Automorphic Forms, Mock Modular Forms and String Theory (17w5097)

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Overview

Automorphic forms in string theory

Over the last years there have been the beginnings of cross-interactions and cross-fertilizations between the theory of automorphic representations and string theory. The well-understood connections are mainly restricted to cases that in string theory go under the name of BPS-protected couplings with at least 8 residual supersymmetries. Such couplings arise for example in the lowest orders in a low-energy expansion of certain string theory scattering amplitudes. These can be represented by terms in an effective low-energy action that is of the schematic form

$$S_{\text{eff}} = \int_M d\mu \left(R + \mathcal{E}_{(0,0)} R^4 + \mathcal{E}_{(1,0)} \nabla^4 R^4 + \mathcal{E}_{(0,1)} \nabla^6 R^4 + \ldots \right) \quad (1)$$

where R, R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ denote certain invariants built from curvature tensors (or their covariant derivatives) on space-time M. The expansion of the effective action is ordered by the number of derivatives (any curvature tensor has two derivatives) and since derivatives translate into momentum p after Fourier transform, and hence energy by Einstein's relation $E = p^2$, this is an expansion with later terms becoming more important when more energy is involved in an interaction.

The coefficient functions $\mathcal{E}_{(p,q)}$ appearing in the effective action (1) are constrained to be automorphic functions as functions of the so-called string theory moduli. For type II string theory on toroidal backgrounds the moduli live on Riemannian symmetric spaces based on split real groups and the automorphy groups are the corresponding Chevalley groups. Supersymmetry (a.k.a. BPS-protection) requires the coefficient functions $\mathcal{E}_{(p,q)}$ to satisfy differential equations and these differential equations can be translated into ideals of the universal envelopping algebra. This is the link to automorphic representations that are also characterised by being annihilated by ideals in the universal envelopping algebra.

This link was reviewed in the first introductory talk of the conference by Michael Green and further elaborated on in the talk by Guillaume Bossard. For example, the first coefficient function $\mathcal{E}_{(0,0)}$ is associated with the minimal representation while the function $\mathcal{E}_{(1,0)}$ is associated with the so-called next-to-minimal representation.

The cross-fertilization now arises from the fact that (i) having identified the right automorphic representations allows the use of representation theory to deduce constraints on the associated physics that is typically revealed by considering the Fourier expansion of the automorphic function. Moreover, (ii) the analysis of the effective action (1) also gives important impetus for mathematical research since the ideal that is determined by the coefficient function $\mathcal{E}_{(0,1)}$ of the $\nabla^6 R^4$ term is not of any type normally considered in the theory of automorphic representations. In mathematical terms, its maximal orbit in the wave-front set is not a single nilpotent orbit. This is a phenomenon that is not expected to arise in the theory of automorphic functions and can be traced back to $\mathcal{E}_{(0,1)}$ not being finite under the action of the center of the universal envelopping algebra. In physical terms, this new feature is due to the fact the BPS-protection of this term is reduced to only four supersymmetries for which many new features arise (like Kontsevich–Soibelman wall-crossing).

Both aspects (i) and (ii) have been discussed in talks at the workshop and also in many lively discussions outside the lecture hall. In particular, the generalisation of the standard definition of automorphic representations that is suggested by (ii) was the subject of numerous discussions. Approaches discussed for understanding the associated automorphic forms were

- Spectral methods
- Poincaré series
- Explicit constrained lattice sum constructions

The automorphic forms appearing (1) are associated with space-time dualities and, in the instances described above, are related to non-perturbative effects in the string coupling. In a different vein, one can also study string theory in perturbation theory. This is the theory of computing certain integrals over the moduli spaces of (punctured) Riemann surfaces that represent the world-sheet of the string. The integrands of these integrals are determined by conformal field theory correlation functions and become complicated quickly as the genus and the number of punctures of the Riemann surface increases. Again, one is often restricted to a low-energy expansion.

One case of particular interest is when the world-sheet of the string is torus, i.e. of genus one. The moduli space of all such tori is $SL(2,\mathbb{Z})\backslash\mathcal{H}$ where $\mathcal{H}$ is the Poincaré upper half-plane describing the ratio of the two periods of the torus. The integrand must be a doubly periodic function on the torus, in other words an automorphic function of $SL(2,\mathbb{Z})$. As mentioned above, this automorphic function is determined by a field theory correlator that is in general not known in closed form. Performing, however, a low-energy expansion one can (often) represent the integrand a given order in this expansion in terms of a graph on the torus world-sheet where the punctures are connected by lines. The associated integrands have been dubbed modular graph functions and have recently a lot of attention as discussed in the talks of Michael Green and Eric D'Hoker.

Modular graph functions on the torus are interesting since they contain interesting elliptic multiple zeta values when one considers certain degeneration limits of the torus. (Elliptic) multiple zeta values represent an active current research field in number theory, in particular their single-valued projection at the one-loop level. The modular graph functions that contain them satisfy interesting systems of differential equations that can sometimes be used to extract relations between modular graph functions and hence between the multiple zeta values they contain. A full understanding of this is an open problem and was one of the topics discussed at the conference.

Automorphic representations

Mathematically, the functions $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$, associated with R^4- and $\nabla^4 R^4$-couplings, are attached to certain special automorphic representations which have unusually small functional dimension. Such small automorphic representations are usually representations of metaplectic groups. These small representations proved important in number theory as well as physics, where for example integral representations of L-functions, from Shimura's 1975 construction of the symmetric square L-function through constructions such as recent work of Cai-Friedberg-Ginzburg-Kaplan. The process of unifying linear algebraic groups and their covers leads to helpful new insights, often tending in the direction of quantum groups. An example is the representation of Fourier coefficients of Eisenstein series as "Tokuyama formulas" that are sums over crystal bases. Very recently work of Brubaker, Buciumas, Bump and Friedberg and others has revealed further unexpected relations between quantum groups and automorphic forms on metaplectic groups. In particular, the Fourier coefficients of metaplectic Eisenstein series are multiple Dirichlet series whose local factors can be interpreted as partition functions of supersymmetric solvable lattice models. Functional equations of such Fourier coefficients, which were studied by Kazhdan and Patterson and by Chinta and Gunnells can be interpreted as R-matrices of quantum groups. These ideas have equal origins in number theory and physics.

Approach of the workshop

Developments in recent years have made it clear that automorphic representations provide a crucial ingredient in our understanding of non-perturbative aspects of string amplitudes. With the advent of Umbral moonshine we have also seen new cross-fertilisations between string theory, conformal field theory, finite groups and mock modular forms. These developments provided ample motivation for having a new channel of communication between researchers in both communities, and the workshop succeeded in doing precisely this.

One of the main goals of the workshop was to provide an opportunity for interaction between mathematicians and physicists working on different aspects of automorphic forms, mock modular forms and string theory. The talks were selected in such a way as to give maximum exposition to current research topics from these fields and the lively discussions during the talks and in the breaks gave us the impression that our concept worked. The feedback we received from individual participants confirmed this impression and we think that the goal of creating new research ideas and projects at this exciting interface between mathematics and physics was certainly met.

The workshop brought together many researchers interested in similar questions and the talks and discussions at the workshop revealed many common questions that need to be researched further. There are some new collaborations that were initiated at the workshop as well as some ongoing collaborations that were propelled further as a consequence of the workshop. We felt the workshop was very successful and the academic concept "works" such that we plan to continue similar events in the future. A follow-up program will take place at the Simons Center during the spring of 2019 and we are aware of number of mutual visits between participants to further advance on research questions identified during the workshop.

The workshop was organized into 4 overall themes, and each theme was introduced during a 1-hour lecture and followed by a selection of shorter research seminars spread throughout the week.

Themes and talk descriptions

Automorphic forms in string theory

The keynote speakers were Michael B. Green and Eric D'Hoker, who each gave a one hour lecture. The first lecture reviewed two main instances of where automorphic forms and automorphic functions occur in string theory, namely the low-energy expansion of graviton amplitudes in a U-duality invariant framework with automorphic forms on exceptional duality groups and, secondly, the integrands of string scattering amplitudes at a fixed loop order. This latter subject has mainly been developed at one-loop order corresponding to toroidal world-sheets and automorphy group $SL(2,\mathbb{Z})$. This was reviewed in the talk by D'Hoker and he also explained recent work on the extension two loops and $Sp(4,\mathbb{Z})$ and even higher loops, work that was considerably extended at the BIRS workshop (see outcome section).

Guillaume Bossard discussed a new approach to the automorphic couplings appearing in the low-energy expansion. This approach is based on considering so-called exceptional field theory amplitudes at loop order where certain classes of supersymmetric excitations (BPS states) circulate in the loop. The resulting sum over such states leads to automorphic forms expressed through constrained lattice sums, different from the usual coset sum description of Langlands. This approach also provides expressions for automorphic functions that violate the usual $\mathcal{Z}$-finiteness condition. Discussions between physicists and mathematicians following this talk led to the proof of a conjecture of the rewriting of orbit sums.

Jeff Harvey presented his recent work with G. Moore on subtleties in the way Weyl reflections act in toroidal compactifications. The hitherto unnoted feature is that the combination of the usual geometric action of the reflection group with an action on the states leads to a cover (eight-fold in cases) of the standard Coxeter group.

Automorphic representations

The keynote speaker was Gordan Savin who gave an overview of the field of small automorphic representations. Global uniqueness of the minimal representation was recently shown by Kobayashi and Savin. He also explained the ubiquity of Weissman's construction of the Fourier-Jacobi functor for the minimal representation. As already mentioned the importance of the minimal representation in string theory has become increasingly clear in recent years, thereby creating a surge of interest in the field also from physics.

Marie-France Vignéras reported on here recent results on the existence of supercuspidal p-adic representations for G_2, D_n and E_n, a project that was initiated as a direct result of her preparation for the workshop.

Dihua Jiang presented recent progress on the theory of endoscopic classification. He reviewed his conjecture with Zhang on the large cuspidal spectrum, called the *Global Large Cuspidal Packet Conjecture* (GLCP-conjecture), and its ramifications.

Birgit Speh discussed her work on symmetry breaking of infinite-dimensional representations which deals with real representations of a group G and their restriction to a subgroup $H \subset G$, in particular problems with estimating multiplicities of representations of H in the case of infinite-dimensional representations. The main focus was for the case $O(n,1) \subset O(n+1,1)$ and unitary representations.

Henry Kim reported on his joint work with Yamauchi on Ikeda-type lifts. Using the Ikeda-type lift they construct a higher level cusp form on $E_{7,3}$ from any Hecke cusp form whose corresponding automorphic representation has no supercuspidal local components.

Ben Brubaker gave a general overview on the connection between quantum groups and Whittaker functions of metaplectic Eisenstein series. Daniel Bump then proceeded to report on recent joint work with Brubaker, Buciumas and Gray. Whittaker functions on the n-fold metaplectic cover of $GL(r)$ over a nonarchimedean local field were studied by Kazhdan and Patterson, who computed the scattering matrix of the intertwining integrals on the Whittaker models. It was shown in 2016 by Brubaker, Buciumas and Bump that this scattering matrix coincides with the R-matrix of a quantum group, a twist of quantum affine $U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n))$, where q is the residue cardinality. Moreover, they showed that the spherical Whittaker functions could be expressed as partition functions of solvable lattice models, whose internal structure is related to the quantum affine Lie superalgebra $U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n|1))$. In recent work, Brubaker, Buciumas, Bump and Gray proved that a second solvable lattice model has the same partition function using Yang-Baxter equations.

Automorphic forms on Kac-Moody groups

The keynote speaker was Manish Patnaik who reviewed the basic notions of Eisenstein series on loop groups, emphasising the different versions (positive, negative, ...) that exist in the Kac–Moody case whereas they coincide in the usual finite-dimensional Lie group case. He proved convergence results for these Eisenstein series in the number field and function field case, building on previous work by Garland. Another interesting aspect was the discussion of the affine version of the Weil representation appearing in the theta functions and a corresponding Siegel–Weil theorem.

Kyu-Hwan Lee presented his work with Carbone and Liu on Eisenstein series on more general Kac–Moody groups, focussing on the results they have obtained for the rank 2 hyperbolic case. These results include convergence, functional relations, constant term formulas and holomorphy results for cusp forms.

Alexander Braverman presented his work with David Kazhdan on the affine Tamagawa number formula. The goal of this work is to provide an affine generalization of Langlands calculation of the volume of the fundamental domain $G(\mathbb{Z})\backslash G(\mathbb{R}))$ for G a split semi-simple simply-connected Lie group. Their proposal is to do this by computing the cohomology of the moduli space of G-bundles for G a loop group over a function field.

Indefinite theta series, moonshine and black holes

The keynote speaker was Stephen Kudla, who reported on the recent flurry of activity on theta series associated with indefinite quadratic forms. There is an extensive theory of theta series for positive definite quadratic forms, and connections with representation theory through theta correspondences and minimal representations. Sparked by results of Zwegers in relation with mock modular forms, and generalizations in string theory due to Alexandrov, Banerjee, Manschot and Pioline, a general theory of indefinite theta series is now emerging. Kudla explained his recent results with Jens Funke which shows that indefinite theta series can be constructed using the method of Kudla-Millson.

Roberto Volpato reported on his work with Natalie Paquette and Max Zimet on counting 1/4-BPS states in $\mathcal{N} = 4$ string theory. The generating function of such BPS-states is famously given by the reciprocal of the Igusa cusp form Φ_{10} for $Sp(4;\mathbb{Z})$. Their results combine consistency conditions on Φ_{10} with wall-crossing in string theory and has potential relevance for our understanding of Mathieu moonshine.

Katrin Wendland explained recent progress in constructing a module for Mathieu moonshine using K3 sigma models. She reported on her results on the refined elliptic genus, and the work of Song on the cohomology of chiral de Rham complex. The conclusion is that the generic space of states of K3-theories is modelled by this cohomology, and appears to agree with what is expected from a Mathieu moonshine module.

Martin Raum reported on recent work with Michael Mertens on the skew-Maass lift. This is a generalization of the classical Maass lift to skew-holomorphic Jacobi forms. Part of the motivation for this work comes from recent developments in umbral moonshine.

Christoph Keller reported on a series of works jointly with several people, including Belin, Maloney, Mühlmann, Castro, Gomes, Kachru and Paquette. This concerns an in-depth analysis of the spectrum of permutation orbifolds for genus 1 and 2 conformal field theories. They investigate relations with black holes, Siegel modular forms and emergent spacetime.

Miscellaneous

Thomas Creutzig explained his recent work with Davide Gaiotto, which gives a new class of vertex algebras related by S-duality in GL-twisted $\mathcal{N} = 4$ super-Yang-Mills. These vertex algebras arise in the intersection of pairs of 3d boundary conditions in the 4d theory.

Siddhartha Sahi presented his work on multivariate hypergeometric functions for tube domains. This generalizes the classical theory of hypergeometric functions to functions with matrix arguments. This theory is related to the theory of Jack- and Macdonald-polynomials.

Howard Garland presented work in progress on the cohomology of arithmetic groups and relations to an arithmetic generalization of Riemannian curvature. He offered speculations on applications to automorphic forms and higher-derivative corrections in string theory.

Hermann Nicolai explained a surprising connection between conjectured Kac–Moody symmetries of $\mathcal{N} = 8$ supegravity in $D = 4$ space-time dimensions and the standard model. Breaking supersymmetry (by an unknown mechanism) it is possible to assign quantum numbers under an $SU(3) \times U(1)$ group to the $\mathcal{N} = 8$ fermions that exactly match the corresponding quantum numbers in the standard model. This subgroup requires the extension to the conjectured Kac–Moody symmetry $K(E_{10})$ and cannot be accommodated in the standard R-symmetry $SU(8)$.

Outcome of the Meeting

Publications linked to the workshop

Modular graph functions can also be defined in principle for higher genus Riemann surfaces, with additional subtleties as was explained in the talk by Eric D'Hoker. Some of the ideas he sketched regarding higher string invariants were clarified in subsequent discussions and the results have now been written up and published online as [1]. Another preprint that was an immediate consequence of the workshop was [2] by Marie-France Vignéras who analysed certain supercuspidal p-adic representations.

Comments from workshop participants

"It is my view that mathematics has two very different main sources: physics and number theory. In automorphic forms, the historical development has been from number theory. Yet automorphic forms also appear in physics. So the Langlands program is a meeting ground for ideas coming from two different directions. I felt that the workshop was extremely helpful in getting number theorists and physicists together for a common cause. The format of mainly 40 minute talks with a few survey lectures worked very well."

- Daniel Bump

"Many thanks again for organizing the recent BIRS workshop, and for inviting me. I enjoyed my stay very much, and I learned a lot.

I had discussions with Birgit Speh and with Martin Raum which may or may not lead to new collaborations.... New ideas are abundant after the worskhop; particularly the discussions with Roberto Volpato and with Jeff Harvey have helped."

- Katrin Wendland

"Thank you organizing an inspiring conference. I found it useful to learn which examples and groups are useful. I intend to look into automorphic forms for $SO(4,4)$ *and* $SO(4,20)$*."*

- Birgit Speh

"I enjoyed the workshop a great deal. It was very exciting to hear from so many distinguished physicists about the way that automorphic forms appear in the study of string scattering amplitudes, and to hear from my colleagues in automorphic forms about their latest progress. And the connections to VOAs and mock modular forms were very striking; this is an area I would appreciate the chance to learn more about.

The lectures were quite interesting and generally of very high quality. Let me add that I have already watched both Friday lectures on-line, as I was not able to stay for them.

For specifics, Guillaume Bossard asked me some interesting questions, and I hope this will be the start of more discussions. I spent time talking to Martin Raum, with whom I am collaborating, and this was a helpful conversation in terms of moving our project forward. I spoke with Ben Brubaker and once again set in motion next steps on further joint work. I had a helpful conversation with Gordan Savin which followed up a prior conversation from our last meeting in Boston.

I had some helpful discussions with Daniel Persson, Axel Kleinschmidt and Dan Bump concerning future steps in facilitating collaborations between mathematicians and physicists."

- Solomon Friedberg

" I really enjoyed the workshop and would like to thank all of you for your hard work.

It was fantastic to have one half of the talks on physics and the other half on mathematics so that we could see the common grounds and interactions of the two disciplines. The organization of talks was very coherent, focusing on one or two themes each day. I could learn a lot each day.

I hope that there may be more workshops and conferences like this one in the future."

- Kuy-Hwan Lee

" First of all: congratulations on putting together a superb workshop in Banff. There was a good balance between time for talks and free time to discuss, especially since everyone was physically staying in the same place, sharing lunch and dinner.

I did not initiate any new collaborations, but got a lot done with present collaborators Boris Pioline and Michael Green on one collaboration, as well as with Piotr Tourkine on another collaboration.

*Having also attended last year's workshop at Simons in Stony Brook, I think this year's workshop had a more balanced set of topics and interests than last year's. The rather informal but efficient organization was perfect for me. Let me add that I view the Banff Center as an *ideal* place for such meetings, so any time you care to invite me again, I will be there !"*

- Eric D'Hoker

"My recent visit to BIRS was a great help in my research. I learned things from the talks of course and met several people whose work I knew but whom I had never met personally including Stephen Kudla and Marie-France Vigneras. Most importantly, I had conversations with Martin Westorholt-Raum and with Thomas Creutzig that had a direct bearing on a problem I was struggling with in a current research project. My conversations with them led me to a new approach and made me familiar with parts of the literature that I would have had difficulty finding on my own. This is exactly the sort of thing one hopes might happen at an interdisciplinary meeting like this. Thank you so much for providing such a pleasant atmosphere for these productive interactions."

- Jeff Harvey

References

[1] E. D'Hoker, M. B. Green and B. Pioline, Higher genus modular graph functions, string invariants, and their exact asymptotics, arXiv:1712.06135 [hep-th].

[2] M.-F. Vignéras, Existence of supersingular representations of p-adic reductive groups, arXiv:1712.10142 [math.NT].