2017 Banff Lecture on Fundamental Groups

π_1 **in topology:** homotopy classes of closed paths starting and ending at point x with natural multiplication.

Classification of covers: a *cover* of X is a space Y equipped with a continuous map $p : Y \to X$ s.t. each point of X has an open neighbourhood V for which $p^{-1}(V) = \coprod U_i$ of Y such that p induces a homeomorphism $U_i \xrightarrow{\sim} V$. Monodromy action on $y \in p^{-1}(x)$ induced by lifting paths and homotopies.

Theorem: Assume X connected, locally connected and locally simply connected. The functor Fib_x sending a cover $p : Y \to X$ to the fibre $p^{-1}(x)$ equipped with the monodromy action induces an equivalence of the category of covers of X with the category of left $\pi_1^{\mathrm{top}}(X,x)$-sets.

In fact, Fib_x is representable by a *universal cover* $\widetilde{X}_x \to X$. It is a connected principal $\pi_1^{\mathrm{top}}(X,x)$-cover.

Recall: for a group G a principal G-cover is a cover $Y \to X$ equipped with a G-action that is locally on sufficiently small open sets U isomorphic to the trivial G-cover $G \times U \to U$.

Classification of principal G-covers: a principal G-cover $Y \to X$ corresponds to a homomorphism $\pi_1^{\mathrm{top}}(X,x) \to G$ such that Y becomes isomorphic to the contracted product $\widetilde{X}_x \times^{\pi_1^{\mathrm{top}}(X,x)} G$. Surjective homomorphisms correspond to connected (Galois) covers.

A cover is *finite* if it has finite fibres.

Corollary: Under the assumptions above, the functor Fib_x induces an equivalence of the category of finite covers of X with the category of left continuous finite $\pi_1^{\mathrm{top}}(X,x)^{\wedge}$-sets, where $\wedge$ denotes profinite completion.

Remark: Drop the assumption of local simply connectedness. One can still construct a profinite group Π such that the category of finite covers of X is with the category of left continuous finite Π-sets, by a projective limit construction.

This says that the category of finite covers has special properties: it is a Galois category. The group Π is the automorphism group of the set-valued *fibre functor* on the category of finite covers sending $p : Y \to X$ to $p^{-1}(x)$ for $x \in X$ fixed.

Situation in algebraic geometry: Grothendieck's π_1 is defined as a group satisfying the analogue of the above corollary.

Let S be a connected scheme. Finite étale cover: finite étale surjection $X \to S$. In particular, each fibre at a point $s \in S$ is the spectrum of a finite étale $\kappa(s)$-algebra.
Fix a geometric point $\bar{s} : \operatorname{Spec}(\Omega) \to S$. For a finite étale cover $X \to S$ we consider the geometric fibre $X \times_S \operatorname{Spec}(\Omega)$ over $\bar{s}$, and denote by $\operatorname{Fib}_{\bar{s}}(X)$ its underlying set.

Definition: $\pi_1^{\mathrm{et}}(S, \bar{s}) := \operatorname{Aut}(\operatorname{Fib}_{\bar{s}})$

Theorem: 1. The group $\pi_1^{\mathrm{et}}(S, \bar{s})$ is profinite, and its action on $\operatorname{Fib}_{\bar{s}}(X)$ is continuous for every finite étale cover $X \to S$.
2. The functor $\operatorname{Fib}_{\bar{s}}$ induces an equivalence of the category of finite étale covers of S with the category of finite continuous left $\pi_1^{\mathrm{et}}(S, \bar{s})$-sets.

In fact, $\pi_1^{\mathrm{et}}(S, \bar{s})$ is constructed 'by hand' by considering the inverse system of all étale *Galois* covers $X \to S$, rigidified along base points, and taking the inverse limit of associated automorphism groups $\operatorname{Aut}(X|S)$.
[Thus the construction rather gives a 'fundamental pro-torsor' $P_{\bar{s}} \to S$ under $\pi_1^{\mathrm{et}}(S, \bar{s})$.]

Examples:

1. For X of finite type over $\mathbf{C}$ and $\bar{x} : \operatorname{Spec}(\mathbf{C}) \to X$ there is a canonical isomorphism

$$\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}}, \bar{x})} \xrightarrow{\sim} \pi_1^{\mathrm{et}}(X, \bar{x}).$$

2. For $X = \operatorname{Spec}(k)$, $\bar{x} : \operatorname{Spec}(k_s) \to \operatorname{Spec}(k)$ we have

$$\pi_1^{\mathrm{et}}(X, \bar{x}) \cong \operatorname{Gal}(k_s|k).$$

Analogue of principal G-cover: torsor under finite constant group scheme.
Recall: If $G \to S$ is an fppf group scheme, a *(left) G-torsor* over S is an fppf scheme $X \to S$ together with a group action $\rho : G \times_S X \to X$ such that the map $(\rho, \mathrm{id}) : G \times_S X \to X \times_S X$ is an isomorphism.

Suppose k is a field, S a k-scheme, G (comes from) a finite étale k-group scheme.
If G is constant, then 'connected G-torsor $X \to S$' = 'G-Galois cover'.
If $k = k_s$, every finite étale k-group scheme is constant.

Conclusion: If $k = k_s$, S a k-scheme, G finite étale k-group scheme, G-torsors correspond to homomorphisms $\pi_1^{\mathrm{et}}(S, \bar{s}) \to G$.

Nori's fundamental group scheme extends the above to G-torsors for arbitrary k-group schemes.

k = field, S = connected geometrically reduced scheme over k, such that there is a point $s \in S(k)$.
Theorem: There is a profinite group scheme $\pi_1^N(S,s)$, the *Nori fundamental group scheme,* with a $\pi_1^N(S,s)$-torsor $\pi : P \to S$ having a trivialization $\pi^{-1}(s) \cong \pi_1^N(S,s)$ such that for every finite k-group scheme G and G-torsor $p : X \to S$ with a trivialization $\alpha : p^{-1}(s) \cong G$ there is a unique morphism of group schemes $\pi_1^N(S,\bar{s}) \to G$ inducing X and α via contracted product with P.

(Contracted product of torsors: $P \times^{\pi} G$ is the quotient of $P \times G$ by the π-action $\alpha(p,g) = (p\alpha, h(\alpha^{-1})g)$. It exists by fppf descent.)

The theorem can be proven by constructing $\pi_1^N(S,s)$ and P as inverse limits of the system of geometrically connected G-torsors $P_\alpha \to S$ with maps preserving a given k-point in the fibre inducing the trivialization.

Corollary: If $k = \overline{k}$ and $s \in S(k)$ is fixed, $\pi_1^{\text{et}}(S,\bar{s})$ is canonically isomorphic to the group of k-points of the maximal pro-étale quotient of $\pi_1^N(S,s)$. In characteristic 0 this quotient map is an isomorphism.

For S proper over k Nori also constructed $\pi_1^N(S,s)$ as a Tannakian fundamental group. Recall:

A *neutral Tannakian category* over a field k is a rigid k-linear abelian tensor category $\mathcal{C}$ whose unit 1 satisfies $\text{End}(1) \cong k$, and is moreover equipped with an exact faithful tensor functor $\omega : \mathcal{C} \to \text{Vecf}_k$ into the category of finite dimensional k-vector spaces. The functor ω is called a (neutral) *fibre functor.*

Theorem: Every neutral Tannakian category $(\mathcal{C}, \omega)$ over k is equivalent to the category Rep_G of finite dimensional representations of an affine group scheme G over k.

The Tannakian category giving rise to $\pi_1^N(S,s)$ is constructed by means of certain vector bundles.

Definition (Weil): A vector bundle $\mathcal{E}$ on S is *finite* if there exist polynomials $f \neq g$ with non-negative integer coefficients such that $f(\mathcal{E}) \cong g(\mathcal{E})$.

Here for $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbf{Z}_{\geq 0}[x]$ with non-negative integer coefficients, we define

$$f(\mathcal{E}) := \bigoplus_{i=0}^{n} (\mathcal{E}^{\otimes i})^{\oplus a_i}.$$

Finite line bundles = torsion line bundles.

[Relation with torsors $\pi : X \to S$ under a finite k-group scheme G: by the torsor property $\mathcal{E} := \pi_* \mathcal{O}_X$ satisfies $\mathcal{E}_X^{\otimes 2} \cong \mathcal{E}_X^{\oplus n}$ with $n = \dim_k \Gamma(G, \mathcal{O}_G)$.]

The category of finite vector bundles on S is a rigid k-linear tensor category with unit, and taking the fibre at $s \in S(k)$ gives a fibre functor. However, it is not abelian in char. $p > 0$. Nori enlarges it to the category EF_S of essentially finite vector bundles: subquotients of finite vector bundles that are semistable of slope 0.

Theorem: If S is proper and geometrically reduced over k, then EF_S is a neutral Tannakian category with Tannakian fundamental group isomorphic to $\pi_1^N(S, S)$.

Generalizations to k-schemes without a k-point: Esnault–Hai (in char. 0, using connections), Borne–Vistoli (the fundamental gerbe).

Notice: $\pi_1^N(S, S)$ is a group scheme over k, not just a group. It is compatible with separable algebraic base change, but *not* with extensions of algebraically closed fields (Mehta–Subramanian, Pauly, Stäbler).

Other examples of Tannakian fundamental groups: let X now be a nice topological space, $X \in x$. Complex local systems ((i.e. locally constant sheaves of $\mathbf{C}$-vector spaces) of finite rank on X form a neutral Tannakian category with the fibre functor 'fibre at x', giving rise to a Tannakian fundamental group $\pi_1^{\mathrm{alg}}(X, x)$.

But we also know by the theory of π_1^{top}: local systems of rank n correspond to linear representations $\pi_1^{\mathrm{top}}(X, x) \to \mathrm{GL}_n(\mathbf{C})$. The linear representation category of the group $\pi_1^{\mathrm{top}}(X, x)$ is thus the same as that of the group scheme $\pi_1^{\mathrm{alg}}(X, x)$.

One can also recover $\pi_1^{\mathrm{alg}}(X, x)$ from $\pi_1^{\mathrm{top}}(X, x)$ directly via a limit construction: it is the *algebraic hull* of $\pi_1^{\mathrm{top}}(X, \overline{x})$, i.e. the $\varprojlim$ of all Zariski closures of images of representations $\pi_1^{\mathrm{top}}(X, \overline{x}) \to \mathrm{GL}_n(\mathbf{C})$.

Assume now X is a complex manifold. We may then use the *Riemann–Hilbert correspondence:* The functor of taking horizontal sections induces an equivalence between holomorphic rank n vector bundles on X with a flat connection and rank n local systems. Thus the Tannakian category of flat connections on X also has fundamental group $\pi_1^{\mathrm{alg}}(X, x)$.

Finally assume $X = V^{\mathrm{an}}$ for a smooth complex algebraic variety V. Then by GAGA (Deligne) the above Tannakian categories are also equivalent to that of *algebraic* vector bundles on V with a flat connection regular at infinity. This gives a definition of $\pi_1^{\mathrm{alg}}(X, x)$ within algebraic geometry; it is the *de Rham fundamental group of* V*,* denoted by $\pi_1^{\mathrm{dR}}(V, x)$. It can be done over any field of char. 0.

Moreover, $\pi_1^{\mathrm{et}}(V, x)$ is a quotient of $\pi_1^{\mathrm{alg}}(V^{\mathrm{an}}, x)$ (corresponding to finite étale group schemes), hence of $\pi_1^{\mathrm{dR}}(V, x)$.

Theorem (Grothendieck) Given a morphism $(V, v) \to (W, w)$ of pointed smooth algebraic varieties over $\mathbf{C}$, the induced map on étale fundamental groups $\pi_1^{\mathrm{et}}(V, v) \to \pi_1^{\mathrm{et}}(W, w)$ is an isomorphism if and only if $\pi_1^{\mathrm{dR}}(V, v) \to \pi_1^{\mathrm{dR}}(W, w)$ is.

Really a theorem about profinite and pro-algebraic completions of finitely generated groups.

In particular, $\pi_1^{\mathrm{et}}(V, v)$ is abelian (resp. trivial) if and only if $\pi_1^{\mathrm{dR}}(V, v)$ is. In other words: $\pi_1^{\mathrm{et}}(V, v)$ is abelian (resp. trivial) if and only if every irreducible vector bundle with a flat connection has rank one (resp. is trivial).

Esnault–Mehta, Esnault–Srinivas, Esnault–Shiho: studied analogues in positive characteristic, crystalline context...

Deligne remarked: $\pi_1^{\mathrm{dR}}(V, v)$ is not compatible with extensions of algebraically closed fields either, already for rank 1 connections on $\mathbf{P}^1 \setminus \{0, \infty\}$.

But its unipotent completion $\pi_1^{\mathrm{dR}}(V, v)^{\mathrm{un}}$ is compatible with arbitrary field extensions!

Definition: $\pi_1^{\mathrm{dR}}(V, v)^{\mathrm{un}}$ is the fundamental group of the neutral Tannakian category of vector bundles with connection on V that are finite successive extensions of trivial connections.

We may thus regard $\pi_1^{\mathrm{dR}}(V,v)^{\mathrm{un}}$ as a (pro-unipotent) $\mathbf{C}$-group scheme. It is related to the pro-unipotent completion of the topological fundamental group $\pi_1 := \pi_1^{\mathrm{top}}(V(\mathbf{C}),v)$ as follows.

Consider the group algebra $\mathbf{Q}[\pi_1]$ with the natural coproduct given by $\Delta(x) = x \otimes x$ and augmentation ideal $I := \ker(\mathbf{Q}[\pi_1] \to \mathbf{Q})$. The quotients $\mathbf{Q}[\pi_1]/I^N$ carry an induced coproduct, and we define $\pi_1^{\mathrm{top}}(V(\mathbf{C}),v)^{\mathrm{un}}$ as the $\mathbf{Q}$-group scheme with Hopf algebra

$$\lim_{\substack{\to \\ N}} (\mathrm{Hom}_{\mathbf{Q}}, \mathbf{Q}[\pi_1]/I^N), \mathbf{Q}).$$

Then we have

$$\pi_1^{\mathrm{dR}}(V,v)^{\mathrm{un}} \cong \pi_1^{\mathrm{top}}(V(\mathbf{C}),v)^{\mathrm{un}} \otimes_{\mathbf{Q}} \mathbf{C}.$$

There are étale and crystalline variants, with comparison isomorphisms.

For $V = \mathbf{P}^1$ minus finitely many rational points or, more generally, V smooth unirational, these all come from a group scheme object in the Tannakian category of mixed Tate motives (Deligne–Goncharov).