Universality for the dimer model

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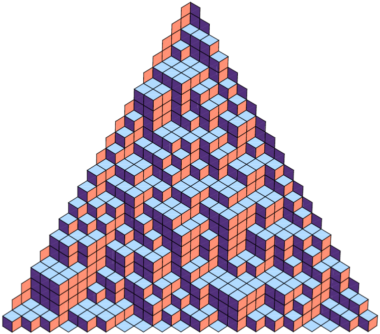

Banff, October 2017

Happy birthday Chris!

Or, more precisely :
Wszystkiego najlepszego

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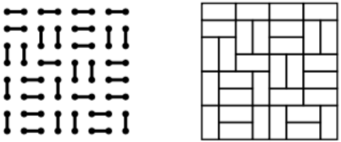
The dimer model

Definition

G = bipartite finite graph, planar
Dimer configuration = perfect matching on G:
each vertex incident to one edge
Dimer model: uniformly chosen configuration

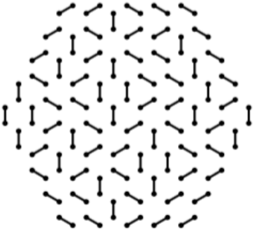


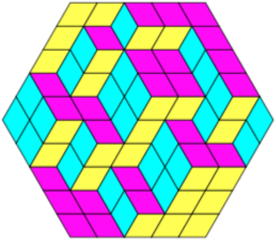
On square lattice, equivalent to domino tiling.

Dimer model as random surface

Example: honeycomb lattice

Dimer = lozenge tiling
Equivalently: stack of 3d cubes.





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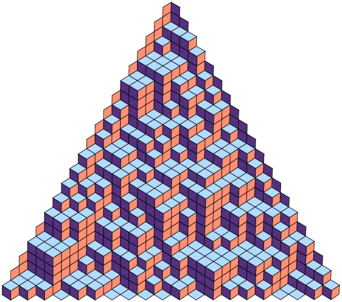
Height function

Introduced by Thurston. Hence view as random surface.

Large scale behaviour?



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Main Question:

Given boundary conditions, what is large scale behaviour?
Universality? Conformal Invariance?

Choose carefully boundary conditions to avoid **frozen regions**.

Background

Classical model of statistical mechanics:

Kasteleyn, Temperley–Fisher 1960s
Kenyon, Propp, Lieb, Okounkov, Sheffield, Dubédat, Borodin, Petrov, Toninelli, Ferrari, Gorin,... 1990s+

"Exactly Solvable": determinantal structure

$$\text{e.g.,}\quad Z_{m,n} = \prod_{j=1}^{m}\prod_{k=1}^{n} \left|2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right|^{1/2}$$

Analysis via: discrete complex analysis, Schur polynomials, Young tableaux, algebraic geometry...

Mapping to other models:

Tilings, 6-vertex, XOR Ising, **Uniform Spanning Trees** (UST)

Theorem

Let $h^{\#\delta}$ = height function on hexagonal lattice, mesh-size = δ.
Let $P \subset \mathbb{R}^3$ a plane.

Theorem (B.–Laslier–Ray 2016)

Let $D \subset \mathbb{C}$ simply connected, ∂D locally connected.
Assume boundary conditions are close to $P \subset \mathbb{R}^3$. Then

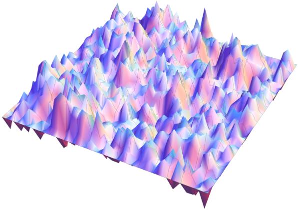
$$(h^{\#\delta} - \mathbb{E}(h^{\#\delta})) \circ \ell \xrightarrow[\delta \to 0]{} \frac{1}{\chi} h_{\mathsf{GFF}},$$

where ℓ = linear map, $\chi = 1/\sqrt{2}$.
h_{GFF} = Gaussian free field with Dirichlet boundary conditions.

(Convergence in distribution in $H^{-1-\varepsilon}$.)

What is the Gaussian free field?

Informally, $\mathbb{P}(f) = \frac{1}{Z} \exp\left(-\frac{1}{2}\int_D |\nabla f|^2\right) df$



GFF = canonical random function on D.
But too rough to be a function
Rigorously: in Sobolev space H^{-s}, $\forall s > 0$

$$(h_{\mathrm{GFF}}, f) \sim \mathcal{N}\left(0, \iint_D G_D(x,y)f(x)f(y)dxdy\right)$$

where $G_D(\cdot,\cdot) = -(1/2\pi)\Delta^{-1}$ is Green's function in D.
$\implies$ **conformal invariance**

Novelty of approach

Universality of fluctuations

Insight as to why GFF universal?
Needed: SRW $\to$ BM on certain graph.
Does not rely on exact solvability
Instead: imaginary geometry and SLE

Robustness

Extends Kenyon 2000 (flat case with smooth D)
Extends to Dimer Model on isoradial graphs
Dimer model in random environment

Work in progress:

Riemann surfaces (see later)
Generic boundary conditions, assuming no frozen region

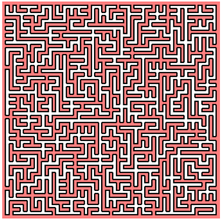
Temperley's bijection; Kenyon–Sheffield

Start with a UST on graph Γ.
Construct associated dimer config. on a modified graph G

Dimer configurations on $G \leftrightarrow$ UST on Γ
Height function $\leftrightarrow$ Winding of branches in tree

New goal:

Study winding of branches in UST.

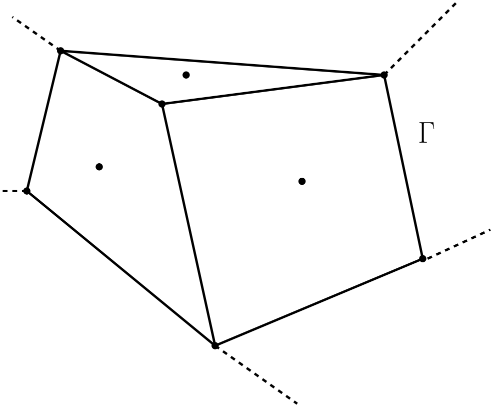


Question

How much do you wind around in a random maze?

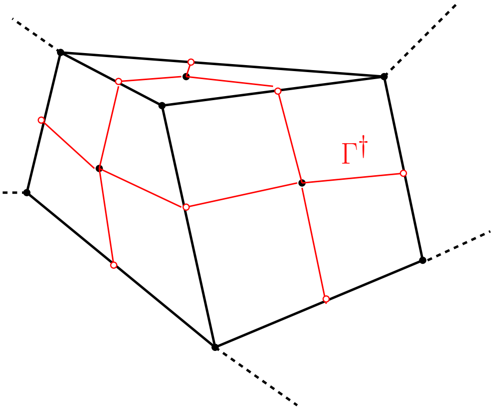
Temperley's bijection: how does it work (1)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.
[Trees = oriented: each vertex has unique outgoing edge, except on boundary (wired). No cycles allowed.]



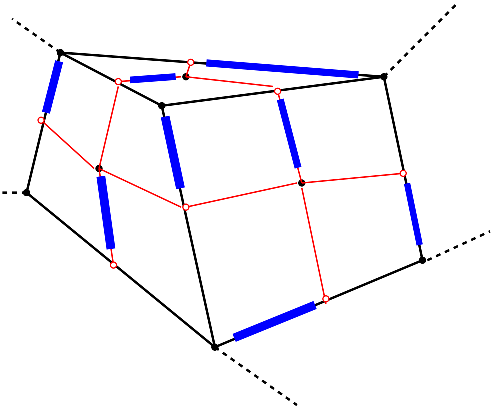
Temperley's bijection: how does it work (2)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



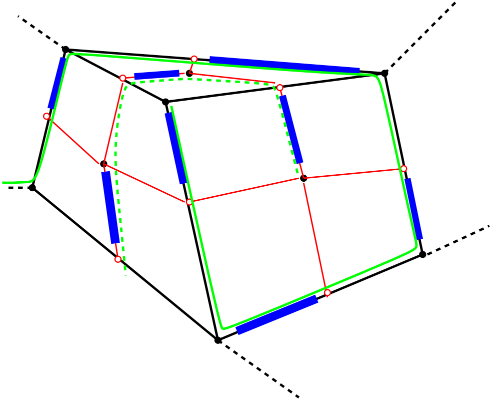
Temperley's bijection: how does it work (3)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



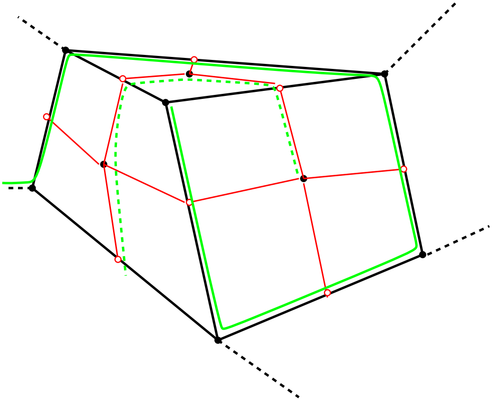
Temperley's bijection: how does it work (4)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



Temperley's bijection: how does it work (5)?

Pair of dual UST on $(\Gamma, \Gamma^\dagger) \leftrightarrow$ dimer on $G = \Gamma \oplus \Gamma^\dagger$.



Collection of green edges must be a tree because: connected, n vertices and $n-1$ edges.

Winding in UST

Question

How much do you wind around in a random maze?
Answer: the GFF !

Let $h^{\#\delta}$ = winding of branches in UST.

Real main theorem

Let $G^{\#\delta}$ be a sequence of graphs. Assume $(\star)$. Then

$$h^{\#\delta} - \mathbb{E}(h^{\#\delta}) \xrightarrow[\delta \to 0]{} \frac{1}{\chi} h_{\mathsf{GFF}},$$

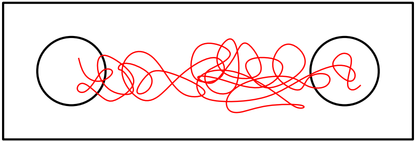
h_{GFF} = Gaussian free field (Dirichlet boundary conditions).
$\chi = 1/\sqrt{2}$.

Note: $\mathbb{E}(h^{\#\delta})$ itself is **not** universal, only fluctuations!

Assumptions for the theorem

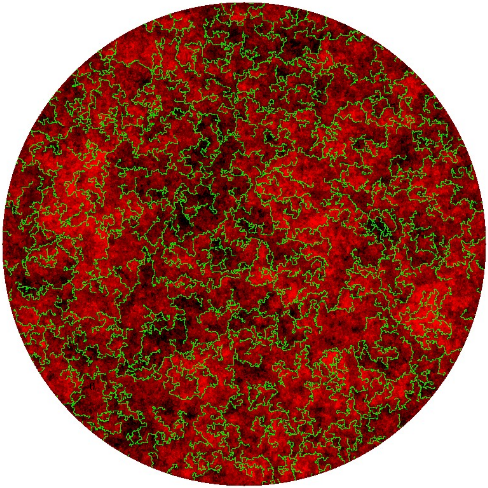
Holds under very general assumptions:

(1) Simple Random Walk on $G^{\#\delta}$ converges to Brownian motion
(2) Uniform crossing condition:



("Russo–Seymour–Welsh" estimate)
(3) Bounded density of vertices; edges have bounded winding

Ideas for the proof: working in the continuum



Scaling limit of Uniform Spanning Tree

Theorem (Lawler, Schramm, Werner '03, Schramm '00)

$D \subset \mathbb{C}$

- *Uniform spanning tree on $D \cap \delta\mathbb{Z}^2 \to$ "A continuum tree" (continuum uniform spanning tree).*
- *Branches of the continuum tree are SLE_2 curves.*

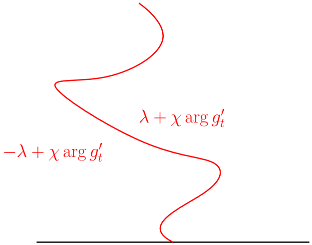
Yadin–Yehudayoff 2010: universality (assuming convergence of SRW to BM).

Imaginary Geometry theorem statement

Dubédat, Miller–Sheffield: "flow lines of GFF/χ are SLE_κ curves", provided:

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

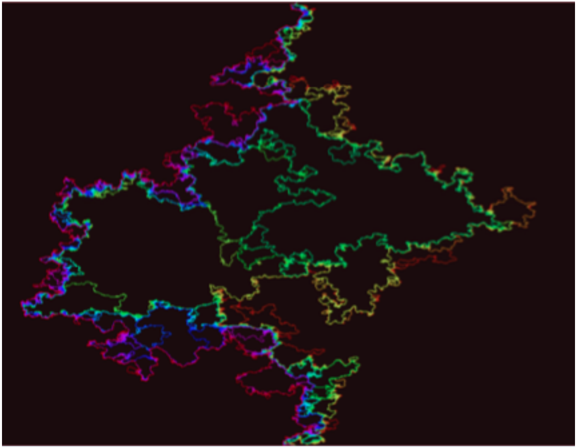
Meaning: coupling (h, η), $h =$ GFF, $\eta = \mathrm{SLE}_\kappa$:



Take-home message

"Values" of h/χ along curve record "winding" of SLE_κ (in sense of $\arg g'$).

Flow lines of GFF: $e^{ih/\chi}$.



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$\chi = 1/\sqrt{2}$, flow lines = SLE_2 (Miller–Sheffield).

Stronger convergence

Suggests "winding" of continuum UST is $(1/\chi)$ GFF.

Extended convergence:

Theorem (B.–Laslier–Ray)

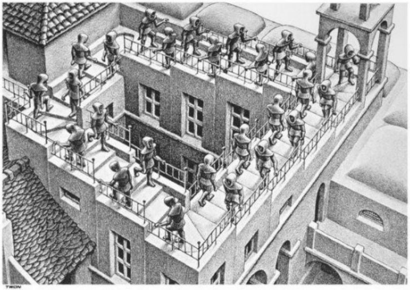
UST and dimer height function converge **jointly** *to* $(\mathcal{T}, h)$ *where* $\mathcal{T}$ *is the tree of flow lines of* $e^{ih/\chi}$.

Dimers on Riemann surfaces

Q: Impact of curvature on such random structures?

Goal: universal limit "height function" + conformal invariance

In fact, "height function" is a closed 1-form:



Hodge decomposition:

h consists of a function together with **instanton component** (a harmonic function on universal cover).

Flavour of results

Temperley's bijection

We extend Temperley's bijection to Riemann surfaces.
Instead of UST, "Temperleyan forests".

Generalising our previous results:

Theorem (BLR '17, in preparation)

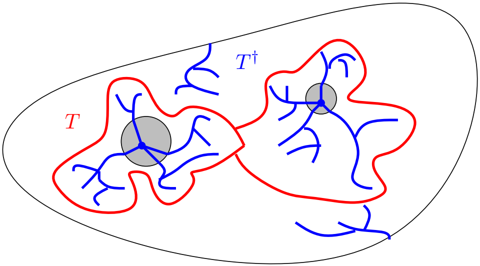
Suppose Temperleyan forest converges. Then both components of dimer height function converges.

Temperleyan forests

Definition: Temperleyan forest

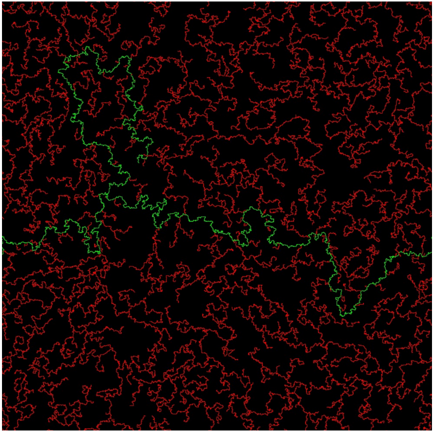
Oriented subgraph $T^\dagger$ of $\Gamma^\dagger$:
– $\forall v \notin \partial\Gamma^\dagger$, unique outgoing edge (except on boundary = wired).
– Every cycle is non-contractible.
– Each connected component of T contains at most one cycle.

Ex: **non**-Temperleyan:



Low Euler characteristic

When Euler's $\chi = 0$ (i.e., annulus or torus) then we show Temperleyan forest reduces to **Cycle Rooted Spanning Forest**.
More precisely: derivative $\frac{d\mathbb{P}_{\mathsf{Temp}}}{d\mathbb{P}_{\mathsf{CRSF}}} = 2^{\#\{dual\ cycles\}}$.



Advantage: **Wilson's algorithm**.

Scaling limit of CRSF

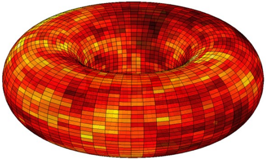
Theorem (B.–Laslier–Ray)

Assume ($\star\star$).
Then CRSF converges in Schramm space to universal scaling limit.
$+$ subexponential tail for $\#$ dual cycles.

(Solves some conjectures by Kassel-Kenyon.)

Corollary

Dimer height function converges when $\chi = 0$.
Both components are **universal, conformally invariant**.



In torus case, proves conjecture by Dubédat

In progress

Can handle Riemann surfaces of low complexity (Euler's $\chi = 0$) at this stage. Work on case $\chi < 0$ in progress.

Many questions remain:

- Law of limit?
- Tilted dimer measures on surfaces?
- Generic boundary conditions in planar domains?
- Gaseous phases?
- Link with exactly solvable approach (Borodin–Gorin–Guionnet, Petrov, etc.)
- ...

THANK YOU!