Anabelian geometry

Grothendieck's programme: K a field, Y/K a smooth connected variety, $y \in Y(K)$ a basepoint. We have the profinite étale fundamental group $\pi_1^{\text{ét}}(Y_{\overline{K}}; y)$, endowed with a Galois action; for $z \in Y(K)$ we also have the profinite torsor of paths $\pi_1^{\text{ét}}(Y_{\overline{K}}; y, z)$, endowed with a compatible Galois action. One can study the Diophantine geometry of Y via the *non-abelian Kummer map*

$$Y(K) \to \mathrm{H}^1(G_K, \pi_1^{\text{ét}}(Y_{\overline{K}}; y)).$$

Kim's variant: $U_n/\mathbb{Q}_\ell$ the n-step $\mathbb{Q}_\ell$-unipotent étale fundamental group of (Y, y). Study the Diophantine geometry of Y via the more computable non-abelian Kummer map

$$Y(K) \to \mathrm{H}^1(G_K, U_n(\mathbb{Q}_\ell)).$$

Unipotent Kummer maps for small n

When $n = 1$ and Y is complete, $U_1 = V_\ell \mathrm{Alb}(Y)$ is the $\mathbb{Q}_\ell$-linear Tate module of the Albanese variety of Y, and the "non-abelian" Kummer map is the composite

$$Y(K) \to \mathrm{Alb}(Y)(K) \to \mathrm{H}^1(G_K, V_\ell \mathrm{Alb}(Y)).$$

The non-abelian Kummer maps for $n > 1$ are thought to see more refined arithmetic information. In the particular case that $n = 2$, the non-abelian Kummer map is thought to see information related to archimedean and ℓ-adic heights.

Local heights as functions on H^1

Theorem (Balakrishnan–Dan-Cohen–Kim–Wewers, 2014)

Let E° be the complement of 0 in an elliptic curve E over a p-adic local field K, and U_2 the 2-step $\mathbb{Q}_\ell$-unipotent fundamental group ($\ell \neq p$) of E°. Then the natural map $\mathbb{Q}_\ell(1) \hookrightarrow U_2$ induces a bijection on H^1, and the composite map

$$E^\circ(K) \to \mathrm{H}^1(G_K, U_2(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

is a $\mathbb{Q}$-valued Néron function on E with divisor $2[0]$, postcomposed with the natural embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$.

Generalisation to abelian varieties: setup

In place of the elliptic curve E, we will consider an abelian variety A over a local field K and a line bundle L/A, and let $L^\times = L \setminus 0$ denote the complement of the zero section. The natural anabelian invariant associated to this setup is the $\mathbb{Q}_\ell$-unipotent fundamental group of $L^\times$ – this is a central extension of the $\mathbb{Q}_\ell$-linear Tate module $V_\ell A$ by $\mathbb{Q}_\ell(1)$.

The role of the local height in this setup is played by the *Néron log-metric*

$$\lambda_L \colon L^\times(K) \to \mathbb{R},$$

namely the unique (up to additive constants) function which scales like the log of a metric on the fibres of L and such that for any/all non-zero section(s) s of L, $\lambda_L \circ s$ is a Néron function on A with divisor $\mathrm{div}(s)$. This is even $\mathbb{Q}$-valued when K is non-archimedean.

Conventions

Notation

Fix (for the rest of the talk) a prime p, a finite extension $K/\mathbb{Q}_p$, and an algebraic closure $\overline{K}/K$, determining an absolute Galois group G_K.

Later, we will denote by B_{dR}, B_{st}, $\mathrm{B}_{\mathrm{cris}}$ etc. the usual period rings constructed by Fontaine, and will fix a choice of p-adic logarithm, giving us an embedding $\mathrm{B}_{\mathrm{st}} \hookrightarrow \mathrm{B}_{\mathrm{dR}}$.

Generalisation to abelian varieties: the theorem

Theorem (B.)

Let A/K be an abelian variety, $L^\times/A$ the complement of zero in a line bundle L, and U the $\mathbb{Q}_\ell$-unipotent fundamental group ($\ell \neq p$) of $L^\times$. Then the natural map $\mathbb{Q}_\ell(1) \hookrightarrow U$ induces a bijection on H^1, and the composite map

$$L^\times(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_\ell)) \xleftarrow{\sim} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

takes values in $\mathbb{Q}$, and is the Néron log-metric on L.*

The p-adic analogue

We will define a certain natural subquotient $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ of the non-abelian Galois cohomology set $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$, allowing us to state a p-adic analogue of the preceding theorem.

Theorem (B.)

Let A/K be an abelian variety, $L^\times/A$ the complement of zero in a line bundle L, and let U be the $\mathbb{Q}_p$-unipotent fundamental group of $L^\times$. Then U is de Rham, the natural map $\mathbb{Q}_p(1) \hookrightarrow U$ induces a bijection on $\mathrm{H}^1_{g/e}$, and the composite map

$$L^\times(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xleftarrow{\sim} \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

is (well-defined and) the Néron log-metric on L.

Local (abelian) Bloch–Kato Selmer groups

- S. Bloch and K. Kato define, for any de Rham representation V of G_K on a $\mathbb{Q}_p$-vector space, subspaces

$$\mathrm{H}^1_e(G_K, V) \leq \mathrm{H}^1_f(G_K, V) \leq \mathrm{H}^1_g(G_K, V)$$

  of the Galois cohomology $\mathrm{H}^1(G_K, V)$.
- Their dimensions are easily computable, and $\mathrm{H}^1_e(G_K, V)$ can be studied via an "exponential" exact sequence

$$0 \to V^{G_K} \to \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(V) \to \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{D}^+_{\mathrm{dR}}(V) \to \mathrm{H}^1_e(G_K, V) \to 0.$$

- When $V = V_pA$ is the $\mathbb{Q}_p$ Tate module of an abelian variety A/K, these are all equal to the $\mathbb{Q}_p$-span of the image of the Kummer map

$$A(K) \to \mathrm{H}^1(G_K, V_pA).$$

Local non-abelian Bloch–Kato Selmer sets

- Following Kim, we will define, for any de Rham representation $U/\mathbb{Q}_p$ of G_K on a unipotent group, pointed subsets

$$\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$$

  of the non-abelian Galois cohomology set $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$.
- We will also make sense of the relative quotients, including $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) = \mathrm{H}^1_g/\mathrm{H}^1_e$, which appears in the p-adic main theorem.
- $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$ can be studied via an "exponential" exact sequence generalising the abelian sequence (see later).
- When U is the $\mathbb{Q}_p$ pro-unipotent* fundamental group of a smooth connected variety Y/K (which is de Rham), H^1_g contains the image of the non-abelian Kummer map

$$Y(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_p)).$$

Basic definitions

Definition (Galois representations on unipotent groups)

A *representation of G_K on a unipotent group $U/\mathbb{Q}_p$* is an action of G_K on U (by algebraic automorphisms) such that the action on $U(\mathbb{Q}_p)$ is continuous.
We say that U is *de Rham* (resp. semistable, crystalline etc.) just when the following equivalent conditions hold:

- $\mathrm{Lie}(U)$ is de Rham;
- $\mathcal{O}(U)$ is ind-de Rham;
- $\dim_K(\mathsf{D}_{\mathrm{dR}}(U)) = \dim_{\mathbb{Q}_p}(U)$, where $\mathsf{D}_{\mathrm{dR}}(U)/K$ is the unipotent group representing the functor

$$\mathsf{D}_{\mathrm{dR}}(U)(A) := U(A \otimes_K \mathsf{B}_{\mathrm{dR}})^{G_K}.$$

Definition (Local non-abelian Bloch–Kato Selmer sets)

Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. We define pointed subsets

$$\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$$

of the non-abelian cohomology $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ to be the kernels

$$\begin{aligned}
\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) &:= \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1}))\right); \\
\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) &:= \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}}))\right); \\
\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) &:= \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{st}}))\right).
\end{aligned}$$

One can use B_{dR} in place of B_{st} in the definition of H^1_g.

Definition (Quotients of Bloch–Kato Selmer sets)

Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. We denote by $\sim_{\mathrm{H}^1_e}$, $\sim_{\mathrm{H}^1_f}$, $\sim_{\mathrm{H}^1_g}$ the equivalence relations on $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ whose equivalence classes are the fibres of

$$\begin{aligned}
\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) &\to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1})); \\
\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) &\to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}})); \\
\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) &\to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{st}})).
\end{aligned}$$

We then define, for instance, the *Bloch–Kato quotient*

$$\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) := \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))/\sim_{\mathrm{H}^1_e}.$$

Why a cosimplicial approach?

The abelian Bloch–Kato exponential for a de Rham representation V arises from tensoring it with the exact sequence

$$0 \to \mathbb{Q}_p \to \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \to \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \to 0$$

and taking the long exact sequence in Galois cohomology. Equivalently, if we consider the cochain complex

$$\mathrm{C}_e^\bullet : \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \to \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+},$$

then the cohomology groups of the cochain $(\mathrm{C}_e^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}$ are canonically identified as

$$\mathrm{H}^j\left((\mathrm{C}_e^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}\right) \cong \begin{cases} V^{G_K} & j=0; \\ \mathrm{H}^1_e(G_K, V) & j=1; \\ 0 & j \geq 2. \end{cases}$$

The advantage of using cochain complexes is that we can perform analogous constructions for H^1_f and H^1_g. For instance, taking the cochain complex

$$\mathrm{C}_g^\bullet : \mathrm{B}_{\mathrm{st}} \to \mathrm{B}_{\mathrm{st}}^{\oplus 2} \oplus \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \to \mathrm{B}_{\mathrm{st}},$$

the cohomology groups of the cochain $(\mathrm{C}_g^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}$ are canonically identified as

$$\mathrm{H}^j\left((\mathrm{C}_g^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}\right) \cong \begin{cases} V^{G_K} & j=0; \\ \mathrm{H}^1_g(G_K, V) & j=1; \\ \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(V^*(1))^* & j=2; \\ 0 & j \geq 3. \end{cases}$$

The cochain complexes $\mathrm{C}_e^\bullet$, $\mathrm{C}_f^\bullet$, $\mathrm{C}_g^\bullet$ themselves cannot be directly be used in the non-abelian setting (as we cannot tensor a group by a vector space), so we have to tweak them slightly to find a non-abelian generalisation of the Bloch–Kato exponential.

For example, in place of $\mathrm{C}_e^\bullet$, we consider the diagram

$$\mathrm{B}_{\mathrm{cris}}^{\varphi=1} \times \mathrm{B}_{\mathrm{dR}}^{+} \rightrightarrows \mathrm{B}_{\mathrm{dR}}$$

of $\mathbb{Q}_p$-algebras. Taking points in U and then G_K-fixed points, we then obtain the diagram

$$\mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) \times \mathrm{D}_{\mathrm{dR}}^{+}(U)(K) \rightrightarrows \mathrm{D}_{\mathrm{dR}}(U)(K).$$

There is an action of $\mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) \times \mathrm{D}_{\mathrm{dR}}^{+}(U)(K)$ on $\mathrm{D}_{\mathrm{dR}}(U)(K)$ by $(x, y)\colon z \mapsto y^{-1}zx$ – we will see later that the orbit space is canonically identified with $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$.

Non-abelian analogy

In order to extend the study of local Bloch–Kato Selmer groups to the non-abelian context, we need to replace three abelian concepts with non-abelian analogues.

- In place of the cochain complexes $\mathrm{C}_*^\bullet$ of G_K-representations, we will use *cosimplicial* $\mathbb{Q}_p$*-algebras* $\mathrm{B}_*^\bullet$ with G_K-action.
- In place of the cochain complexes $(\mathrm{C}_*^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}$, we will examine the *cosimplicial groups* $U(\mathrm{B}_*^\bullet)^{G_K}$.
- In place of the cohomology groups of these cochain complexes, we will calculate the *cohomotopy groups/sets* of the corresponding cosimplicial groups.

Cosimplicial groups

Definition (Cosimplicial objects)

A *cosimplicial object* of a category $\mathcal{C}$ is a covariant functor $X^\bullet\colon \Delta \to \mathcal{C}$ from the simplex category Δ of non-empty finite ordinals and order-preserving maps. We think of this as a collection of objects X^n together with *coface* maps $d^\bullet$

$$X^0 \rightrightarrows X^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} X^2 \cdots$$

and *codegeneracy* maps $s^\bullet$

$$X^0 \leftarrow X^1 \leftleftarrows X^2 \cdots$$

satisfying certain identities.

Definition (Cohomotopy groups/sets)

Let $U^\bullet$ be a cosimplicial group

$$U^0 \rightrightarrows U^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} U^2 \cdots .$$

We define the 0*th cohomotopy group* $\pi^0(U^\bullet)$ to be

$$\pi^0\left(U^\bullet\right) := \{u^0 \in U^0 \mid d^0(u^0) = d^1(u^0)\} \leq U^0.$$

We also define the pointed set of 1-*cocycles* to be

$$\mathrm{Z}^1(U^\bullet) := \{u^1 \in U^1 \mid d^1(u^1) = d^2(u^1)d^0(u^1)\} \subseteq U^1$$

and the 1*st cohomotopy (pointed) set* $\pi^1(U^\bullet) := \mathrm{Z}^1(U^\bullet)/U^0$ to be the quotient of $\mathrm{Z}^1(U^\bullet)$ by the twisted conjugation action of U^0, given by $u^0\colon u^1 \mapsto d^1(u^0)^{-1}u^1d^0(u^0)$.

Definition (Cohomotopy groups/sets (cont.))

When $U^\bullet$ is abelian, $\pi^0(U^\bullet)$ and $\pi^1(U^\bullet)$ are abelian groups, and we can define the higher cohomotopy groups $\pi^j(U^\bullet)$ to be the cohomology groups of the cochain complex

$$U^0 \to U^1 \to U^2 \cdots$$

with differential $\sum_k (-1)^k d^k$.

Example (Non-abelian group cohomology)

Suppose G is a topological group acting continuously on another topological group U. Then $\mathrm{C}^n(G, U) := \mathrm{Map}_{\mathrm{cts}}(G^n, U)$ can be given the structure of a cosimplicial group. Its cohomotopy $\pi^j(\mathrm{C}^\bullet(G, U))$ is canonically identified with the group cohomology $\mathrm{H}^j(G, U)$ for $j = 0, 1$, and for all j when U is abelian.

Long exact sequences in cohomotopy

Notation

When we assert that a sequence

$$\cdots \to U^{r-1} \to U^r \overset{\curvearrowright}{\to} U^{r+1} \to U^{r+2} \to \cdots$$

is *exact*, we shall mean that:

- $\cdots \to U^{r-1} \to U^r$ is an exact sequence of groups (and group homomorphisms);
- $U^{r+1} \to U^{r+2} \to \cdots$ is an exact sequence of pointed sets;
- there is an action of U^r on U^{r+1} whose orbits are the fibres of $U^{r+1} \to U^{r+2}$, and whose point-stabiliser is the image of $U^{r-1} \to U^r$.

Cosimplicial groups give us many ways of producing long exact sequences of groups and pointed sets. For example:

Theorem (Bousfield–Kan, 1972)

Let

$$1 \to Z^\bullet \to U^\bullet \to Q^\bullet \to 1$$

be a central extension of cosimplicial groups. Then there is a cohomotopy exact sequence

$$1 \longrightarrow \pi^0(Z^\bullet) \longrightarrow \pi^0(U^\bullet) \longrightarrow \pi^0(Q^\bullet) \longrightarrow \pi^1(Z^\bullet) \curvearrowright \pi^1(U^\bullet) \longrightarrow \pi^1(Q^\bullet) \longrightarrow \pi^2(Z^\bullet).$$

The cosimplicial models

Our general method for studying local Bloch–Kato Selmer sets and their quotients will be to define various cosimplicial $\mathbb{Q}_p$-algebras $\mathrm{B}^\bullet_e$, $\mathrm{B}^\bullet_f$, $\mathrm{B}^\bullet_g$, $\mathrm{B}^\bullet_{g/e}$, $\mathrm{B}^\bullet_{f/e}$ with G_K-action such that, for any de Rham representation of G_K on a unipotent group $U/\mathbb{Q}_p$, we have a canonical identification

$$\pi^1\left(U(\mathrm{B}^\bullet_*)^{G_K}\right) \cong \mathrm{H}^1_*(G_K, U(\mathbb{Q}_p)).$$

Cohomotopy of the cosimplicial Dieudonné functors

In fact, we can give a complete description of the cohomotopy groups/sets of each $U(\mathrm{B}_*^\bullet)^{G_K}$. For instance, we have

$$\pi^j\left(U(\mathrm{B}_e^\bullet)^{G_K}\right) \cong \begin{cases} U(\mathbb{Q}_p)^{G_K} & j=0; \\ \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) & j=1; \\ 0 & j\geq 2 \text{ and } U \text{ abelian}; \end{cases}$$

$$\pi^j\left(U(\mathrm{B}_{g/e}^\bullet)^{G_K}\right) \cong \begin{cases} \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) & j=0; \\ \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) & j=1; \\ \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U(\mathbb{Q}_p)^*(1))^* & j=2 \text{ and } U \text{ abelian}; \\ 0 & j\geq 3 \text{ and } U \text{ abelian}. \end{cases}$$

Construction of Bloch–Kato algebras

The cosimplicial algebras required to make this work are all built from standard period rings. For example, the diagram

$$\mathrm{B}_{\mathrm{cris}}^{\varphi=1}\times\mathrm{B}_{\mathrm{dR}}^+ \rightrightarrows \mathrm{B}_{\mathrm{dR}}$$

(which we saw earlier) is a semi-cosimplicial $\mathbb{Q}_p$-algebra (that is, a cosimplicial algebra without codegeneracy maps). $\mathrm{B}_e^\bullet$ is then the universal cosimplicial $\mathbb{Q}_p$-algebra mapping to this semi-cosimplicial algebra (the cosimplicial algebra cogenerated by it). Concretely, this has terms

$$\mathrm{B}_e^n = \mathrm{B}_{\mathrm{cris}}^{\varphi=1}\times\mathrm{B}_{\mathrm{dR}}^+\times\mathrm{B}_{\mathrm{dR}}^n.$$

The non-abelian Bloch–Kato exponential

The description of the cohomotopy of $U(\mathrm{B}_e^\bullet)^{G_K}$ in degrees 0 and 1 is equivalent to the existence of a *non-abelian exponential exact sequence*

$$1 \longrightarrow U(\mathbb{Q}_p)^{G_K} \longrightarrow \mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) \times \mathsf{D}_{\mathrm{dR}}^{+}(U)(K) \curvearrowright \mathsf{D}_{\mathrm{dR}}(U)(K) \xrightarrow{\exp} \mathrm{H}_e^1(G_K, U(\mathbb{Q}_p)) \longrightarrow 1.$$

Remark

Concretely, the exponential exact sequence provides a canonical identification of H_e^1 as a double-coset space

$$\mathrm{H}_e^1(G_K, U(\mathbb{Q}_p)) \cong \mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p)\backslash \mathsf{D}_{\mathrm{dR}}(U)(K)/\mathsf{D}_{\mathrm{dR}}^{+}(U)(K).$$

Construction of the non-abelian Bloch–Kato exponential

By induction along the central series of U, we see quickly that $\pi^0\left(U(\mathrm{B}_e^\bullet)\right) = U(\mathbb{Q}_p)$ and $\pi^1\left(U(\mathrm{B}_e^\bullet)\right) = 1$. Unpacking the definition of $\mathrm{B}_e^\bullet$, this says that

$$1 \to U(\mathbb{Q}_p) \to U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1}) \times U(\mathrm{B}_{\mathrm{dR}}^{+}) \overset{\curvearrowright}{\to} U(\mathrm{B}_{\mathrm{dR}}) \to 1$$

is exact (i.e. the action is transitive with point-stabiliser $U(\mathbb{Q}_p)$). We then obtain a long exact sequence in Galois cohomology

$$1 \longrightarrow U(\mathbb{Q}_p)^{G_K} \longrightarrow \mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) \times \mathsf{D}_{\mathrm{dR}}^{+}(U)(K) \curvearrowright \mathsf{D}_{\mathrm{dR}}(U)(K) \xrightarrow{\exp} \mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \longrightarrow \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1}) \times U(\mathrm{B}_{\mathrm{dR}}^{+})),$$

which is already most of the desired exponential sequence.

Construction of the non-abelian Bloch–Kato exponential (cont.)

It remains to show that the image of exp is exactly $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$. The exact sequence shows that the image is exactly the kernel of

$$\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1})) \times \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{dR}}^+)),$$

which certainly is contained in $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$.

It is then not too hard to prove that in fact the kernel is exactly $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$, using the fact that the map

$$\mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{dR}}^+)) \to \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{dR}}))$$

has trivial kernel (we omit the diagram-chase in the interests of brevity). This establishes the desired exact sequence, and hence the description of the cohomotopy of $U(\mathrm{B}_e^\bullet)^{G_K}$. □

Lemma

Let

$$1 \to Z \to U \to Q \to 1$$

be a central extension of de Rham representations of G_K on unipotent groups over $\mathbb{Q}_p$. Then there is an exact sequence

$$1 \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(Z)(\mathbb{Q}_p) \xrightarrow{z} \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(\mathbb{Q}_p) \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(Q)(\mathbb{Q}_p) \to$$
$$\hookrightarrow \mathrm{H}^1_{g/e}(G_K, Z(\mathbb{Q}_p)) \curvearrowright \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \longrightarrow \mathrm{H}^1_{g/e}(G_K, Q(\mathbb{Q}_p)) \to$$
$$\hookrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(Z(\mathbb{Q}_p)^*(1))^*.$$

Proof of lemma.

From the construction of $\mathrm{B}^\bullet_{g/e}$ (out of B_{st}), it follows that

$$1 \to Z(\mathrm{B}^\bullet_{g/e})^{G_K} \to U(\mathrm{B}^\bullet_{g/e})^{G_K} \to Q(\mathrm{B}^\bullet_{g/e})^{G_K} \to 1$$

is a central extension of cosimplicial groups. The desired exact sequence is then the cohomotopy exact sequence for these cosimplicial groups. □

If, as in the main theorem, $U/\mathbb{Q}_p$ is the $\mathbb{Q}_p$-unipotent fundamental group of $L^\times = L \setminus 0$, where L is a line bundle on an abelian variety A/K, then U is a central extension

$$1 \to \mathbb{Q}_p(1) \to U \to V_pA \to 1.$$

Applying the preceding lemma shows that $\mathbb{Q}_p(1) \hookrightarrow U$ induces a bijection on $\mathrm{H}^1_{g/e}$, so that $\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \cong \mathbb{Q}_p$.

Showing that the $\mathrm{H}^1_{g/e}$-valued non-abelian Kummer map is then identified with the Néron log-metric requires some extra work, but is largely straightforward.