Curvature Effects in Surface Superconductivity

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Phase Transitions Models
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Outline

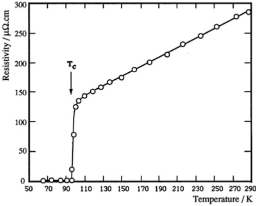
1. Introduction [FH]:
   - Ginzburg-Landau (GL) theory and response of a superconductor to an applied magnetic field.
   - Surface superconductivity: GL asymptotics and Pan's conjecture.
- Main results & work in progress [CR1–3,CG]:
  2. Energy and density asymptotics between H_{c2} and H_{c3} [CR1–2];
  3. Curvature effects on surface superconductivity [CR3].
  4. Effects of boundary singularities (corners) [CG].

Main References

- [FH] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, Progr. Nonlinear Diff. Eqs. Appl. **77**, 2010.
- [CR1] M. C., N. Rougerie, *Commun. Math. Phys.* **332** (2014).
- [CR2] M. C., N. Rougerie, *Arch. Rational Mech. Anal.* **219** (2015).
- [CR3] M. C., N. Rougerie, *Lett. Math. Phys.* **106** (2016).
- [CG] M. C., E.L. Giacomelli, *Rev. Math. Phys.* **29** (2017).

Superconductivity

Certain materials which behave like *metals* at *room temperature* become superconductors (zero resistivity) below a certain $T_c > 0$ (ceramic compound $YBa_2Cu_3O_7$ in fig.).



- When a *type-II* superconductor is immersed in a magnetic field, the field is *expelled* from the bulk.
- *Strong* magnetic fields can penetrate the sample and eventually *destroy* superconductivity.
- The response of a superconductor to a magnetic field can be described by the Ginzburg-Landau theory.

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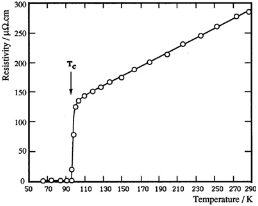

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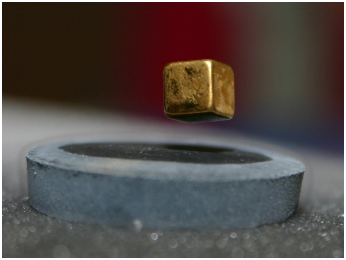

Ginzburg-Landau Theory

GL energy functional

The energy per unit length of a very long superconducting wire of (*smooth* and simply connected) cross section $\Omega \subset \mathbb{R}^2$ is obtained by minimizing

$$\mathcal{G}_\kappa^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ \left| (\nabla + i\mathbf{A}) \Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + \left| \mathrm{curl}\mathbf{A} - h_{\mathrm{ex}} \right|^2 \right\}$$

- Variational equations

$$\begin{cases} -(\nabla + i\mathbf{A})^2 \Psi = \kappa^2 \left(1 - |\Psi|^2\right) \Psi, & \text{in } \Omega, \\ -\nabla^\perp \mathrm{curl}\mathbf{A} = \mathbf{j}_\mathbf{A}[\Psi], & \text{in } \Omega, \\ \mathbf{n} \cdot (\nabla + i\mathbf{A}) \Psi = 0, & \text{on } \partial\Omega, \\ \mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}, & \text{on } \partial\Omega. \end{cases}$$

- $|\Psi|^2$ relative density of superconducting electrons (Cooper pairs).
- $\mathbf{A}$ magnetic potential with magnetic field $h = \mathrm{curl}\mathbf{A}$.
- κ^{-1} penetration depth ($\kappa \to \infty$ = extreme type-II superconductors).
- Uniform applied magnetic field $\perp$ to Ω of size h_{ex}.

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Perfectly superconducting state

In absence of applied field, the superconducting state $|\Psi| \equiv 1$, $\mathbf{A} = 0$ (Meissner state) is the *unique minimizer* of the GL energy.

Normal state

If $h_{\mathrm{ex}} \gg 1$ and κ fixed (huge applied field), the normal state $\Psi \equiv 0$ with $\mathrm{curl}\mathbf{A} = h_{\mathrm{ex}}$ is the unique minimizer of the GL energy.

Mixed state

For intermediate applied fields, any minimizer (possibily non-unique) is a mixed state satisfying $0 \leqslant |\Psi| \leqslant 1$.

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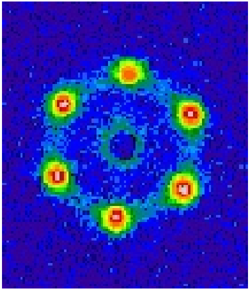
Phenomenology (physics)

- Superconductivity is first *lost* at *isolated defects* (vortices).
- For larger magnetic fields the number of vortices increases and eventually vortices arrange in a triangular lattice, which was predicted by Abrikosov in 1957 and later observed by Essmann, Trauble in 1967.



Vortices in Nb crystal
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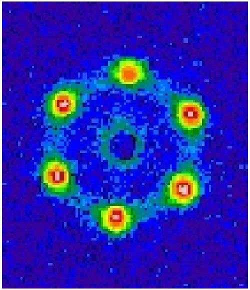

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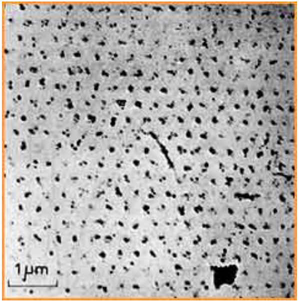
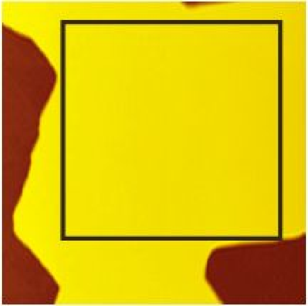


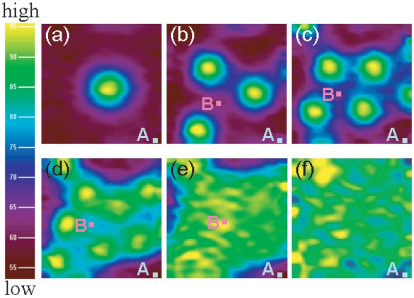
Vortices in Pb at 1.1 K [Essmann, Trauble '67].

Phenomenology (physics)

- Before being totally *lost*, superconductivity survives at the boundary (surface superconductivity) as predicted by Saint-James, de Gennes in 1963 and observed by Strongin *et al* in 1964.



Pb island of superconductor at 4.32 K [Ning et al '09].



Vortices and surface superconductivity on a Pb island [Ning et al '09].

Critical Magnetic Fields

As $\kappa \to \infty$, one can identify three bifurcation values (critical fields) for h_{ex}:

First critical field

If $h_{\mathrm{ex}} < H_{c1}(\kappa) \approx C_\Omega \log \kappa$, one has $|\Psi^{\mathrm{GL}}| > 0$, $\mathbf{A}^{\mathrm{GL}} \simeq 0$. Above H_{c1} isolated defects of Ψ^{GL}(vortices), where the superconductivity is lost, start to appear [Sandier, Serfaty '00].

Second critical field

At $H_{c2}(\kappa) \approx \kappa^2$, superconductivity disappears in the bulk and becomes a boundary phenomenon (surface superconductivity).

Third critical field

If $h_{\mathrm{ex}} > H_{c3}(\kappa) \approx \Theta_0^{-1}\kappa^2$ with $\Theta_0 < 1$ a universal constant (actually $\Theta_0^{-1} \simeq 1.6946$), the superconductivity is totally lost and $\Psi^{\mathrm{GL}} \equiv 0$ with $h = h_{\mathrm{ex}}$ is the unique minimizer [Fournais, Helffer '06].

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Second Critical Field

Second critical field (mathematical definition)

- *No* precise mathematical definition of H_{c2}, but only the idea of a vague transition from bulk to boundary behavior.
- $H_{c2}(\kappa) = \kappa^2$ (can be taken as a *definition*).
- Agmon estimates yield an *exponential decay* of Ψ^{GL} far from $\partial\Omega$, provided $h_{ex} > H_{c2}$.

Proposition (Agmon Estimates [Helffer, Morame '01])

If $h_{ex} = b\kappa^2$ *for some* $b > 1$ *and* κ *large enough,* $\exists A > 0$ *such that*

$$\int_\Omega \mathrm{d}\mathbf{r}\, e^{A\kappa\, \mathrm{dist}(\mathbf{r},\partial\Omega)} \left|\Psi^{GL}(\mathbf{r})\right|^2 = \mathcal{O}(\kappa^{-1}),$$

$$\left|\Psi^{GL}(\mathbf{r})\right| = \mathcal{O}(\kappa^{-\infty}), \qquad \text{for } \mathrm{dist}(\mathbf{r},\partial\Omega) \gg \kappa^{-1}.$$

Between H_{c2} and H_{c3}

Change of units

$$\mathcal{G}_{\kappa}^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i h_{\mathrm{ex}} \mathbf{A} \right) \Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2} \kappa^2 |\Psi|^4 + h_{\mathrm{ex}}^2 \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}$$

- We are interested in the regime $H_{c2} < h_{\mathrm{ex}} < H_{c3}$, i.e.,

$$\boxed{h_{\mathrm{ex}} = b\kappa^2}, \qquad \boxed{1 < b < \Theta_0^{-1}}$$

- $\mathbf{A}$ measured in units h_{ex}, i.e., $\mathbf{A} \to h_{\mathrm{ex}} \mathbf{A}$.
- Change of units to (ε, b) with $\varepsilon \ll 1$:

$$\boxed{\varepsilon = \left(b\kappa^2 \right)^{-1/2}}$$

- $E_{\varepsilon}^{\mathrm{GL}} = \min_{(\Psi,\mathbf{A}) \in H^1 \times H^1} \mathcal{E}_{\varepsilon}^{\mathrm{GL}}[\Psi, \mathbf{A}]$ and $(\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ any minimizing pair.

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$$\mathcal{E}^{\mathrm{GL}}_{\varepsilon}[\Psi, \mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \frac{1}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}$$

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Heuristics (between H_{c2} and H_{c3})

- Restriction to a neighborhood of $\partial\Omega$ & tubular coordinates there: $(s, \varepsilon t)$ tangential and normal coordinates (rescaled).
- Gauge choice [Fournais, Helffer '10]

$$\Psi^{\mathrm{GL}}(\mathbf{r}) = e^{i\phi_\varepsilon(s,t)}\psi(s,t), \qquad \mathbf{A}^{\mathrm{GL}}(\mathbf{r}) \longrightarrow \left(-t + \mathcal{O}(\varepsilon|\log\varepsilon|)\right)\boldsymbol{\tau}(s)$$

where $\boldsymbol{\tau}(s)$ is the unit vector tangential to $\partial\Omega$.

- In the regime $1 < b < \Theta_0^{-1}$, $|\Psi^{\mathrm{GL}}|$ is approx. constant in the tangential direction, i.e.,

$$\psi(s,t) \simeq f(t)\, e^{-i\frac{\alpha}{\varepsilon}s}$$

- The GL energy becomes up to $o(1)$ error terms

$$\frac{1}{\varepsilon}\int_0^{|\partial\Omega|} \mathrm{d}s \int_0^{C|\log\varepsilon|} \mathrm{d}t \left\{ |\partial_t\psi|^2 + |(\varepsilon\partial_s - it)\,\psi|^2 + \frac{1}{b}|\psi|^4 - \frac{2}{b}|\psi|^2 \right\}$$

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$$\frac{|\partial\Omega|}{\varepsilon}\int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}\left(2f^2 - f^4\right) \right\}$$

Effective 1D Functional

$$\mathcal{E}^{\rm 1D}_{0,\alpha}[f] := \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \tfrac{1}{2b} \left(2f^2 - f^4 \right) \right\}$$

- $\exists!$ minimizer $f_{0,\alpha} \geqslant 0$ with energy $E^{\rm 1D}_{0,\alpha}$.
- $f_{0,\alpha}$ is non-trivial iff $b^{-1} > \mu_0(\alpha)$, where $\mu_0(\alpha)$ is the ground state energy of $H_\alpha = -\partial_t^2 + (t+\alpha)^2$ in $L^2(\mathbb{R}^+, \mathrm{d}t)$ with Neumann b.c..
- $\Theta_0 = \min_{\alpha \in \mathbb{R}} \mu_0(\alpha)$.
- For any $1 \leqslant b < \Theta_0^{-1}$, $f_{0,\alpha}$ is non-trivial and $\exists$ a phase $\alpha_0 < 0$ minimizing $E^{\rm 1D}_{0,\alpha}$ over $\alpha \in \mathbb{R}$. The corresponding profile is $f_0 := f_{0,\alpha_0}$ and

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Past Results

COND-MATH

GL energy asymptotics

- [Pan '02] $E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_b}{\varepsilon} + o(\varepsilon^{-1})$ for $1 < b < \Theta_0^{-1}$ and $E_b < 0$.
- [Almog, Helffer '07; Fournais, Helffer, Persson '11]:

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(1)$$

for $1.25 \leqslant b < \Theta_0^{-1}$ by perturbative methods.

Order parameter asymptotics

- [Fournais, Helffer, Persson '11] If $1.25 \leqslant b < \Theta_0^{-1}$ the density $|\Psi^{\mathrm{GL}}|^2$ is close to f_0^2, i.e., ($\tau = \mathrm{dist}(\mathbf{r}, \partial\Omega)$, $\tau = \varepsilon t$)

$$\left\| |\Psi^{\mathrm{GL}}|^2 - |f_0(t)|^2 \right\|_{L^2(\Omega)} \ll \left\| f_0^2(t) \right\|_{L^2(\Omega)} \propto \varepsilon^{1/2}$$

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Open Problems

- Extend the GL energy asymptotics to the whole surface superconductivity regime, i.e., for $1 < b < \Theta_0^{-1}$.

Pan's conjecture

[Pan '02] The density $|\Psi^{\mathrm{GL}}|^2$ is close to $f_0^2(0)$ in $L^\infty(\partial\Omega)$, i.e.,

$$\left\| |\Psi^{\mathrm{GL}}|(\mathbf{r}) - f_0(0) \right\|_{L^\infty(\partial\Omega)} = o(1)$$

- A stronger version of Pan's conjecture is $\left\| |\Psi^{\mathrm{GL}}| - f_0 \right\|_{L^\infty(\mathcal{A}_\varepsilon)} = o(1)$ in any boundary layer $\mathcal{A}_\varepsilon$ containing the bulk of superconductivity.
- Since $f_0 > 0$, Pan's conjecture would imply no vortices in $\mathcal{A}_\varepsilon$.

Energy and Density Asymptotics

Theorem (GL asymptotics [MC, Rougerie '13])

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 \leqslant b < \Theta_0^{-1}$ *in the limit* $\varepsilon \to 0$, *one has*

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(1), \qquad \left\| |\Psi^{\mathrm{GL}}|^2 - f_0^2(t) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon)$$

- For $1 \leqslant b < \Theta_0^{-1}$, $f_0 > 0$ and $\left\| f_0^2(t) \right\|_{L^2(\mathcal{A}_\varepsilon)} \propto \varepsilon^{1/2}$.
- The above result is still *compatible* with vortices in $\mathcal{A}_\varepsilon$, as it is the error $\mathcal{O}(1)$ in the energy asymptotics.

Refined 1D Effective Model

- To prove a *stronger* estimate of Ψ^{GL}, one has to refine the error term $\mathcal{O}(1) \Longrightarrow$ the $\varepsilon-$dependent terms must be retained up to order ε.
- If the terms of order ε are retained, the 1D effective energy becomes (in the *disc* case, i.e., $k(s) \equiv k$ constant)

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Refined Energy Asymptotics

Theorem (Energy asymptotics [MC, Rougerie '14])

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain with boundary curvature $k(s)$. For any fixed $\boxed{1 < b < \Theta_0^{-1}}$ *in the limit $\varepsilon \to 0$, one has*

$$\boxed{E_\varepsilon^{\mathrm{GL}} = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} \mathrm{d}s\, E_\star(k(s)) + \mathcal{O}(\varepsilon |\log \varepsilon|^\infty)}$$

- Expanding further $E_\star(k(s))$, one gets [MC, Rougerie '15]

$$\boxed{E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega|\, E_0^{\mathrm{1D}}}{\varepsilon} - \mathcal{E}_{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}s\, k(s) + o(1)}$$

$$\mathcal{E}_{\mathrm{corr}}[f_0] = \int_0^\infty \mathrm{d}t\, t \left\{ \left(f_0'\right)^2 + \left(-\alpha_0(t+\alpha_0) - \tfrac{1}{b} + \tfrac{1}{2b} f_0^2\right) f_0^2 \right\}.$$

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Proof of Pan's Conjecture

Theorem (Density asymptotics [MC, Rougerie '14])

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$ *in the limit $\varepsilon \to 0$, one has*

$$\left\| |\Psi^{\mathrm{GL}}| - f_0\left(0\right) \right\|_{L^\infty(\partial\Omega)} = \mathcal{O}(\varepsilon^{1/4}|\log \varepsilon|)$$

- Stronger result $\left\| |\Psi^{\mathrm{GL}}| - f_0\left(\varepsilon t\right) \right\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} = o(1)$ in any suitable boundary layer $\mathcal{A}_{\mathrm{bl}} \subset \{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant C\varepsilon\sqrt{|\log \varepsilon|}\}$.
- Since $f_0(0) > 0$, the degree of Ψ^{GL} along $\partial\Omega$ is well defined and

$$\deg\left(\Psi^{\mathrm{GL}}, \partial\Omega\right) = \frac{|\Omega|}{\varepsilon^2} - \frac{\alpha_0}{\varepsilon} + \mathcal{O}(\varepsilon^{-3/4}|\log \varepsilon|^\infty)$$

Curvature Corrections

- To leading order $|\Psi^{\mathrm{GL}}| \simeq f_0(\mathrm{dist}(\mathbf{r}, \partial\Omega)/\varepsilon)$ and superconductivity is uniformly distributed in the boundary layer. Any lower order effect of the curvature?
- Recalling that $E_{\star}^{1\mathrm{D}}(k) = E_0^{1\mathrm{D}} + \varepsilon k \mathcal{E}_{\mathrm{corr}}[f_0] + \mathcal{O}(\varepsilon^{3/2}|\log\varepsilon|^{\infty})$.

Theorem (Curvature corrections [MC, Rougerie '15])

For any $1 < b < \Theta_0^{-1}$ *as* $\varepsilon \to 0$ *and for any "rectangular" set* D

$$\int_D \mathrm{d}\mathbf{r}\, |\Psi^{\mathrm{GL}}|^4 = \varepsilon C_1(b)|\partial\Omega \cap \partial D| + \varepsilon^2 C_2(b) \int_{\partial D \cap \partial\Omega} \mathrm{d}s\, k(s) + o(\varepsilon^2)$$

with $C_1(b) = -2bE_0^{1\mathrm{D}} \geqslant 0$ *and* $C_2(b) = 2b\mathcal{E}_{\mathrm{corr}}[f_0]$.

Sign of the Correction

- The sign of $C_2(b)$ determines whether superconductivity is attracted or repelled by points of large curvature.
- As $b \to \left(\Theta_0^{-1}\right)^-$, $f_0 \to 0$ and $\mathcal{E}_{\mathrm{corr}}[f_0] \to 0$ but $\mathcal{E}_{\mathrm{corr}}[f_0] > 0$.
- For any $1 < b < \Theta_0^{-1}$, only numerics [Bharathiganesh, MC, Rougerie in progress]:

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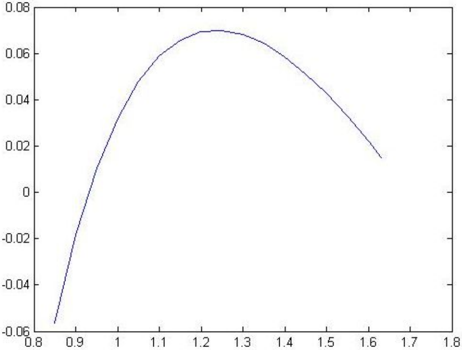
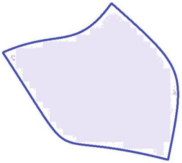


EFFECT OF CORNERS

- So far we have considered only domains with smooth boundary. What happens if the boundary is not smooth but contains corners?

- The presence of corners might affect the boundary distribution of superconductivity.



- The third critical field H_{c3} can also be shifted because of corners.
- From now on we will assume that the boundary of Ω is a Lipschitz boundary with finitely many corners.
- The normal $\mathbf{n}(s)$ as well as tubular coordinates and the curvature $k(s)$ are all defined only a.e., with jumps at corners.

H_{c3} with Corners

- If we decrease h_{ex} from huge values:

$$\mathcal{E}_{\varepsilon}^{GL}[\Psi] \simeq \int_{\Omega} d\mathbf{r} \left\{ \left| \left(\nabla + i \tfrac{\mathbf{F}}{\varepsilon^2} \right) \Psi \right|^2 - \tfrac{1}{b\varepsilon^2} |\Psi|^2 \right\} = \left\langle \Psi \left| H_{\varepsilon} - \tfrac{1}{b\varepsilon^2} \right| \Psi \right\rangle$$

- The ground state ψ_{ε} of H_{ε} is localized on a scale ε and blowing up a new effective problem emerges, i.e., the magnetic Laplacian on a sector with opening angle ϑ.
- The ground state energy $\gamma(\vartheta)/\varepsilon^2$ of H_{ε} is mostly unknown, but
  - $\gamma(\vartheta) \to 0$ as $\vartheta \to 0$ [Bonnaillie-Noël, Dauge '06];
  - $\gamma(\pi) = \Theta_0$ and $\gamma(\vartheta) < \Theta_0$ if $\vartheta \leqslant \frac{\pi}{2} + \delta$ [Bonnaillie-Noël '05];

Conjecture (($\star$) [Bonnaillie-Noël, Dauge '07])

Motivated by numerical computations, one expects that

- $\gamma(\vartheta)$ *is increasing in* ϑ*;*
- $\gamma(\vartheta) < \Theta_0$ *for* $\vartheta < \pi$*;*
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A New Critical Field H_{corner}?

H_{c3} with corners [Bonnaillie-Noël, Fournais '07]

Assuming ($\star$), in presence of corners of angles $\vartheta_j < \pi$

$$\boxed{H_{c3} = \lambda_\star^{-1}\varepsilon^{-2} + \mathcal{O}(1)}$$

with $\lambda_\star = \min_j \lambda(\vartheta_j)$.

- According to the conjecture ($\star$), $\lambda_\star < \Theta_0$ and therefore H_{c3} is larger in presence of corners.
- Before disappearing, superconductivity gets concentrated near the corner with smallest opening angle and Ψ^{GL} decays exponentially in the distance from that corner.
- What happens to surface superconductivity? is there another field $H_{c2} < H_{\mathrm{corner}} < H_{c3}$ marking the transition from boundary to corner concentration?

Surface Superconductivity

Theorem (GL asymptotics [MC, Giacomelli '16])

Let $\Omega \subset \mathbb{R}^2$ *be a bounded simply connected domain, whose boundary is a curvilinear polygon, then for any* $1 < b < \Theta_0^{-1}$, *as* $\varepsilon \to 0$,

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|^2)$$

$$\left\| \left|\Psi^{\mathrm{GL}}(\mathbf{r})\right|^2 - f_0^2\left(\tfrac{\mathrm{dist}(\mathbf{r},\partial\Omega)}{\varepsilon}\right) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon|\log\varepsilon|) \ll \left\| f_0^2\left(\tfrac{\mathrm{dist}(\mathbf{r},\partial\Omega)}{\varepsilon}\right) \right\|_{L^2(\Omega)}$$

- The presence of corners has no effect to leading order.
- H_{c2} is unaffected but, if $(\star)$ is correct, one would expect that, in presence of at least one acute angle,

$$H_{\mathrm{corner}} = \Theta_0^{-1}\varepsilon^{-2} + \mathcal{O}(1).$$

Effect of Corners

- The curvature $k(s)$ for a Lipschitz boundary is still bounded and integrable, and therefore we might expect the same energy asymptotics up to order 1:

$$E_{\varepsilon}^{\mathrm{GL}} \stackrel{?}{=} \frac{|\partial\Omega|\, E_0^{\mathrm{1D}}}{\varepsilon} - \mathcal{E}_{\mathrm{corr}}[f_0] \int_{\partial\Omega \ \mathrm{smooth}} \mathrm{d}s\, k(s) + o(1)$$

Theorem (GL refined asympt. [MC, Giacomelli '16])

Under the same hypothesis above, as $\varepsilon \to 0$,

$$\boxed{E_{\varepsilon}^{\mathrm{GL}} = \frac{|\partial\Omega|\, E_0^{\mathrm{1D}}}{\varepsilon} - \mathcal{E}_{\mathrm{corr}}[f_0] \int_0^{|\partial\Omega|} \mathrm{d}s\, k(s) + \sum_j E_{\mathrm{corners}}(\vartheta_j) + o(1)}$$

where ϑ_j are the opening angles of the corners and the integral of $k(s)$ is meant in Lebesgue sense.

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THEOREM (GL REFINED ASYMPT. [MC, GIACOMELLI '16])

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Corner Energy

- The corner energy is defined *implicitly* as

$$\boxed{E_{\mathrm{corners}}(\vartheta) := \liminf_{\ell \to \infty} \left(E^{\mathrm{GL}}_{\Gamma_\ell} - 2\ell E^{\mathrm{1D}}_0 \right)}$$

where Γ_ℓ is a sector of angle ϑ and side length ℓ, and

$$E^{\mathrm{GL}}_{\Gamma_\ell} := \inf_{\Psi \in \tilde{H}^1(\Gamma_\ell)} \int_{\Gamma_\ell} \mathrm{d}\mathbf{r} \, \left\{ \left| \left(\nabla + \tfrac{1}{2} i \mathbf{r}^\perp \right) \Psi \right|^2 - \tfrac{1}{2b} \left(2|\Psi|^2 - |\Psi|^4 \right) \right\}.$$

- Ψ satisfies mixed boundary conditions, i.e., the support of Ψ does not intersect the arc of Γ_ℓ.
- We can show that $E_{\mathrm{corners}}(\vartheta)$ is bounded above and below but we can not prove that the limit $\ell \to \infty$ exists, although we do expect it.
- The energy $2\ell E^{\mathrm{1D}}_0$ has to be subtracted because each edge of the sector gives a contribution ℓE^{1D}_0 to the energy, due to Neumann boundary conditions there.
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- The Gauss-Bonnet theorem suggests that the additional contributions due to the presence of corners sum up to reconstruct 2π and, since $\int_{\partial\Omega\,\mathrm{smooth}} \mathrm{d}s\, k(s) + \sum_j (\pi - \vartheta_j) = 2\pi$,

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We conjecture that for any $\vartheta \in [0, 2\pi)$

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Perspectives

- Derive the first order corrections to the GL order parameter in presence of corners [MC, Giacomelli in progress];
- Proof of Pan's conjecture in presence of corners $\Longrightarrow$ existence and asymptotic value of H_{corner}.

Thank you for the attention!

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- Suppose we decrease h_{ex} from huge values: above H_{c3} the normal state is the unique minimizer and $\mathrm{curl}\mathbf{A}^{GL} \equiv 1$, with $\mathcal{E}_\varepsilon^{GL} = 0$.
- When h_{ex} is lowered below H_{c3}, in first approximation $\mathrm{curl}\mathbf{A}^{GL} = 1$ and Ψ^{GL} is small, so that the energy to minimize is *linear*

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2. Upper bound (trivial): test $\mathcal{E}_{\mathrm{hp}}$ on $\psi_{\mathrm{trial}}(\sigma, t) \simeq f_0(t) e^{-i\alpha_0 \sigma}$.

- Lower bound:
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  4. Use of the potential function.
  5. Positivity of the cost function.

❶ MAGNETIC FIELD REPLACEMENT [FOURNAIS, HELFFER '10]

- Agmon estimates $\Longrightarrow$ restriction to the boundary layer (with $\varepsilon t = \mathrm{dist}(\mathbf{r}, \partial\Omega)$, $\sigma = \varepsilon s$) $\mathcal{A}_\varepsilon = \left\{ 0 \leqslant \sigma \leqslant \frac{|\partial\Omega|}{\varepsilon}, 0 \leqslant t \leqslant c_0 |\log \varepsilon| \right\}$.
- Gauge choice + elliptic estimates $\Longrightarrow$ up to error terms of order $\mathcal{O}(\varepsilon)$, E^{GL} is given by (with $\psi = e^{-i\phi_\varepsilon} \Psi^{\mathrm{GL}}$)

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- Gauge choice + elliptic estimates $\Longrightarrow$ up to error terms of order $\mathcal{O}(\varepsilon)$, E^{GL} is given by (with $\psi = e^{-i\phi_\varepsilon} \Psi^{\mathrm{GL}}$)

$$\boxed{\mathcal{E}_{\mathrm{hp}}[\psi] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \left\{ \left| (\nabla - it\mathbf{e}_\sigma) \psi \right|^2 - \frac{1}{b} |\psi|^2 + \frac{1}{2b} |\psi|^4 \right\}}$$

1. Restriction to the boundary layer + magnetic field replacement.
2. Upper bound (trivial): test $\mathcal{E}_{\mathrm{hp}}$ on $\psi_{\mathrm{trial}}(\sigma, t) \simeq f_0(t) e^{-i\alpha_0 \sigma}$.

- Lower bound:
  3. Energy splitting.
  4. Use of the potential function.
  5. Positivity of the cost function.

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Energy Asymptotics: Lower Bound

$$\mathcal{E}_{\mathrm{hp}}[\psi] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \left\{ \left| (\nabla - it\mathbf{e}_\sigma)\,\psi \right|^2 - \frac{1}{b}|\psi|^2 + \frac{1}{2b}|\psi|^4 \right\}$$

❸ Energy splitting

- If $1 \leqslant b < \Theta_0^{-1}$, one can set $\psi(\sigma, t) = f_0(t) e^{-i\alpha_0\sigma} v(\sigma, t)$.
- Using the variational equation of f_0 and its boundary conditions

$$\boxed{\mathcal{E}_{\mathrm{hp}}[\psi] = \frac{|\partial\Omega|}{\varepsilon} E_0^{\mathrm{1D}} + \mathcal{E}[v]}$$

with $\mathbf{j}(v) = \frac{i}{2}\left(v\nabla v^* - v^*\nabla v\right)$ the superconducting current and

$$\boxed{\mathcal{E}[v] = \int \mathrm{d}\sigma \mathrm{d}t\, f_0^2 \left\{ |\nabla v|^2 - 2(t + \alpha_0)\mathbf{e}_\sigma \cdot \mathbf{j} + \frac{1}{2b} f_0^2 \left(1 - |v|^2\right)^2 \right\}}$$

- It remain to bound $\mathcal{E}[v]$ and we will eventually show that $\mathcal{E}[v] \geqslant 0$.

Energy Asymptotics: Lower Bound

COMB-MATH

$$\mathcal{E}[v] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \, f_0^2 \left\{ |\nabla v|^2 - 2(t+\alpha_0)\mathbf{e}_\sigma \cdot \mathbf{j} + \frac{1}{2b} f_0^2 \left(1 - |v|^2\right)^2 \right\}$$

❹ Use of the potential function

- The field $2(t+\alpha_0)f_0^2\mathbf{e}_\sigma$ is divergence free so that one can find F such that $\nabla^\perp F = 2(t+\alpha_0)f_0^2\mathbf{e}_\sigma$, e.g., the potential function

$$F_0(t) = 2\int_0^t \mathrm{d}\eta \, (\eta+\alpha_0) f_0^2(\eta).$$

- $F_0(0) = F_0(+\infty) = 0$ (by optimality of α_0), $F_0'(0) < 0$ and F_0 has a unique extreme point $\Longrightarrow F_0 \leqslant 0$.
- Stokes formula yields

$$\mathcal{E}[v] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \left\{ f_0^2(t) \, |\nabla v|^2 + F_0(t)\mu + \frac{1}{2b} f_0^4(t) \left(1 - |v|^2\right)^2 \right\}$$

with $\mu = \mathrm{curl}(\mathbf{j})$ the vorticity measure, satisfying $|\mu| \leqslant |\nabla v|^2$.

Energy Asymptotics: Lower Bound

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- Stokes formula yields

$$\mathcal{E}[v] \geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \; \left\{ f_0^2(t) \, |\nabla v|^2 + F_0(t) |\mu| \right\}$$

with $\mu = \mathrm{curl}(\mathbf{j})$ the vorticity measure, satisfying $|\mu| \leqslant |\nabla v|^2$.

$$\mathcal{E}[v] = \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \, f_0^2 \left\{ |\nabla v|^2 - 2(t+\alpha_0)\mathbf{e}_\sigma \cdot \mathbf{j} + \frac{1}{2b} f_0^2 \left(1 - |v|^2\right)^2 \right\}$$

❹ USE OF THE POTENTIAL FUNCTION

- The field $2(t+\alpha_0) f_0^2 \mathbf{e}_\sigma$ is divergence free so that one can find F such that $\nabla^\perp F = 2(t+\alpha_0) f_0^2 \mathbf{e}_\sigma$, e.g., the potential function

$$F_0(t) = 2 \int_0^t \mathrm{d}\eta \, (\eta + \alpha_0) f_0^2(\eta).$$

- $F_0(0) = F_0(+\infty) = 0$ (by optimality of α_0), $F_0'(0) < 0$ and F_0 has a unique extreme point $\Longrightarrow F_0 \leqslant 0$.
- Stokes formula yields

$$\mathcal{E}[v] \geqslant \int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}\sigma \mathrm{d}t \, \left(f_0^2(t) + F_0(t)\right) |\nabla v|^2$$

with $\mu = \mathrm{curl}(\mathbf{j})$ the vorticity measure, satisfying $|\mu| \leqslant |\nabla v|^2$.

❺ Positivity of the cost function

- We define the vortex cost function as

$$K_0(t) = f_0^2(t) + F_0(t)$$

- If $1 \leqslant b < \Theta_0^{-1}$, $K_0(t) \geqslant 0$, for any $t \in \mathbb{R}^+$, which allows to conclude that $\mathcal{E}[u] \geqslant 0$ and the lower bound is proven.
- Optimality condition + variational equation for f_0 imply a remarkable identity for $F_0(t)$ yielding

$$K_0(t) = \left(1 - \tfrac{1}{b}\right) f_0^2(t) + (t + \alpha_0)^2 f_0^2(t) + \tfrac{1}{2b} f_0^4(t) - {f_0'}^2(t)$$

- $K_0(0) > 0$ and $K_0(+\infty) = 0 \Longrightarrow$ if $K < 0$ somewhere $\exists t_0 > 0$ *global minimum* for K_0 and $K_0'(t_0) = 0$. Since $K_0' = 2 f_0 f_0' + 2(t + \alpha_0) f_0^2$ one has $f_0'(t_0) = -(t_0 + \alpha_0) f_0(t_0)$ and

$$K_0(t_0) = \left(1 - \tfrac{1}{b}\right) f_0^2(t_0) + \tfrac{1}{2b} f_0^4(t_0) \geqslant 0$$

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❺ POSITIVITY OF THE COST FUNCTION

- We define the vortex cost function as

$$K_0(t) = f_0^2(t) + F_0(t)$$

- If $1 \leqslant b < \Theta_0^{-1}$, $K_0(t) \geqslant 0$, for any $t \in \mathbb{R}^+$, which allows to conclude that $\mathcal{E}[u] \geqslant 0$ and the lower bound is proven.
- Optimality condition + variational equation for f_0 imply a remarkable identity for $F_0(t)$ yielding

$$K_0(t) = \left(1 - \tfrac{1}{b}\right) f_0^2(t) + (t + \alpha_0)^2 f_0^2(t) + \tfrac{1}{2b} f_0^4(t) - {f_0'}^2(t)$$

- $K_0(0) > 0$ and $K_0(+\infty) = 0 \Longrightarrow$ if $K < 0$ somewhere $\exists t_0 > 0$ *global minimum* for K_0 and $K_0'(t_0) = 0$. Since $K_0' = 2 f_0 f_0' + 2\,(t + \alpha_0)\, f_0^2$ one has $f_0'(t_0) = -(t_0 + \alpha_0) f_0(t_0)$ and

$$K_0(t_0) = \left(1 - \tfrac{1}{b}\right) f_0^2(t_0) + \tfrac{1}{2b} f_0^4(t_0) \geqslant 0$$