Computing zeta functions of nondegenerate toric hypersurfaces

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Joint work with David Harvey (UNSW) and Kiran Kedlaya (UCSD)

Slides available at `edgarcosta.org` under Research

The zeta function problem

Let X be a smooth variety over a finite field $\mathbb{F}_q$ of characteristic p, consider

$$\begin{aligned}\zeta_X(t) &:= exp\left(\sum_{i\geq 1} \#X(\mathbb{F}_{q^i})\frac{t^i}{i}\right)\\ &= \prod_i \det\left(1 - t\,\mathrm{Frob}\,|H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}} \in \mathbb{Q}(t)\end{aligned}$$

Problem

Compute ζ_X from an *explicit* description of X.

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This approach is only practical for very few classes of varieties, e.g., low genus curves and p small.

Real life applications

- Cryptography/Coding Theory, we are often interested in $\#X(\mathbb{F}_q)$

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- arithmetic statistics
  - Birch–Swinnerton-Dyer
  - Sato–Tate
  - Lang–Trotter

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- p-adic methods based on Monsky–Washnitzer cohomology

Today

New p-adic method to compute $\zeta_X(t)$ that achieves a striking balance between practicality and generality.

Outline

Nondegenerate toric hypersurfaces

p-adic Cohomology

Some examples

Nondegenerate toric hypersurfaces

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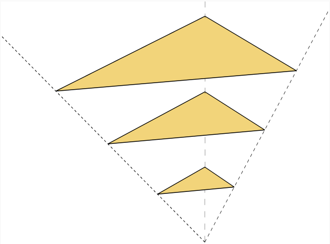
$$P_\Delta := \bigoplus_{d \geq 0} P_d, \quad P_d := R[d\Delta \cap \mathbb{Z}^n]$$

$$X_\Delta := \operatorname{Proj} P_\Delta$$

and $V(f)$ is an hypersurface in the toric variety X_Δ.



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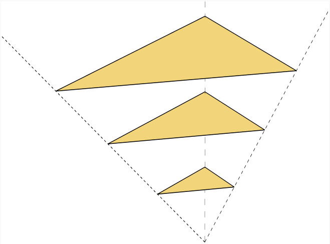

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Examples

Δ	X_Δ
$\operatorname{Conv}(0, e_1, \ldots, e_n)$	$\mathbb{P}^n$
$\operatorname{Conv}(0, e_1, \ell e_2, \ldots, \ell e_n)$	$\mathbb{P}^n(\ell, 1, \ldots, 1)$
$\operatorname{Conv}(0, e_1, e_2, e_1 + e_2)$	$\mathbb{P}^1 \times \mathbb{P}^1$

Nondegenerate toric hypersurfaces

Definition

We say that f is nondenegerate if the ideal

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$\implies$ Nondegeneracy is a **generic** condition.

Nondegenerate toric hypersurfaces

Geometric definition

The hypersurface defined by f is nondegenerate if for every face $\sigma \subset \Delta$ (including Δ itself) f restricted to the torus associated to σ is nonsingular of codimension 1.

Example

Let C be a plane curve in $\mathbb{P}^2$, then C is nondegenerate if:

- C does not pass through the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$;
- C intersects the coordinate axes $x = 0$, $y = 0$, $z = 0$ transversally;
- C is smooth on the complement of the coordinate axes.

Examples

Vertices of Δ	Resulting hypersurface
$0, de_1, de_2$	Smooth plane curve of genus $\binom{d-1}{2}$
$0, (2g+1)e_1, 2e_2$	Odd hyperelliptic curve of genus g
$0, ae_1, be_2$	$C_{a,b}$-curve
$0, 4e_1, 4e_2, 4e_3$	Quartic K3 surface
$0, 2e_1, 6e_2, 6e_3$	Degree 2 K3 surface
$0, 5e_1, \ldots, 5e_5$	Quintic Calabi-Yau threefold
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Remark

There are 4319 reflexive polyhedra that give rise to K3 surfaces.

p-adic Cohomology

Goal

Setup

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^\pm, \dots, x_n^\pm]$ a Laurent polynomial
- $Z := V(f) = \{x \in X_\Delta : f(x) = 0\}$ a nondegenerate hypersurface

Goal

Compute

$$\zeta_Z(t) := exp\left(\sum_{i \geq 1} \#Z(\mathbb{F}_{q^i}) t^i / i\right) \in \mathbb{Q}(t)$$

$$= \prod_i \det\left(1 - t\,\mathrm{Frob}\,|H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}}$$

$$= Q(t)^{(-1)^n} \prod_{i=0}^{n-1} \left(\frac{1}{1 - q^i t}\right)^{b_i},$$

where $Q(t) = \det\left(1 - q^{-1} t\,\mathrm{Frob}\,|H^n_{\mathrm{et}}(\overline{X_\Delta \backslash Z}, \mathbb{Q}_\ell)\right) \in 1 + \mathbb{Z}[t]$.

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where $Q(t) = \det\left(1 - p^{-1} t \operatorname{Frob} | H^{\dagger, n}(X_\Delta \backslash Z)\right) \in 1 + \mathbb{Z}[t]$.

Master plan

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- $Z := V(f) = \{x \in X_\Delta : f(x) = 0\}$ a nondegenerate hypersurface
- $U := X_\Delta \backslash Z$
- σ := p-th power Frobenius

Goal

Compute the matrix representing the action of σ in $H^{\dagger,n}(U)$ with enough p-adic precision.

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explicit description over $\mathbb{C}$
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de Rham cohomology
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cohomology relations
+
commutative algebra

$\Longrightarrow$

basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q}) = \left\{ x^\beta \omega / f^i \right\}_\beta$
with $\omega := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \in \Omega^n$
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reduction algorithm

Generic algorithm – Abbott–Kedlaya–Roe type

$$\begin{tikzcd} H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q}) \arrow[r, "\sim", "\mathrm{id}"'] & H^{\dagger,n}(U) \arrow[loop above, "\sigma"] \end{tikzcd}$$

1. Compute $\left\{ \frac{x^\beta}{f^m}\omega \right\}_\beta$ a monomial basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q})$ with $\omega := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \in \Omega^n$
2. In $H^{\dagger,n}$ compute a series approximation for

$$\sigma\left(\frac{x^\beta}{f^m}\omega\right) = p^n \frac{x^{p\beta}}{f^{pm}}\omega \sum_{i\geq 0} \binom{-m}{i} \left(\frac{\sigma(f) - f^p}{f^p}\right)^i$$

3. Write the approximation in terms of basis elements, i.e., apply the de Rham relations

Explicit description of H^n_{dR}

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For $g \in P_m$ we have

$$\begin{aligned} \frac{g\,f}{f^{m+1}}\omega &= \frac{g}{f}\omega \\ m\frac{g\,\partial_i f}{f^{m+1}}\omega &\equiv \frac{(\partial_i g)f}{f^{m+1}}\omega && \text{for } i = 1, \ldots, n \end{aligned}$$

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Same equations hold for $U_{\mathbb{Q}_q}$, if $g \in P_m^{\mathrm{int}} := R[\mathrm{int}(m\Delta) \cap L]$.

Generic algorithm – Abbott–Kedlaya–Roe

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1. Compute $\left\{ \dfrac{x^\beta}{f^m}\omega \right\}_{\beta \in P_m^{\mathrm{int}} \setminus (J_f)_m,\, m \leq n}$ a monomial basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_p})$
2. In $H^{\dagger,n}$ compute a series approximation for
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3. Write the approximation in terms of basis elements, i.e., apply the reduction algorithm

A sparse representation of Frobenius

Unfortunately, the truncation of the series expansion to K terms

$$\sigma\left(\frac{x^{\beta}}{f^{m}}\omega\right) \approx p^{n}\frac{x^{p\beta}\omega}{f^{pm}}\sum_{i=0}^{K-1}\binom{-m}{i}\left(\frac{\sigma(f)-f^{p}}{f^{p}}\right)^{i}$$

involves dense polynomials of degree $p(K-1)$ in n variables, and thus an unavoidable factor of p^n in the runtime.

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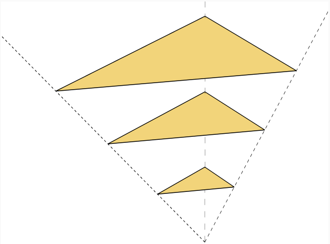
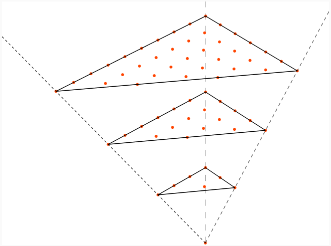
$$\sum_{i=0}^{K-1} \binom{-m}{i} \left(\frac{\sigma(f) - f^p}{f^p}\right)^i = \cdots = \sum_{i=0}^{K-1} \binom{-m}{i} \binom{m+K-1}{K-i-1} \sigma(f)^i f^{-p(m+i)}$$

Schematically

Abbott–Kedlaya–Roe	vs	C.–Harvey–Kedlaya
$\sum_{i=0}^{K-1} \binom{-m}{i} \left(\frac{\sigma(f)-f^p}{f^p}\right)^i$		$\sum_{i=0}^{K-1} \binom{-m}{i} \binom{m+K-1}{K-i-1} \sigma(f)^i f^{-p(m+i)}$
$(pdK)^{n+O(1)}$ terms		$(dK)^{n+O(1)}$ terms





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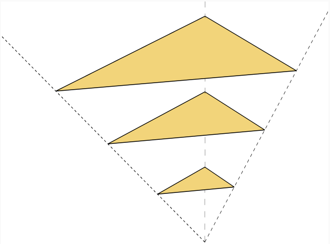
$\sum_{i=0}^{K-1} \binom{-m}{i} \left(\frac{\sigma(f)-f^p}{f^p}\right)^i$		$\sum_{i=0}^{K-1} \binom{-m}{i} \binom{m+K-1}{K-i-1} \sigma(f)^i f^{-p(m+i)}$

$(pdK)^{n+O(1)}$ terms



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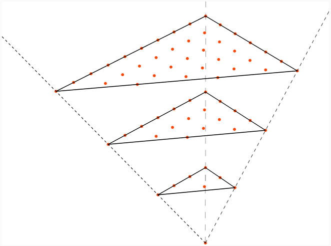

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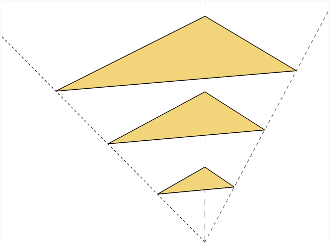

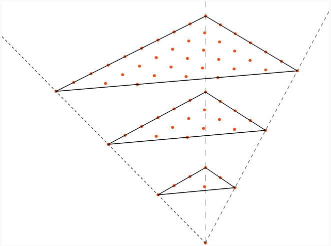

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$$\pi : P_n \longmapsto P_n$$

$$\ell x^{\alpha+\beta} \frac{g\omega}{f^{\ell+1}} \equiv x^\beta \frac{\pi(g)\omega}{f^\ell}, \; x^\alpha \in P_1$$

"slice" $\mapsto$ "slice"

"dot" $\mapsto$ "dot"

Generic algorithm – C.–Harvey–Kedlaya

$$
\begin{tikzcd}
H^n_{\mathrm{dR}}(U) \arrow[r, "\sim", "\mathrm{id}"'] & H^{\dagger,n}(U_{\mathbb{F}_p}) \arrow[loop above, "\sigma"]
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1. Compute $\left\{\frac{x^\beta}{f^m}\omega\right\}_\beta$ a monomial basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q})$
2. In $H^{\dagger,n}$ compute a **sparse** approximation for

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3. Apply **sparse** reduction algorithm to reduce expansion to basis elements.
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For large p, all the work is in step 3

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  - Toric hypersurfaces: beta version in C++ with NTL

Some examples

Example: random dense K3 surface

$X \subset \mathbb{P}^3_{\mathbb{F}_p}$ for $p = 49999$ given by

$$\begin{aligned}
&-9x^4 - 10x^3y - 9x^2y^2 + 2xy^3 - 7y^4 + 6x^3z + 9x^2yz - 2xy^2z + 3y^3z \\
&+ 8x^2z^2 + 6y^2z^2 + 2xz^3 + 7yz^3 + 9z^4 + 8x^3w + x^2yw - 8xy^2w - 7y^3w \\
&+ 9x^2zw - 9xyzw + 3y^2zw - xz^2w - 3yz^2w + z^3w - x^2w^2 - 4xyw^2 \\
&- 3xzw^2 + 8yzw^2 - 6z^2w^2 + 4xw^3 + 3yw^3 + 4zw^3 - 5w^4 = 0
\end{aligned}$$

In 1h5m5s, we obtain

$$Z_X(t) = ((1-t)(1-pt)(1-p^2t)Q(t)$$

where

$$\begin{aligned}
pQ(t/p)p =& (1-t)(p + 63115t + 14796t^2 + 42361t^3 + 49443t^4 \\
&+ 11718t^5 + 42046t^6 + 51501t^7 + 20534t^8 + 27146t^9 \\
&+ 38370t^{10} + 27146t^{11} + 20534t^{12} + \cdots + pt^{20})
\end{aligned}$$

Example: a quartic surface in the Dwork pencil

Consider the surface X in $\mathbb{P}^3_{\mathbb{F}_p}$ for $p = 49999$ given by

$$x_0^4 + \cdots + x_3^4 + x_0x_1x_2x_3 = 0.$$

In 4.3s, we compute that

$$Z_X(t) = \frac{1}{(1-t)(1-pt)(1-p^2t)R_1(pt)^3R_2(pt)^6S(t)}$$

where the “interesting” factor

$$S(t) = (1-pT)(1+95902t+p^2t^2).$$

In this case, the monomials generate a sublattice of index 4^2 in $\mathbb{Z}^3$. The polynomials R_1 and R_2 arise from the action of Frobenius on the Picard lattice; by a p-adic formula of de la Ossa–Kadir,

$$R_1(t) = (1 \pm t)(1 \pm t), \quad R_2(t) = 1 - T^2.$$

Example: a quintic threefold in the Dwork pencil

Consider the threefold X in $\mathbb{P}^4_{\mathbb{F}_p}$ for $p = 1000003$ given by

$$x_0^5 + \cdots + x_4^5 + x_0 x_1 x_2 x_3 x_5 = 0.$$

In 667s, we compute that

$$Z_X(t) = \frac{R_1(pt)^{20} R_2(pt)^{30} S(t)}{(1-t)(1-pt)(1-p^2 t)(1-p^3 t)}$$

where the “interesting” factor

$$S(t) = 1 + 74132440T + 748796652370pT^2 + 74132440p^3T^3 + p^6T^4.$$

and R_1 and R_2 are the numerators of the zeta functions of certain curves (given by a formula of Candelas–de la Ossa–Rodriguez Villegas).

Example: another family of K3 surfaces

Consider now the surface X in the weighted projective space $\mathbb{P}(8,5,4,3)_{\mathbb{F}_p}$ for $p = 49999$ given by taking the closure of the affine surface

$$yz^5 + xz^4 + y^4 + z^4 + x^2 + 1 = 0.$$

In 120s, we compute that

$$Z_X(t) = \frac{1}{(1-t)(1-pt)(1-p^2t)R(pt)S(t)}$$

where

$$pS(p^{-1}t) = p - 14662t - 31559t^2 - 5620t^3 - 31559t^4 - 14662t^5 + pt^6.$$

This example is from Miles Reid's list of 95 families of nondegenerate toric surfaces which are K3 surfaces.

Example: a hypergeometric motive (also a K3 surface)

Consider the appropriate completion of the toric surface over $\mathbb{F}_p$ with $p = 71$ given by

$$x^3 y + y^4 + z^4 - 12xyz + 1 = 0.$$

In 0.14s, we compute that the “interesting” factor of $\zeta_X(t)$ is

$$1 - 75t - 55pt^2 + 134p^2t^3 - 55p^3t^4 - 75p^4t^5 + p^6t^6.$$

This example (from arXiv:1612.09249) can be confirmed using Magma:

```
EulerFactor(HypergeometricData([1/12,1/6,5/12,7/12,
10/12,11/12],[0,0,0,1/3,1/2,2/3]),2^10 * 3^6, 71);
```

However, we can handle much larger p (e.g., p = 49999), for which Magma can only compute the coefficient of t.

Example: another Calabi–Yau threefold

Let X be the closure in the weighted projective space $\mathbb{P}(10,11,16,19,21)_{\mathbb{F}_p}$ for $p = 49999$ of the affine threefold

$$y^7 + x^2zw + zyzw + y^2zw + z^3w + w^3 + xz + yz = 0.$$

In 401s, we we compute that the "interesting" factor of ζ_X is

$$1 + 6423186t + 2211095838pt^2 - 127485903944p^2t^3$$
$$+2211095838p^4t^4 + 6423186p^6T^5 + p^9T^6$$

By analogy with the Reid list, one can classify Calabi–Yau threefolds arising as hypersurfaces in weighted projective spaces; there are 7555 such families. See `http://hep.itp.tuwien.ac.at/~kreuzer/CY/`.

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  Given an hypersurface defined over $\mathbb{Q}$, we may compute the zeta functions of its reductions modulo various primes at once. The average time complexity for each prime $p < N$ is
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These have not been implemented yet.

Questions?