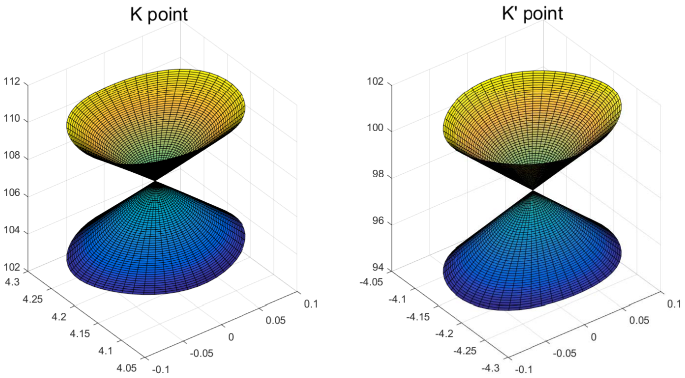
1. **Stable under $\mathcal{PC}$-preserving perturbations**: The Dirac point $(\mathbf{K}_\star, E_D)$ is protected by $\mathcal{PC}$ symmetry (i.e., $\mathcal{PC}\mathcal{L}^B = \mathcal{L}^B\mathcal{PC}$).
2. **Unstable under $\mathcal{PC}$-breaking perturbations**: Breaking $\mathcal{PC}$ symmetry (e.g., $\mathcal{PC}\mathcal{L}^B = -\mathcal{L}^B\mathcal{PC}$) destroys the Dirac point and opens a local spectral gap.

Using the operator perturbation theory and computing the bifurcation matrix $\left(\left\langle \Phi_j, \mathcal{L}^B\Phi_l \right\rangle\right)_{j,l=1,2}$ lead to the conclusions.

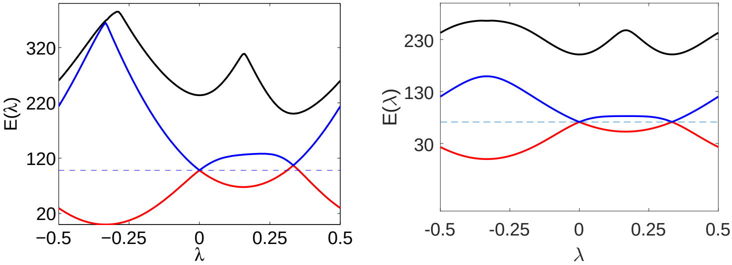
Energy surfaces and Dirac points



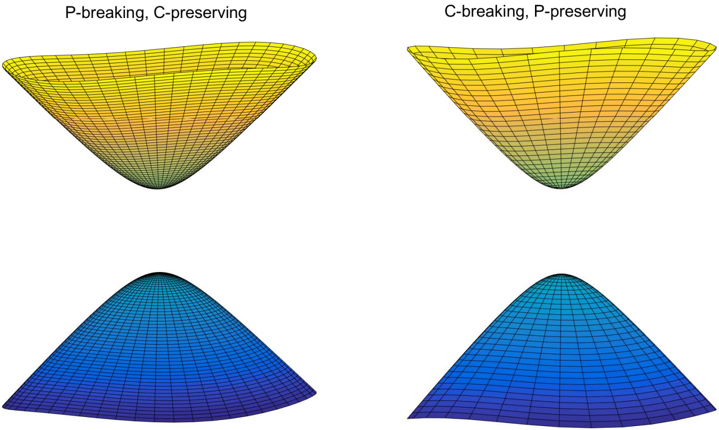
Dirac cones (local)



Dirac points (along $\mathbf{k}_1 - \mathbf{k}_2$ direction)



Instability



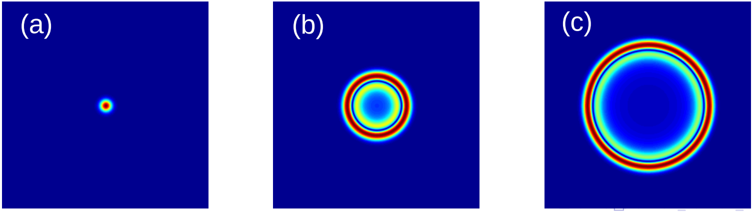
Massless Dirac Equation

Effective dynamics of the wave packet at the Dirac point is governed by the massless Dirac equation. Namely,

$$\begin{cases} \psi(\mathbf{x},0) = \alpha_1(\delta\mathbf{x})\Phi_1 + \alpha_2(\delta\mathbf{x})\Phi_2 \\ \partial_t\psi(\mathbf{x},0) = i\sqrt{E_D}(\alpha_1(\delta\mathbf{x})\Phi_1 + \alpha_2(\delta\mathbf{x})\Phi_2) \end{cases}$$

where Φ_1, Φ_2 are the eigen mode corresponding to a Dirac point $(\mathbf{K}_*, E_D)$, $|\delta| \ll 1$. The field $\psi(x,t) \approx e^{i\sqrt{E_D}t}\left(\alpha_1(\delta\mathbf{x},\delta t)\Phi_1 + \alpha_2(\delta\mathbf{x},\delta t)\Phi_2\right)$ with

$$i\,\sqrt{E_D}\partial_T \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \ v_F \begin{pmatrix} 0 & i\partial_{X_1} - \partial_{X_2} \\ i\partial_{X_1} + \partial_{X_2} & 0 \end{pmatrix} \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\ .$$



Honeycomb structured media with an edge

$A(\mathbf{x})$: honeycomb with Dirac point $(\mathbf{K}_*, E_D)$, $B(\mathbf{x})$: anti-$\mathcal{PC}$, i.e., $\mathcal{PC}\mathcal{L}^B = -\mathcal{L}^B\mathcal{PC}$.

Bulk: (" Insulator") Two (topologically) different bulk materials: $W_+(\mathbf{x}) = A(\mathbf{x}) + \delta\eta_\infty B(\mathbf{x})$ and $W_-(\mathbf{x}) = A(\mathbf{x}) - \delta\eta_\infty B(\mathbf{x})$. Then the band structure of the operators $\mathcal{L}^\pm = -\nabla \cdot W_\pm(\mathbf{x})\nabla$ have local gaps around the Dirac point $(\mathbf{K}_*, E_D)$.

material Edge: Connect the two bulk materials by a *domain wall function:*

$$\eta(0) = 0, \; \eta(\zeta) \to \pm\eta_\infty \text{ as } \zeta \to \pm\infty.$$

Namely, we consider the operator

$$\mathcal{L}^d \equiv -\nabla \cdot W(\mathbf{x})\nabla, \quad \text{where} \quad W(\mathbf{x}) = A(\mathbf{x})I + \delta\eta(\delta\mathfrak{K}_2 \cdot \mathbf{x})B(\mathbf{x}).$$

$\mathfrak{K}_2$ is the normal direction of the edge.

Edge state

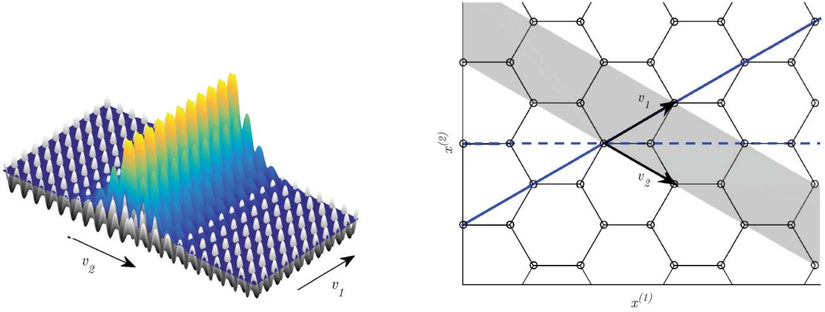
We seek the edge mode associated with the Dirac point $(\mathbf{K}_\star, E_D)$. For $k_\parallel = \mathbf{K}_\star \cdot \mathfrak{v}_1$, we shall seek solutions of the eigenvalue problem

$$\mathcal{L}^D \Psi(\mathbf{x}; k_\parallel) = E(k_\parallel)\Psi(\mathbf{x}; k_\parallel),$$

$$\Psi(\mathbf{x} + \mathfrak{v}_1; k_\parallel) = e^{ik_\parallel}\Psi(\mathbf{x}; k_\parallel) \qquad \text{(propagation parallel to the edge)},$$

$$\Psi(\mathbf{x}; k_\parallel) \to 0 \quad \text{as} \quad |\mathbf{x} \cdot \mathbf{k}_2| \to \infty \qquad \text{(localization tranverse to the edge)}.$$

Zigzag edge and corresponding edge states



The weight matrix is periodic in $\mathbf{v}_1$ direction, but modulated by a domain wall in $\mathbf{v}_2$ direction.

Multiple scale construction of the edge state

For $\delta \ll 1$, we use a multi-scale expansion to construct the edge state. We re-express the eigenvalue problem in terms of fast ($\mathbf{x}$) and slow/transverse ($\zeta = \delta \mathbf{k}_2 \cdot \mathbf{x}$) spatial scales, and seek a solution in the form:

$$E^\delta = E^{(0)} + \delta E^{(1)} + \cdots ,$$
$$\Psi^\delta = \Psi^{(0)}(\mathbf{x}, \zeta) + \delta \Psi^{(1)}(\mathbf{x}, \zeta) + \cdots ,$$

We are interested in the edge state associated with the Dirac point $(\mathbf{K}_\star, E_*)$ of the $\mathcal{L}$ with the $L^2(\mathbb{R}^2/\Lambda)$ eigenvalue E_*, then

$$E^{(0)} = E_D, \quad \Psi^{(0)} = \alpha_1^{\mathbf{K}_\star}(\zeta)\Phi_1^{\mathbf{K}_\star}(\mathbf{x}) + \alpha_2^{\mathbf{K}_\star}(\zeta)\Phi_2^{\mathbf{K}_\star}(\mathbf{x}),$$

where $\zeta \mapsto \alpha_j^{\mathbf{K}_\star}(\zeta)$, $j = 1, 2$ is determined by the solvability conditions.

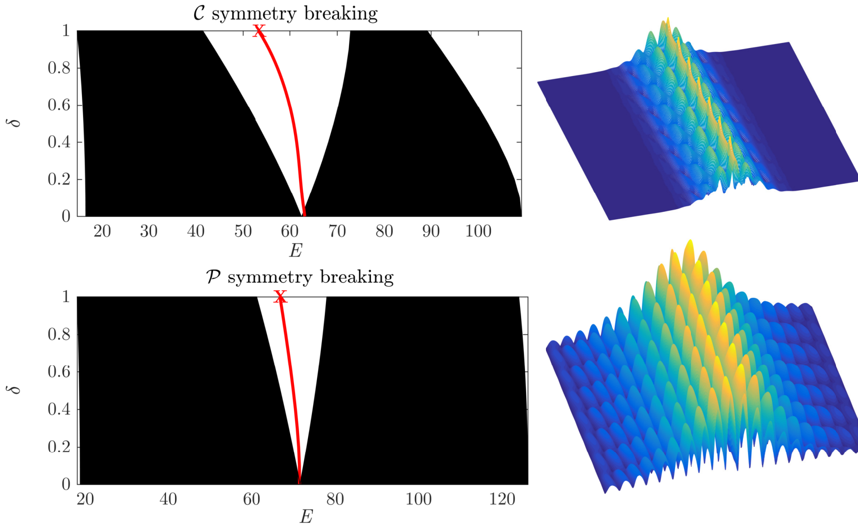
Zero energy edge states

For $\mathcal{L}_d$, for fixed parallel quasi-momentum $k_{\parallel} = \mathbf{K}_\star \cdot \mathfrak{v}_1$, has the formal, topologically protected eigenpair solution $(E^\delta_{\mathbf{K}_\star}, \Psi^\delta_{\mathbf{K}_\star})$, corresponding to a state which propagates in the $\mathfrak{v}_1-$ direction with parallel quasi-momentum $k_{\parallel} = \mathbf{K}_\star \cdot \mathfrak{v}_1$, and is exponentially decaying in the transverse direction, as $\mathfrak{K}_2 \cdot \mathbf{x} \to \pm\infty$.
Furthermore, the eigenpair $(E^\delta_{\mathbf{K}_\star}, \Psi^\delta_{\mathbf{K}_\star})$ can be expanded to any finite order in δ in powers of δ. To leading order $E^\delta_{\mathbf{K}_\star} = E_D + \mathcal{O}(\delta^2)$ and

$$\Psi^\delta_{\mathbf{K}_\star}(\mathbf{x}) = \begin{cases} \delta^{1/2}\, \gamma \left[\chi_-(\mathfrak{K}_2^\perp) \cdot \left(\Phi_1^{\mathbf{K}_\star}(\mathbf{x}), \Phi_2^{\mathbf{K}_\star}(\mathbf{x})\right)\right] e^{-\frac{\left|\vartheta_\sharp{}^{\mathbf{K}_\star}\right|}{v_F|\mathfrak{K}_2|}\int_0^{\delta\mathfrak{K}_2\cdot\mathbf{x}}\eta(s)ds} \\ +\mathcal{O}(\delta) \quad \text{if} \quad \vartheta_\sharp{}^{\mathbf{K}_\star} > 0; \\ \delta^{1/2}\, \gamma \left[\chi_+(\mathfrak{K}_2^\perp) \cdot \left(\Phi_1^{\mathbf{K}_\star}(\mathbf{x}), \Phi_2^{\mathbf{K}_\star}(\mathbf{x})\right)\right] e^{-\frac{\left|\vartheta_\sharp{}^{\mathbf{K}_\star}\right|}{v_F|\mathfrak{K}_2|}\int_0^{\delta\mathfrak{K}_2\cdot\mathbf{x}}\eta(s)ds} \\ +\mathcal{O}(\delta) \quad \text{if} \quad \vartheta_\sharp{}^{\mathbf{K}_\star} < 0. \end{cases}$$

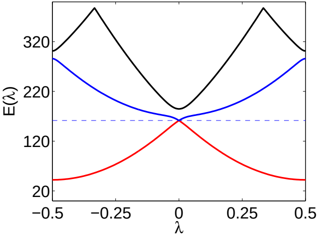
$$\vartheta_\sharp = <\Phi_1, \mathcal{L}^B\Phi_1> = - <\Phi_2, \mathcal{L}^B\Phi_2>.$$

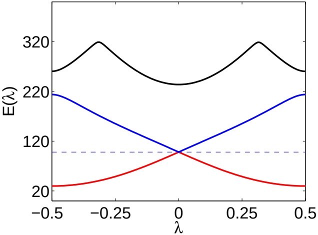
Bifurcation diagrams



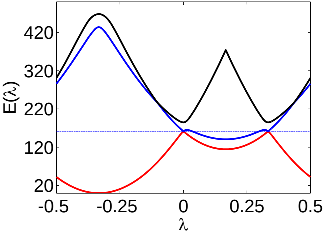
No-fold condition

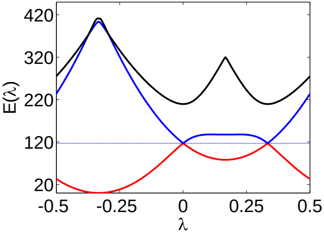
In Zigzag edge, the no-fold condition can be easily satisfied.





Armchair edge: no-fold condition can not be satisfied for isotropic media, but can be easily satisfied for anisotropic media.





Wave packet dynamics

Consider the following equation

$$\partial_{tt}\psi - \nabla \cdot [A(\mathbf{x}) + \delta\eta(\delta\mathbf{x})B(\mathbf{x})]\nabla\psi = 0.$$

A Multiscale expansion leads to Dirac equation with non-trivial mass term

$$i\,\sqrt{E_D}\partial_T \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \vartheta_\sharp \eta(X_1, X_2) & \upsilon_F (i\partial_{X_1} - \partial_{X_2}) \\ \upsilon_F (i\partial_{X_1} + \partial_{X_2}) & -\vartheta_\sharp \eta(X_1, X_2) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} .$$

Compact form

$$i\partial_T \boldsymbol{\alpha} = \left[\frac{\upsilon_F}{\sqrt{E_D}} \partial_{X_1}\sigma_1 + \frac{\upsilon_F}{\sqrt{E_D}} \partial_{X_2}\sigma_2 + \frac{\vartheta_\sharp}{\sqrt{E_D}} \eta(X_1, X_2)\sigma_3 \right] \boldsymbol{\alpha}.$$

This effective equation reveals the existence and dynamics of the edge states.

Consider the effective envelope equation

$$i\partial_T \boldsymbol{\alpha} = \left[\frac{v_F}{\sqrt{E_D}} \partial_{X_1} \sigma_1 + \frac{v_F}{\sqrt{E_D}} \partial_{X_2} \sigma_2 + \frac{\vartheta_\sharp}{\sqrt{E_D}} \eta(X_1, X_2) \sigma_3 \right] \boldsymbol{\alpha}.$$

Suppose $\eta(X_1, X_2)$ is a domain wall function, e.g., $\eta = \eta(X_1) = tanh(X_1)$. Taking harmonic solution $\alpha(X_1, X_2, T) = e^{ikX_2 - i\omega T} \alpha(X_1; k)$, at $k = 0$, there exist a zero energy state satisfying

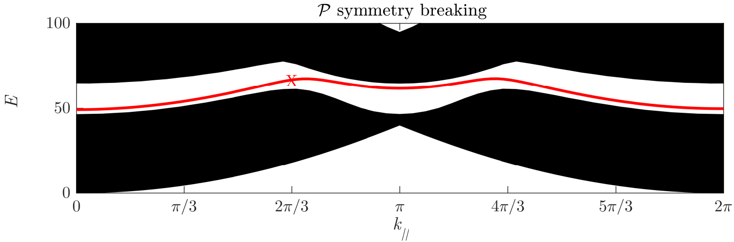
$$v_F \partial_{X_1} \sigma_1 + \theta_\sharp \eta(X_1, X_2) \sigma_3 = 0$$

This is exactly the zero energy edge state at the Dirac point. For $\eta = \eta(\mathfrak{K}_2 \cdot \mathbf{X})$, similar analysis can be done which lead the results we did before.

Evolution of edge state wave packet

Edge State

Bi-directional propagation

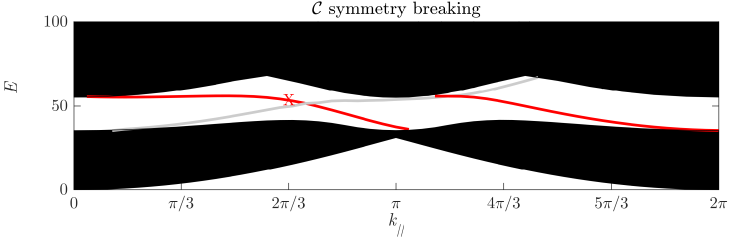


$\mathcal{C}$-preserving, $\mathcal{P}$-breaking perturbations lead to

$$\partial_{k_\parallel} E(k_\parallel)\Big|_{k_\parallel = \mathbf{K}\cdot\mathfrak{v}_1} = -\partial_{k_\parallel} E(k_\parallel)\Big|_{k_\parallel = \mathbf{K}'\cdot\mathfrak{v}_1}.$$

Wave packets with quasimomentum centered at $k_\parallel = \mathbf{K}\cdot\mathfrak{v}_1$ and $k_\parallel = \mathbf{K}'\cdot\mathfrak{v}_1$, respectively, have the same frequency but opposite group velocities; they travel in opposite directions.

Uni-directional propagation



$\mathcal{C}$-breaking, $\mathcal{P}$-preserving perturbations lead to

$$\partial_{k_\parallel} E(k_\parallel)\Big|_{k_\parallel = \mathbf{K}\cdot\mathfrak{v}_1} = \partial_{k_\parallel} E(k_\parallel)\Big|_{k_\parallel = \mathbf{K}'\cdot\mathfrak{v}_1}.$$

Wave packets with quasimomentum centered at $k_\parallel = \mathbf{K}\cdot\mathfrak{v}_1$ and $k_\parallel = \mathbf{K}'\cdot\mathfrak{v}_1$, respectively, have the same frequency and same group velocities. The wave packets are uni-directional - they travel in the same direction.

Apply our analysis to the two examples

- In Haldane & Raghu's setup,

$$A(\mathbf{x}) = \epsilon^{-1} + \gamma\epsilon^{-2}\sigma_2.$$

ϵ^{-1} is a honeycomb structured media. If Faraday rotation parameter γ is a domain wall function, we obtain uni-directional edge state.

- In Khanikaev et al.'s set up,

$$A(\mathbf{x}) = \epsilon^{-1} \pm \chi\epsilon^{-2}\sigma_2.$$

ϵ^{-1} is a honeycomb structured media. If the bi-anisotropy parameter χ is a domain wall, we obtain uni-directional edge states for both spin component and the two spin components travel oppositely.

Summaries

- Photonic graphene: honeycomb structured media
  - $\mathcal{R}-$ and $\mathcal{PC}$ invariant ensures Dirac points.
  - Stable under $\mathcal{R}$-breaking perturbations;
  - Unstable under $\mathcal{PC}$-breaking perturbations;
- Domain-wall modulated photonic graphene: two (topologically) different materials ("insulators") are connected by a domain wall function
  - A formal multi-scale justification of the existence of topologically protected edge states;
  - Breaking $\mathcal{P}$ and preserving $\mathcal{C}$ $\Rightarrow$ bi-directional;
  - Breaking $\mathcal{C}$ and preserving $\mathcal{P}$ $\Rightarrow$ uni-directional (Topological photonic crystals).