Stability in martingale inequalties *

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HAPPY BIRTHDAY, Chris!

Rosa and I wish you a long and very happy young life.

- Sharp inequalities in analysis, geometry, and probability have been investigated for a long, long, time $\cdots$
- What do extremals, or "near" extremals, (those that make the inequality an equality, or "near" equality) look like?
- The aim of "stability/deficit/quantitatively sharp" inequalities is to measure, in terms of an appropriate distance from the extremals, how far an admissible quantity is from attaining equality.
- The martingale results here are motivated form problems in analysis.

Stability (quantitatively sharp/deficit) inequalities

Optimal/sharp inequalities

Suppose you have two functionals $\mathcal{E}$ and $\mathcal{F}$ on some normed (real) linear space $\mathcal{M}$ satisfying the functional inequality $\mathcal{E} \leqslant \mathcal{F}$ in the sense that

$$\mathcal{E}(x) \leqslant \mathcal{F}(x), \quad \forall x \in \mathcal{M}.$$

$\mathcal{E} \leqslant \mathcal{F}$ is sharp if $\forall\ \lambda < 1$, $\exists\ x \in \mathcal{M}$ such that

$$\mathcal{E}(x) > \lambda \mathcal{F}(x)$$

$$\mathcal{M}_0 = \{x \in \mathcal{M} : \mathcal{E}(x) = \mathcal{F}(x)\}$$

is called the set of optimizers (extremals). When $\mathcal{M}_0 \neq \emptyset$, the inequality is said to be optimal. (Note: An optimal inequality is sharp but not vice–versa.)

One question we may ask: Suppose $\{x_n\}$ is a sequence in $\mathcal{M}$ such that $\mathcal{F}(x_n) - \mathcal{E}(x_n) \to 0$. Is it true that $d(x_n, \mathcal{M}_0) \to 0$ also (some metric d) ?

Definition

Let d be a metric on $\mathcal{M}$ (not necessarily the norm metric) and Φ a "rate function." The optimal functional inequality $\mathcal{E} \leqslant \mathcal{F}$ is (d, Φ)– stable if

$$\mathcal{F}(x) - \mathcal{E}(x) \geqslant \Phi(d(x, \mathcal{M}_0)), \quad \forall x \in \mathcal{M}$$

In various examples, $\Phi(t) = ct^2$ and $d(x, y) = \|x - y\|_{\mathcal{M}}$ and

$$\mathcal{F}(x) - \mathcal{E}(x) \geqslant c \inf_{z \in \mathcal{M}_0} \|x - z\|_{\mathcal{M}}^2.$$

The quantity

$$\delta(x) = \mathcal{F}(x) - \mathcal{E}(x)$$

is offen called the deficit.

- Classical Sobolev in $\mathbb{R}^n$ $(n \geqslant 3)$. Optimality: Aubin (1976), Talenti (1976).

$$k_n^2 = \frac{n(n-2)}{4}|\mathbb{S}^{n-1}|$$

$$k_n^2\|f\|^2_{\frac{2n}{n-2}} \leqslant \|\nabla f\|_2^2, \quad \forall f \in H_0^1(\mathbb{R}^n) = \mathcal{M},$$

$$\mathcal{M}_0 = \{x \to c(a + b|x - x_0|^2)^{-(n-2)/2}, a, b > 0, x_0 \in \mathbb{R}^n, c \in \mathbb{R}\}$$

Stability: Biachi-Egnell (1990)

$$\|\nabla f\|_2^2 - k_n^2\|f\|^2_{\frac{2n}{n-2}} \geqslant C \inf_{g \in \mathcal{M}_0} \|\nabla(f-g)\|_2^2$$

- General Sobolev $(0 < \alpha < n/2)$.

$$\|f\|_{\frac{2n}{n-2\alpha}} \leqslant k_{n,\alpha}\|(-\Delta)^{\alpha/2} f\|_2$$

Optimality E. Lieb (1983), Stability S. Cheng, R. Frank, T. Weth (2013)

- Hardy-Littlewood-Sobolev (fractional integrals), $0 < \alpha < n$

$$I_\alpha(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} P_t f(x)\, dt,$$

$$\|I_\alpha f\|_p \leqslant C\|f\|_q, \qquad q = \frac{np}{n - \alpha p}, \quad p > 1.$$

Optimality E. Lieb (1983), Stability E. Carlen (2016):

- Log-Sobolev Gross (1975): Stability M. Fathi, E. Indrei, M. Ledoux (2015), Indrei, D. Kim (2017). Stability measured with Kantorovich–Wasserstein distance.

- Housdorff-Young inequality: Optimality: W. Beckner 1975 (Lieb 1990) $1 \leqslant p \leqslant 2$, $q = \frac{p}{p-1}$

$$\|\hat{f}\|_q \leqslant A_p^n \|f\|_p \quad A_p = p^{1/2p} q^{-1/2q}$$

$A_p < 1$ is best contacts. Extremizers are general Gaussians: $g(x) = ce^{Q(x)+x\cdot v}$.

Stability: M. Christ (2015, 2016): Let $\mathcal{G}$ represent all Gaussian.

$$\|\hat{f}\|_q - A_p^n \|f\|_p \geqslant c \inf_{g\in\mathcal{G}} \|f - g\|_p^2$$

Isoperimetric principle of exit time of BM (one of several)

Let $D \subset \mathbb{R}^n$ be a domain of finite volume. Let D^* be the ball of same volume. Let B_t be Brownian motion starting in D and τ_D be its exit time from D.

$$\int_D \mathbb{E}_z(\tau_D)dz \leqslant \int_{D*} \mathbb{E}_z(\tau_{D^*})dz,$$

with equality if and only if $D = D^*$.

Brasco & De Philippis (2016).

$$\int_{D*} \mathbb{E}_z(\tau_{D^*})dz - \int_D \mathbb{E}_z(\tau_D)dz \geqslant C_n \mathcal{A}(D)^2$$

(*Fraenkel Asymmetry*) $\mathcal{A}(D) := \inf\{\frac{|D \triangle B|}{|D|} : B \text{ is a ball with } |B| = |D|\}.$

Remark

The "Isoperimetric principle" holds for very general Lévy processes (R.B.& P. Méndez–Hénandez 2010). Stability, even for rotationally symmetric stables (fractional Laplacian), is an interesting problem.

Sharp but not optimal (i.e., $\mathcal{M}_0 = \emptyset$)) Martingales inequalities.

Doob's inequality

$\{f_n\}$ an L^p, $1 < p \leqslant \infty$ martingale. $f^* = \sup_n |f_n|$ maximal function.

$$\|f^*\|_p \leqslant \frac{p}{p-1}\|f\|_p$$

- D. Burkholder (1984): The constant $\frac{p}{p-1}$ is best possible. But inequality is not optimal, i.e., $\mathcal{M}_0 = \emptyset$.
- G. Wang (1991): Constant is also best possible in class of Brownian (and dyadic) martingales.

Burkholder (1966) $S(f) = (\sum_n (f_n - f_{n-1})^2)^{1/2}$

There exists constants a_p and b_p such that

$$a_p\|f\|_p \leqslant \|S(f)\|_p \leqslant b_p\|f\|_p \quad 1 < p < \infty$$

Burgess Davis (1976) proved sharp version (BM). But inequality is not optimal, i.e., $\mathcal{M}_0 = \emptyset$, outside of the trivial case of $p = 2$.

X, Y cádlág (right continuous/left limits) martingales:

- Y is differentially subordinate to X ($Y << X$), if the process $\{[X,X]_t - [Y,Y]_t\}_{t\geqslant 0}$ is a.s. nonnegative and nondecreasing in t.

Example:

- $Y_t = \int_0^t K_s \cdot dBs$, $X_t = \int_0^t H_s \cdot dB_s$ with $|K_s| \leqslant |H_s|$, a.s.
- $g_n = \sum_{k=1}^n e_k$, $f_n = \sum_{k=1}^n d_k$ with $|e_k| \leqslant |d_k|$, a.s.

Burkholder (1984)

Suppose $Y << X$. For $1 < p < \infty$, set $p^* = \max\{p, q\}$ where p and q are conjugate exponents.

$$p^* - 1 = \begin{cases} p-1, & 2 \leqslant p < \infty, \\ \frac{1}{p-1}, & 1 < p \leqslant 2. \end{cases}$$

$$\Rightarrow \quad ||Y||_p \leqslant (p^*-1)||X||_p.$$

Inequality is sharp and strict, unless $p = 2$ and $[X,X]_t = [Y,Y]_t$ a.s for all $t \geqslant 0$.

The dyadic maximal function in $\mathbb{R}^n$ (dyadic martingales).

$$M_d(f)(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| dy$$

Sup over dyadic cubes in $[0,1]^n$ containing x.

Here we may restrict to non-negative functions.

Theorem (A. Melas 2015)

Fix $2 < p < \infty$, $\epsilon > 0$ (small enough). Suppose $f \geqslant 0$ (in L^p) is such that

$$\|M_d(f)\|_p \geqslant \left(\frac{p}{p-1} - \varepsilon\right) \|f\|_p.$$

Then

$$\|M_d(f) - \frac{p}{p-1} f\|_p \leqslant c_p \varepsilon^{1/p} \|f\|_p$$

for some constant c_p depending only on p.

For $1 < p \leqslant 2$, ???

Theorem (R.B. & A.Osękowski (2016): Assume $Y << X$)

(i) Let $1 < p < 2$ *and* $\varepsilon > 0$. $||Y||_p \geqslant (\frac{1}{p-1} - \varepsilon)||X||_p$. *Then*

$$\Big|\big||Y| - \frac{1}{(p-1)}|X|\big|\Big|_p \leqslant c_p \varepsilon^{1/2} ||X||_p.$$

$O(\varepsilon^{1/2})$ *as* $\varepsilon \to 0$ *is sharp.* $c_p = O((2-p)^{-1/2})$ *as* $p \uparrow 2$ *and this is sharp.*

(ii) Let $2 < p < \infty$ *and* $\varepsilon > 0$. $||Y||_p \geqslant (p - 1 - \varepsilon)||X||_p$.

$$\Big|\big||Y| - (p-1)|X|\big|\Big|_p \leqslant c_p \varepsilon^{1/p} ||X||_p,$$

$O(\varepsilon^{1/p})$ *as* $\varepsilon \to 0$ *is sharp.* c_p *is* $O((p-2)^{-1/p})$ *as* $p \downarrow 2$ *and* $O(p)$ *as* $p \to \infty$. *These orders are sharp.*

(iii) For $p = 2$, *no* c_2 *and* κ *exist such that* $||Y||_2 \geqslant (1 - \varepsilon)||X||_2$ *implies* $\Big|\big||Y| - |X|\big|\Big|_2 \leqslant c_2 \varepsilon^{\kappa} ||X||_2$. *In fact, there exist martingales* Y *and* X, $Y << X$, *such that*

$$\|Y\|_2 = \|X\|_2, \quad \text{and} \quad \frac{\||Y| - |X|\|_2}{\|X\|_2} > 0 \ \ (\textit{independent of } \varepsilon)$$

Beurling-Ahlfors operator in complex plane $\mathbb{C} = \mathbb{R}^2$

$$Bf(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,\mathrm{d}w$$

Calderón-Zygmund: $\exists$ constant C_p (depending only on p)

$$\|Bf\|_p \leqslant C_p \|f\|_p, \quad 1 < p < \infty \tag{1}$$

$$\partial = \frac{1}{2}\left(\partial_x - i\partial_y\right), \;\; \overline{\partial} = \frac{1}{2}\left(\partial_x + i\partial_y\right) \Rightarrow \;\; B = 4\partial^2\Delta^{-1}, \;\; B \circ \overline{\partial} = \partial$$

In fact, equivalent to (BA):

$$\|\partial f\|_p \leqslant C_p \left\|\overline{\partial} f\right\|_p, \;\; 1 < p < \infty, \;\; f \in C_0(\mathbb{R}^2) \tag{2}$$

Problem

Find norm of B, $\|B\|_{p\to p}$, on $L^p(\mathbb{C})$.

O. Lehto 1965

$$\|B\|_{p\to p} \geqslant (p^* - 1)$$

Conjecture: T. Iwaniec 1984

$$\|B\|_{p\to p} = (p^* - 1), \;\; 1 < p < \infty$$

Known upper bound (R.B & P. Janakiraman 2008)

$$\|B\|_{p\to p} \leqslant 1.575(p^* - 1)$$

Lehto: Consider $f = |z|^\beta \chi_D$, D unit disk. With the right choice of β,

$$\|Bf\|_p > ((p^* - 1) - \varepsilon)\,\|f\|_p.$$

For such $f's$ one computes and finds that

$$|Bf(z)| \approx (p^* - 1)|f(z)|$$

(i.e., they are "near eigenfunctions")

$$\widehat{Bf}(\xi) \;=\; \frac{\bar{\xi}}{\xi}\widehat{f}(\xi) = \frac{\bar{\xi}^2}{|\xi|^2}\hat{f}(\xi) = \frac{\xi_1^2 - 2i\xi_1\xi_2 - \xi_2^2}{|\xi|^2}\hat{f}(\xi)$$

$$\Rightarrow \;\; B = R_1^2 - R_2^2 + 2iR_1R_2 = \mathrm{Re}(B) + i\,\mathrm{Im}(B)$$

where R_1 and R_2 are the Riesz transforms in $\mathbb{R}^2$: $R_j f = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2} f$

1. R. B. & Wang (1995): Both $\|Re(B)\|_{p\to p}$ and $\|Im(B)\|_{p\to p} \leqslant 2(p^*-1)$

$$\Rightarrow \quad \|B\|_p \leqslant 4(p^*-1)$$

2. Nazarov and Volberg (2004) (R. B & Méndez (2004)) improved bounds to $\leqslant (p^*-1)$

$$\Rightarrow \quad \|B\|_{p,p} \leqslant 2(p^*-1)$$

3. Geiss, Montgomery-Smith and Saksman (2009): Riesz transforms on $\mathbb{R}^n$:

$$\|R_j^2 - R_k^2\|_{p\to p} = (p^*-1), \quad \|2R_jR_k\|_{p\to p} = (p^*-1), \;\; j \neq k$$

Theorem (R.B. & A.Osękowski 2016)

T either $\mathrm{Re}(B)$ or $\mathrm{Im}(B)$ or more generally, $R_j^2 - R_k^2$ or $2R_jR_k$, $j \neq k$ in $\mathbb{R}^n$.
(i) Let $1 < p < 2$, $\varepsilon > 0$. If $f \in L^p(\mathbb{R}^n)$ is such that

$$||Tf||_p \geqslant ((p-1)^{-1} - \varepsilon)||f||_p,$$

then

$$|||Tf| - (p-1)^{-1}|f|||_p \leqslant c_p \varepsilon^{1/2}||f||_p.$$

Same constants as in martingale inequalities and also sharp.
(ii) Let $2 < p < \infty$, $\varepsilon > 0$. If $f \in L^p(\mathbb{R}^n)$ is such that

$$||Tf||_p \geqslant (p - 1 - \varepsilon)||f||_p,$$

then

$$|||Tf| - (p-1)|f|||_p \leqslant c_p \varepsilon^{1/p}\|f\|_p,$$

(iii) For $p = 2$, no such estimates: There are no finite constants c_2 and $\kappa > 0$ such that

$$|||Tf| - |f|||_p \leqslant c_2 \varepsilon^{\kappa}||f||_{L^2(\mathbb{R}^d)}$$

Idea of Proof for martingale Inequality: Burkholder's method (AoP 1984)

$$f_n = \sum_{k=1}^{n} d_k, \quad g = \sum_{k=1}^{n} e_k, \quad |e_k| \leqslant |d_k|, \; a.s. \; \forall k$$

Considers the function $V_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$V_p(x,y) = |y|^p - (p^* - 1)^p |x|^p.$$

Goal: show that $EV_p(f_n, g_n) \leqslant 0$. Burkholder then "introduces" the function

$$U_p(x,y) = p\left(1 - \frac{1}{p^*}\right)^{p-1} \left(|y| - (p^* - 1)|x|\right)\left(|x| + |y|\right)^{p-1},$$

and proves: (i)

$$V_p(x,y) \leqslant U_p(x,y) \text{ for all } x, y \in \mathbb{R}$$

and (ii)

$$EU_p(f_n, g_n) \leqslant EU_p(f_{n-1}, g_{n-1}) \leqslant \cdots \leqslant EU_p(f_0, g_0) = 0$$

Lemma ("Basic Lemma" R.B & G. Wang (1995))

Suppose $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is "smooth" and for all* h, $k \in \mathbb{R}$, *it satisfies:*

$$U_{xx}(x,y)|h|^2 + 2U_{xy}(x,y)hk + U_{yy}(x,y)|k|^2 \leqslant c(x,y)(|k|^2 - |h|^2).$$

$c(x,y) \geqslant 0$.

Then if $Y << X$, $U(X_t, Y_t)$ *is a supermartingale and*

$$\mathbb{E}U(X_t, Y_t) \leqslant \mathbb{E}U(X_0, Y_0).$$

Example (Burkholder's function)

$$U_p(x,y) = \beta_p \left(|y| - (p^* - 1)|x|\right)\left(|x| + |y|\right)^{p-1},$$

$$\beta_p = p\left(1 - \frac{1}{p^*}\right)^{p-1}$$

For $1 < p < 2$, set

$$\widetilde{U_p}(x,y) = (p-1)^p|y|^p - |x|^p + \left(1 - p\left(1-\frac{1}{p}\right)^{p-1}\right)\frac{((p-1)|y|-|x|)^2}{(|x|+|y|)^{2-p}}$$

Lemma

$$\widetilde{U_p}(x,y) \leqslant U_p(x,y), \forall x,y \in \mathbb{R}^n.$$

Corollary

Suppose $Y << X$*. Then* $E(\widetilde{U_p}(X,Y) \leqslant 0$*.*

Thus if in addition, $||Y||_p \geqslant (\frac{1}{p-1} - \varepsilon)||X||_p$, we have

$$\begin{aligned}\left(1 - p\left(1-\frac{1}{p}\right)^{p-1}\right)\mathbb{E}\frac{((p-1)|Y|-|X|)^2}{(|X|+|Y|)^{2-p}} &\leqslant ||X||_p^p - (p-1)^p||Y||_p^p \\ &\leqslant (1-(1-(p-1)\varepsilon)^p)\,||X||_p^p \\ &\leqslant p(p-1)\varepsilon||X||_p^p.\end{aligned}$$

$$||(p-1)|Y|-|X|||_p \leqslant \left(\mathbb{E}\left\{\frac{((p-1)|Y|-|X|)^2}{(|X|+|Y|)^{2-p}}\right\}\right)^{1/2} (|||X|+|Y|||_p^{\frac{(2-p)}{2}})$$

$$\leqslant \left(\frac{p(p-1)\varepsilon}{1-p\left(1-\frac{1}{p}\right)^{p-1}}\right)^{1/2} ||X||_p^{p/2}\left(|||X|+|Y|||_p^{\frac{(2-p)}{2}}\right)$$

$$\leqslant \left(\frac{p(p-1)\varepsilon}{1-p\left(1-\frac{1}{p}\right)^{p-1}}\right)^{1/2} ||X||_p^{p/2}\cdot\left(\frac{p}{p-1}||X||_p\right)^{\frac{(2-p)}{2}}.$$

First inequality is Hölder with $\overline{p}=p/2$ and $\overline{q}=2/(2-p)$, second is the Corollary and third is Minkowski and Burkholder.

$2 < p < \infty$, consider:

$$\widehat{U_p}(x,y) = \begin{cases} p\left(1-\frac{1}{p}\right)^{p-1}(|y|-(p-1)|x|)(|x|+|y|)^{p-1}, & \text{if } |y| \geqslant (p-2)|x|, \\ -\frac{(p-1)^{2p-2}}{p^{p-2}}|x|^p, & \text{if } |y| < (p-2)|x|. \end{cases}$$

Lemma

(i)

$$\widehat{U_p}(x,y) \geqslant |y|^p - (p-1)^p|x|^p + \alpha_p\big||y|-(p-1)|x|\big|^p,$$

$$\alpha_p = \frac{p-2}{p-1}\left(\frac{1}{2}-\frac{1}{e}\right).$$

(ii) $\widehat{U_p}$ *satisfies the "Basic Lemma."*

$$\begin{aligned} \alpha_p\big\||Y_\infty|-(p-1)|X_\infty|\big\|_p^p &\leqslant (p-1)^p\|X\|_p^p - \|Y\|_p^p \\ &\leqslant \left[(p-1)^p-(p-1-\varepsilon)^p\right]\|X\|_p^p \\ &\leqslant p(p-1)^{p-1}\varepsilon\|X\|_p^p. \end{aligned}$$

Thank you!

Sharpness

Assume $1 < p < 2$. Let $x > 0$ and let $w > p$ satisfy

$$x^p + pw^{p-1} - w^p = 0$$

Set

$$\theta = 1 - 1/w, \quad \text{and} \quad \beta_k = 1 - \frac{w\delta}{x + k\delta}, \quad k \geqslant 1,$$

where $0 < \delta < x/w$. Using the same notation for an interval $[a, b)$ and its indicator function, set

$$\begin{aligned} d_1 &= x[0,1) \\ d_2 &= \delta[0,\beta_1) + (\theta(x+\delta) - x)[\beta_1, 1] \\ d_3 &= \delta[0,\beta_1\beta_2) + (\theta(x+2\delta) - (x-\delta))[\beta_1\beta_2, \beta_1) \end{aligned}$$

and so forth. Then

$$\lim_{x\to 0} \lim_{\theta\to 0} \lim_{n\to\infty} \| \sum_{k=1}^{n} (-1)^k d_k \|_p = 1$$

$$\lim_{x\to 0} \lim_{\theta\to 0} \lim_{n\to\infty} \| \sum_{k=1}^{n} d_k \|_p = p - 1$$