Probabilistic Tools for Lattice Path Enumeration

Kilian Raschel

Lattice walks at the Interface of Algebra, Analysis and Combinatorics
September 18, 2017
BIRS

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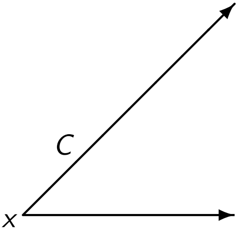
Random processes (RW & BM) in cones

First exit time from a cone C



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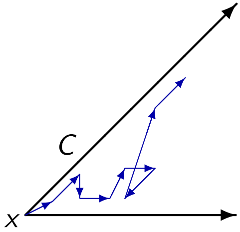

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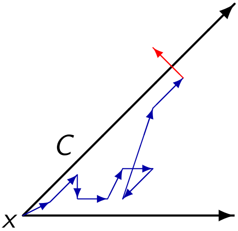

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- $\tau_C = \inf\{n \in \mathbf{N} \ : S(n) \notin C\}$ (S RW)
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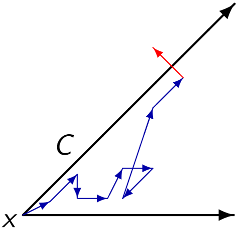

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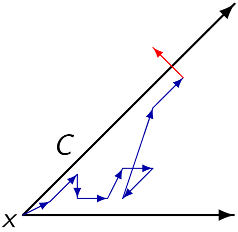

Persistence probabilities $\rightsquigarrow$ total number of walks

- $\mathbf{P}_x[\tau_C > n] \sim \kappa \cdot V(x) \cdot \rho^n \cdot n^{-\alpha}$

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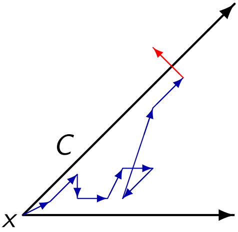

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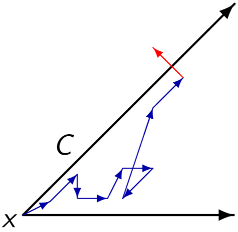

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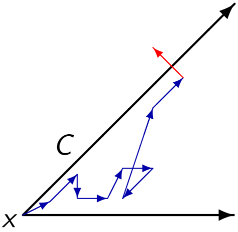

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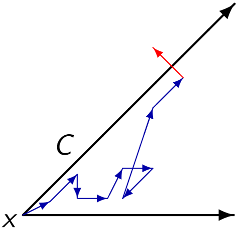

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- $\mathbf{P}_x[\tau_C > n, S(n) = y] \sim \kappa \cdot V(x, y) \cdot \rho^n \cdot n^{-\alpha}$

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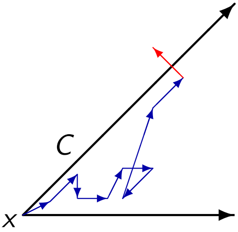

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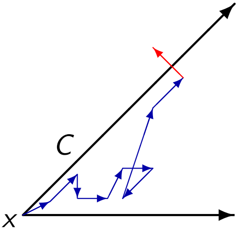

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Aim of the talk: understanding the critical exponents α

Definition of random walks & example of Dyck paths

Random walk on $\mathbf{Z}^d$

- A *random walk* $\{S(n)\}_{n\geqslant 0}$ is

$$S(n) = x + X(1) + \cdots + X(n),$$

where the $X(i)$ are i.i.d.

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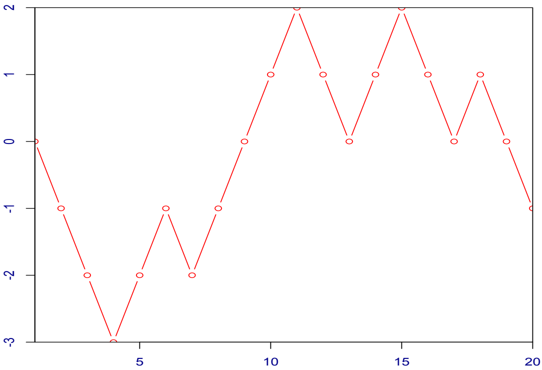

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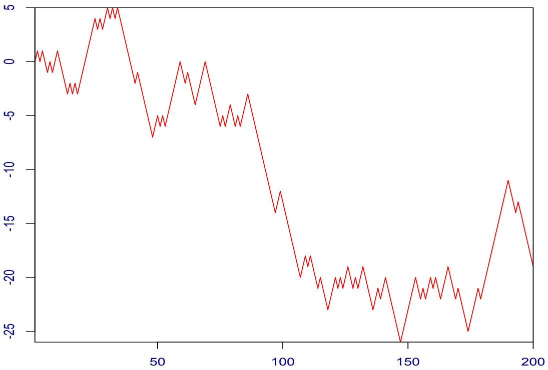

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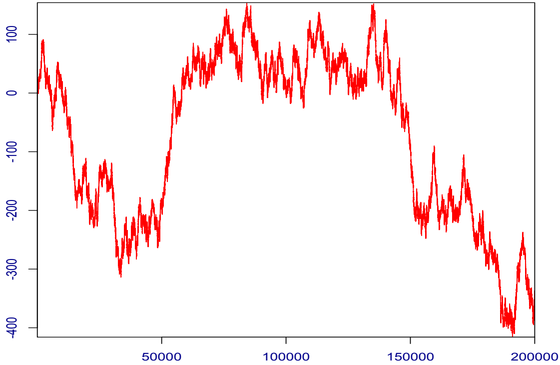
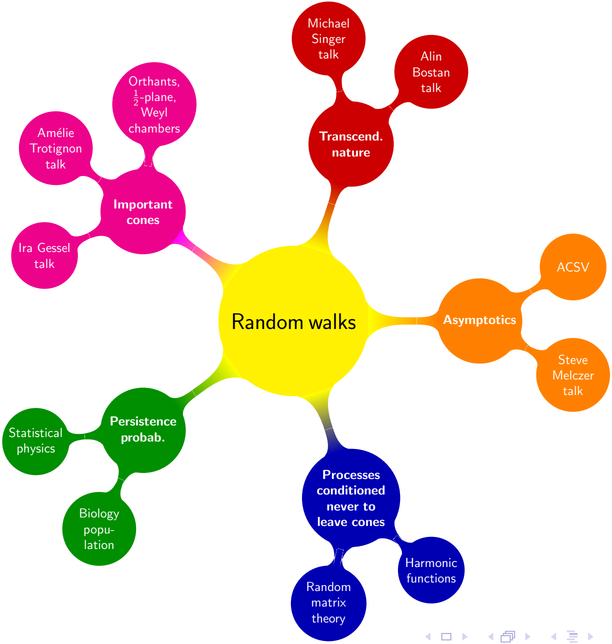
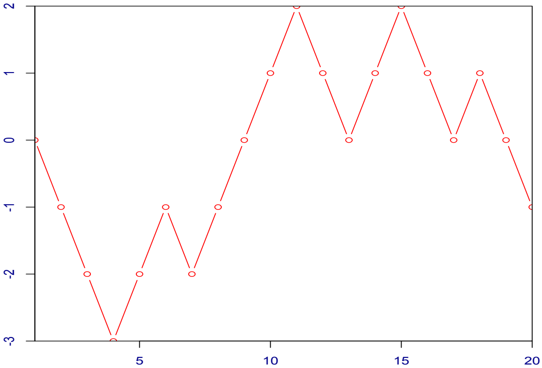


The ubiquity of random walks



Non-constrained walk with $\mathfrak{S} = \{-1, +1\}$



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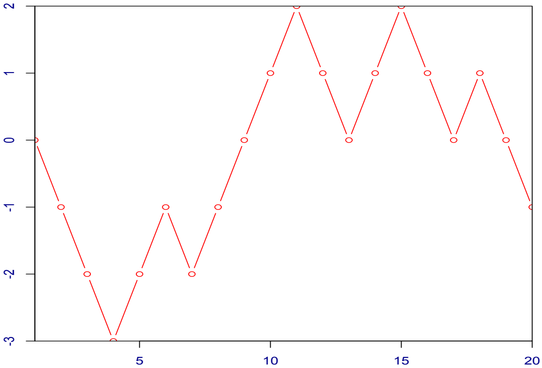

▷ $\#\{x \xrightarrow{n} \mathbf{Z}\} = 2^n$ Walk ⇝ Exponent 0

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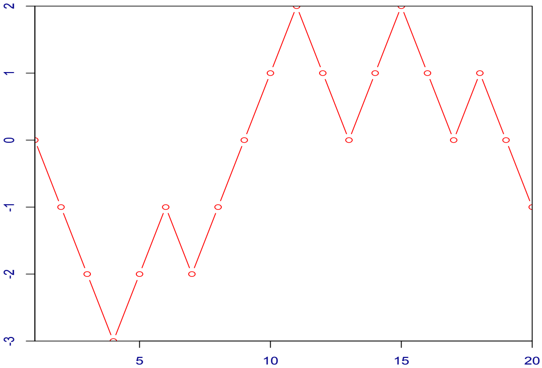

- $\#\{x \xrightarrow{n} \mathbf{Z}\} = 2^n$ — Walk $\rightsquigarrow$ Exponent 0
- $\#\{x \xrightarrow{n} y\} = \binom{n}{\frac{n+(y-x)}{2}} \sim \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$ — Bridge $\rightsquigarrow$ Exponent $\frac{1}{2}$

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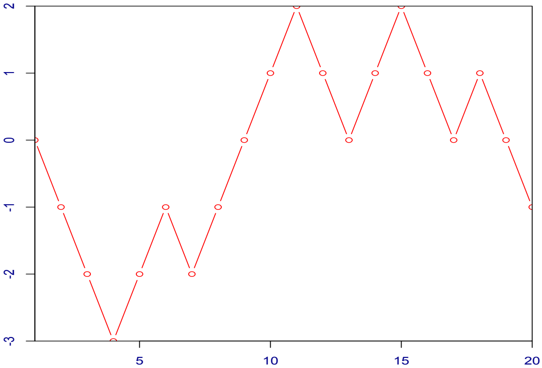

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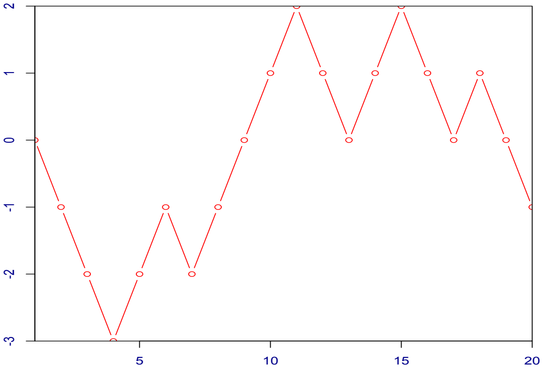

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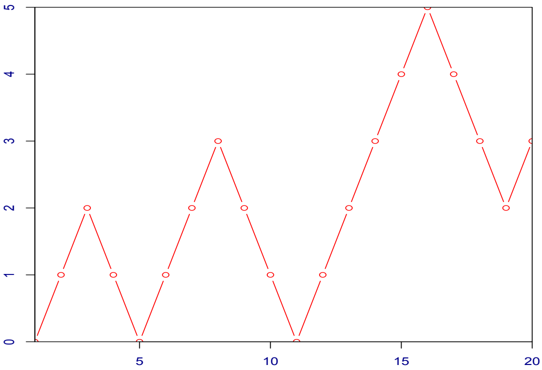
- $\sum \frac{1}{\sqrt{n}} = \infty$: *recurrence* of the simple random walk in $\mathbf{Z}$
- Constant $\sqrt{\frac{2}{\pi}}$ *independent of x & y* in the asymptotics

Constrained walk with $\mathfrak{S} = \{-1, +1\}$ (Dyck paths)



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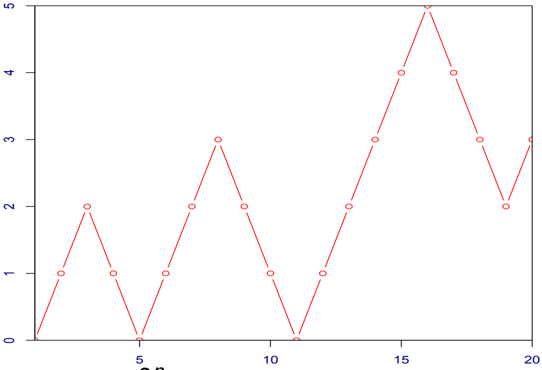

▷ $\#_{\mathbf{N}}\{x \xrightarrow{n} \mathbf{N}\} \sim \dfrac{2^n}{\sqrt{n}}$ Meanders $\rightsquigarrow$ Exponent $\dfrac{1}{2}$

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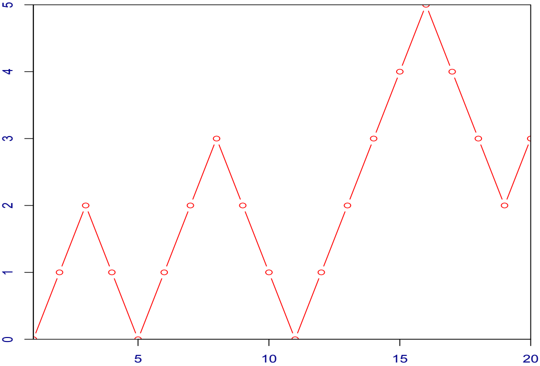

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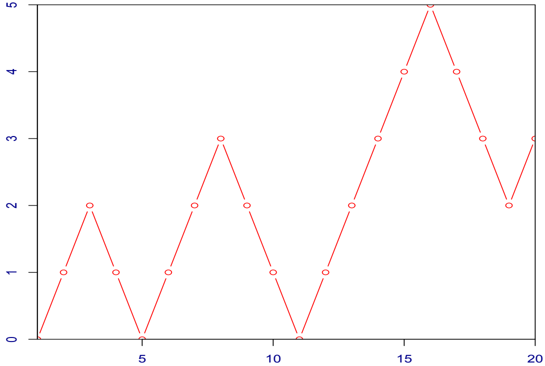

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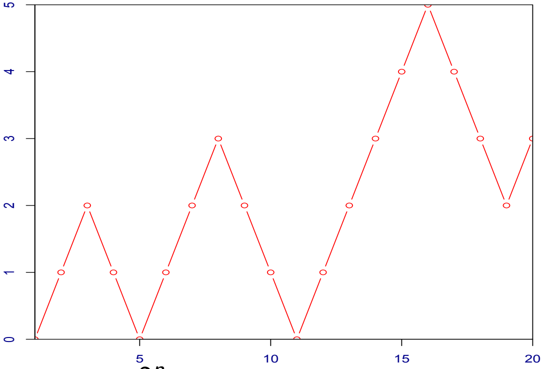

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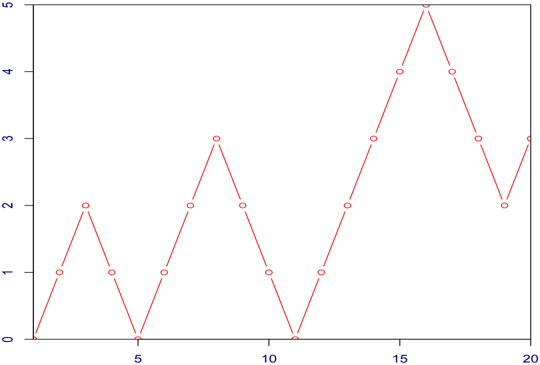

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- ▷ *Wiener-Hopf* techniques in probability theory
- ▷ See ✎ Bousquet-Mélou & Petkovšek '00; Banderier & Flajolet '02; Banderier & Wallner '17

Beyond the classical exponents 0, $\frac{1}{2}$ & $\frac{3}{2}$

Weighted models in dimension 1

Drift $\sum_{s\in\mathfrak{S}} s$ governs the exponents, which are still 0, $\frac{1}{2}$ & $\frac{3}{2}$

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The simple walk in two-dimensional wedges

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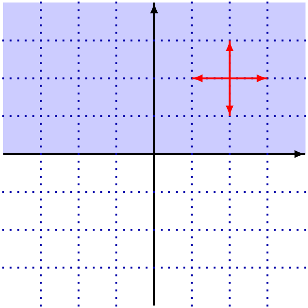

- Half-plane:
  one-dimensional case
- Dyck paths
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{1}{2}$
- Excursions:
  $\rightsquigarrow$ Exponent $2 = \frac{3}{2} + \frac{1}{2}$

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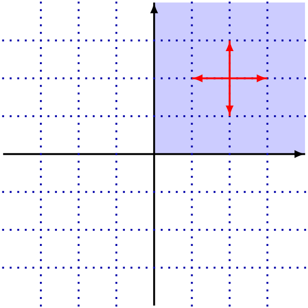

- Quarter plane: product of two one-dimensional cases
- Reflection principle
- Total number of walks:
  $\rightsquigarrow$ Exponent $1 = \frac{1}{2} + \frac{1}{2}$
- Excursions:
  $\rightsquigarrow$ Exponent $3 = \frac{3}{2} + \frac{3}{2}$

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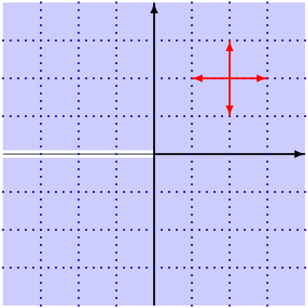

- Slit plane:
  ✎ Bousquet-Mélou & Schaeffer '00
- Highly non-convex cone
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{1}{4}$
- Excursions:
  $\rightsquigarrow$ Exponent $\frac{3}{2}$

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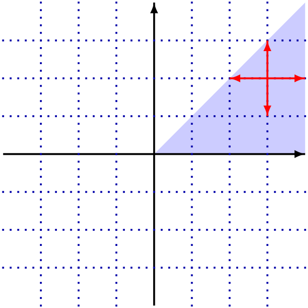

- 45°: ✎ Gouyou-Beauchamps '86
- See
  ✎ Bousquet-Mélou & Mishna '10
- Total number of walks:
  ⇝ Exponent 2
- Excursions:
  ⇝ Exponent 5

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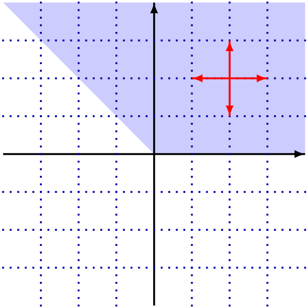

- 135°: Gessel
- See ✎ Kauers, Koutschan & Zeilberger '09; etc.
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{2}{3}$
- Excursions:
  $\rightsquigarrow$ Exponent $\frac{7}{3}$

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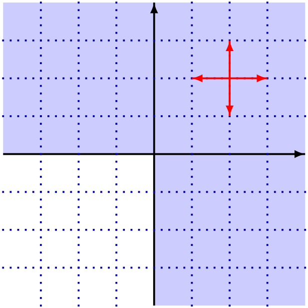

- Walks avoiding a quadrant
- See ✎ Bousquet-Mélou '15; Mustapha '15; Trotignon *et al.* '17+
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{1}{3}$
- Excursions:
  $\rightsquigarrow$ Exponent $\frac{5}{3}$

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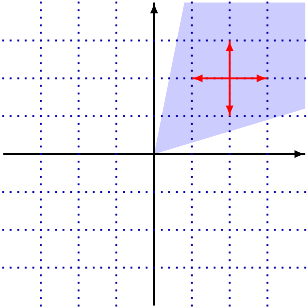

- Arbitrary angular sector θ
- See ✎ Varopoulos '99; Denisov & Wachtel '15

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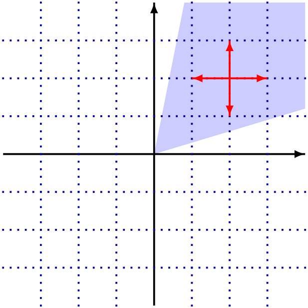
The simple walk in two-dimensional wedges



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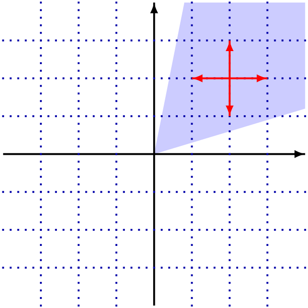

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Conclusion: 1D case not enough

- ▷ Dramatic change of behavior: every exponent is possible!
- ▷ *Non-D-finite* behaviors (first observed by Varopoulos '99)

Brownian motion on R

Law of large numbers

$$\frac{X(1) + \cdots + X(n)}{n^{1}} \xrightarrow{\text{a.s.}} \mathbf{E}[X(1)]$$

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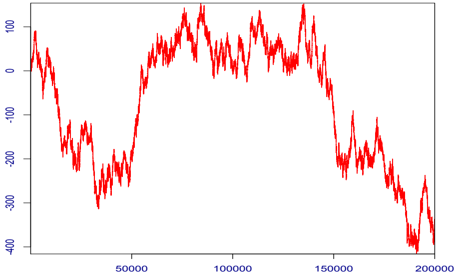

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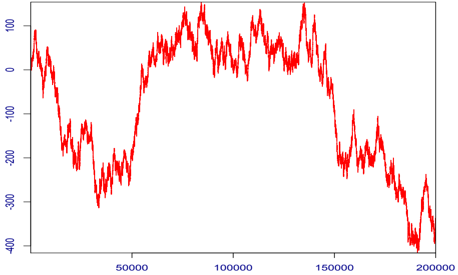

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- Cone C becomes $M\cdot C$

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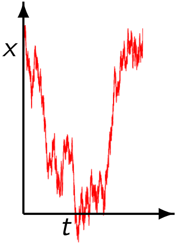
Remainder of this section: computing $\alpha\{\text{BM}\}$ (easier)

Two derivations of the BM persistence probability in R

Reflection principle



$$\mathbf{P}_x[T_{(0,\infty)} > t] = \mathbf{P}_0[\min_{0\leqslant u\leqslant t} B(u) > -x]$$
$$= \mathbf{P}_0[|B(t)| < x]$$
$$= \frac{2}{\sqrt{2\pi t}} \int_0^x e^{-\frac{y^2}{2t}} \mathrm{d}y$$

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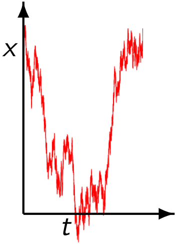

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Heat equation

Function $g(t;x) = \mathbf{P}_x[T_{(0,\infty)} > t]$ satisfies

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) g(t;x) = 0, & \forall x \in (0,\infty),\ \forall t \in (0,\infty) \\ g(0;x) = 1, & \forall x \in (0,\infty) \\ g(t;0) = 0, & \forall t \in (0,\infty) \end{cases}$$

Dimension d: explicit expression for $\mathbf{P}_x[T_C > t]$

Heat equation

✎ Doob '55

For essentially *any domain* C in *any dimension* d, $\mathbf{P}_x[T_C > t]$ & $p^C(t; x, y)$ ($\mathbf{P}_x[T_C > t] = \int_C p^C(t; x, y)\mathrm{d}y$) satisfy *heat equations*

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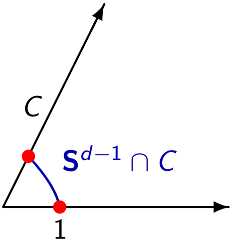
Dirichlet eigenvalues problem

✎ Chavel '84



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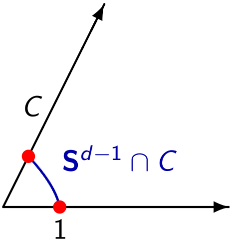

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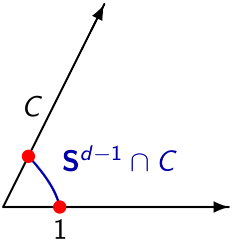

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Asymptotics of the non-exit probability

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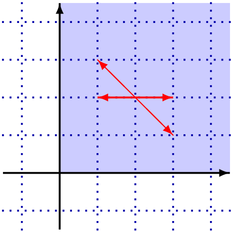
Exercise

Recover the exponent $\frac{\pi}{2\theta}$ of the persistence probability for a simple random walk in a two-dimensional wedge of opening angle θ

Example #1: Gouyou-Beauchamps model

In the quarter plane



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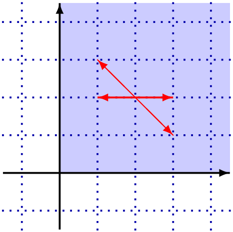

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$$\mathbf{E}[\text{GB}] = (1,0)+(1,-1)+(-1,0)+(-1,1)$$
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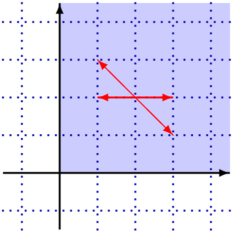

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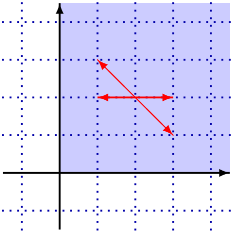

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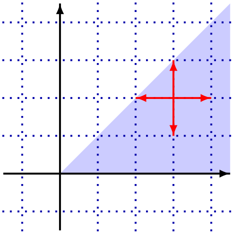

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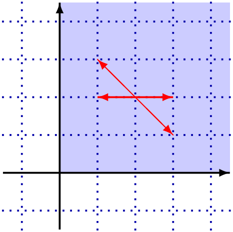

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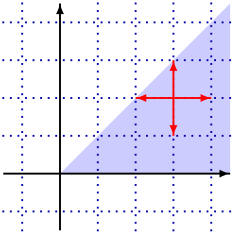
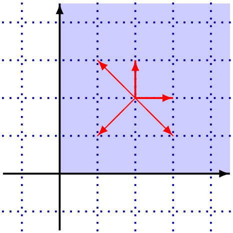


- Wedge of angle $\theta = \frac{\pi}{4}$
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{\pi}{2\theta} = 2$
- Excursions:
  $\rightsquigarrow$ Exponent $\frac{\pi}{\theta} + 1 = 5$

Example #2: quadrant walks

A scarecrow



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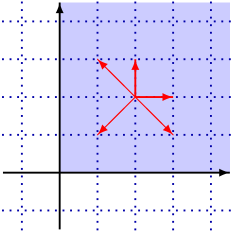

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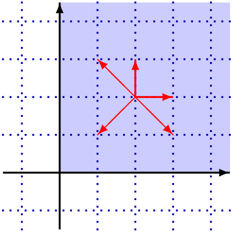

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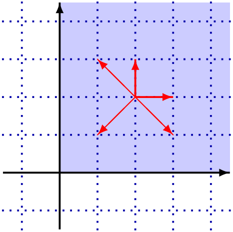

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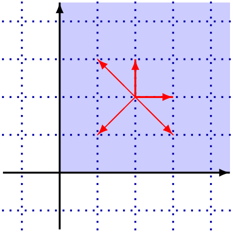

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In dimension 2 (excursions only)

✎ Bostan, R. & Salvy '14

- ▷ Systematic computation of $\alpha = \arccos\{\text{algebraic number}\}$

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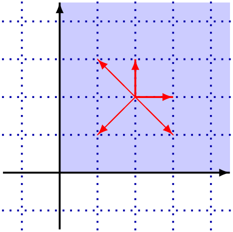

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In dimension 2 (excursions only)

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- Systematic computation of $\alpha = \arccos\{\text{algebraic number}\}$
- Walks with small steps:
  - $\frac{\pi}{\alpha} \in \mathbf{Q}$ iff
  - *generating function* of the excursions is D-finite iff
  - the *group of the model* is finite

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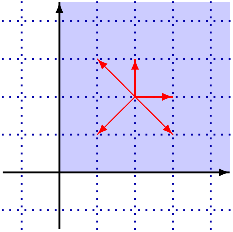


- $\mathbf{E} = (0,0)$ & $\mathbf{V} = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \neq \text{id}$
- $\theta = \arccos\left(-\frac{1}{4}\right) \implies \alpha = \frac{\pi}{\theta} + 1 \notin \mathbf{Q}$
- $\sum_{n=0}^{\infty} \#_{\mathbf{N}^2}\{(0,0) \xrightarrow{n} (0,0)\} t^n$ non-D-finite

In dimension 2 (excursions only)

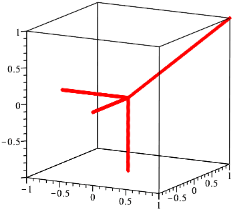
✎ Bostan, R. & Salvy '14

- Systematic computation of $\alpha = \arccos\{\text{algebraic number}\}$
- Walks with small steps:
  - $\frac{\pi}{\alpha} \in \mathbf{Q}$ iff
  - *generating function* of the excursions is D-finite iff
  - the *group of the model* is finite
- If $\sum_{s \in \mathfrak{S}} s \neq 0$, first perform a *Cramér transform*

Three-dimensional models

Example: Kreweras 3D

Model with jumps:



Three-dimensional models

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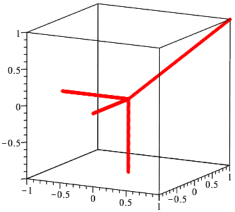

Exponent $\alpha = \frac{1}{2}\sqrt{\lambda_1 + \frac{1}{4}} - \frac{1}{4}$

Three-dimensional models

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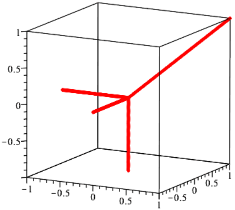

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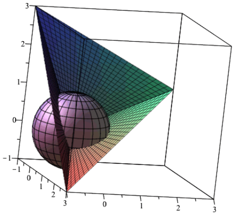

Three-dimensional models

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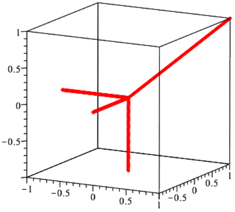

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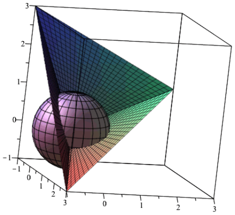

Value of λ_1? $\lambda_1 \in \mathbf{Q}$?

Three-dimensional models

Example: Kreweras 3D

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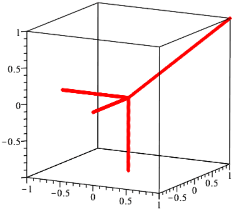

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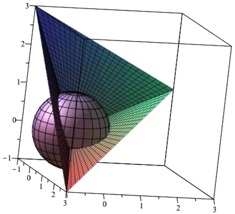


Value of λ_1? $\lambda_1 \in \mathbf{Q}$?

General theory (still to be done!)

- ▷ Classification & resolution of some finite group models

  ✎ Bostan, Bousquet-Mélou, Kauers & Melczer '16

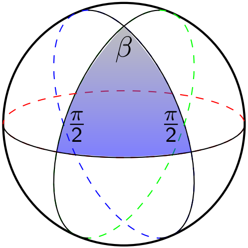
- ▷ Asymptotic simulation ✎ Bacher, Kauers & Yatchak '16; Guttmann '16

  $\rightsquigarrow$ Conjectured Kreweras exponent ≈ 3.3257569

- ▷ Equivalence finite group iff D-finite generating functions?

Eigenvalues of spherical triangles and 3D models

A (the?) soluble case



- Dirichlet problem

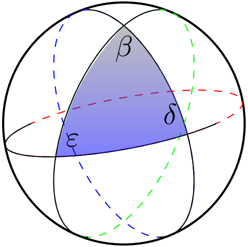
$$\begin{cases} \Delta_{\mathbf{S}^2} m = -\lambda m & \text{in } \mathbf{S}^2 \cap C \\ m = 0 & \text{in } \partial(\mathbf{S}^2 \cap C) \end{cases}$$

- Smallest eigenvalue: $\lambda_1 = (\frac{\pi}{\beta}+1)(\frac{\pi}{\beta}+2)$

✎ Walden '74

- SRW in 3D: $\beta = \frac{\pi}{2}$ and $\lambda_1 = 12$

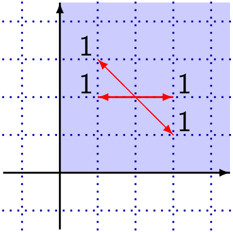
Generic case



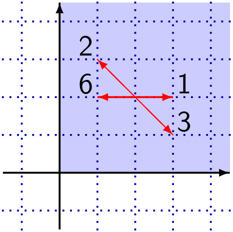
- No closed-form formula known
- Even for a flat triangle in $\mathbf{R}^2$, no closed-form formula for smallest eigenvalue...
- Is there a miracle for Kreweras?
$(\beta = \delta = \varepsilon = \frac{2\pi}{3})$
⇝ *Tetrahedral tiling of the sphere*

Central weightings and stability of the exponent

Critical exponents for weighted GB model



$\alpha = 5$



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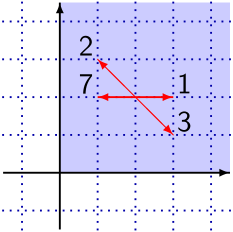
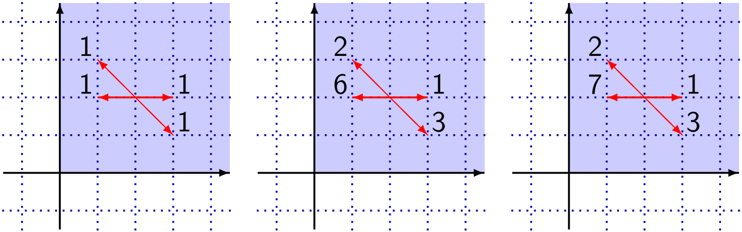


$$\alpha = \frac{\pi}{\arctan\left\{\left(\frac{7}{6}\right)^{1/4}\right\}} + 1$$

$$\approx 4.9042377$$

Central weightings and stability of the exponent

Critical exponents for weighted GB model



$\alpha = 5$ $\quad$ $\alpha = 5$ $\quad$ $\alpha = \frac{\pi}{\arctan\left\{\left(\frac{7}{6}\right)^{1/4}\right\}} + 1 \approx 4.9042377$

Central weightings

- *Replace the initial weight* 1 of jump (i,j) by $c \cdot a^i \cdot b^j$
- *Critical exponent* for the excursions *unchanged*
- Second example above: $a = \frac{1}{\sqrt{6}}$, $b = \frac{1}{3}$, $c = \sqrt{6}$
- Third example is not a central weighting

Much more in Julien Courtiel's talk!

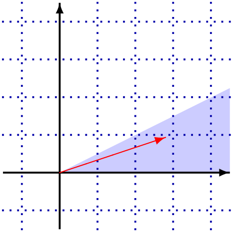
Non-universal exponents: six cases

Excursions: formula for α independent of the drift $\sum_{s\in\mathfrak{S}} s$

Non-universal exponents: six cases

Excursions: formula for α independent of the drift $\sum_{s\in\mathfrak{S}} s$

Case #1: interior drift



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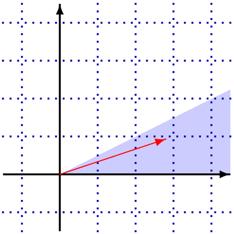

- Law of large numbers: $\mathbf{P}[\forall n, S(n) \in C] > 0$
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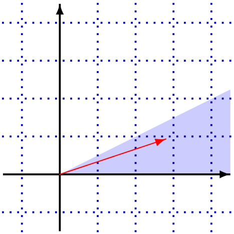

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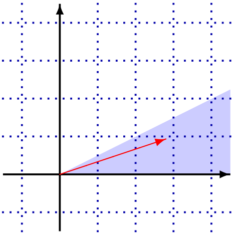

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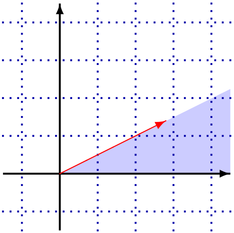

- Half-plane case
- Exponent $\alpha = \frac{1}{2}$

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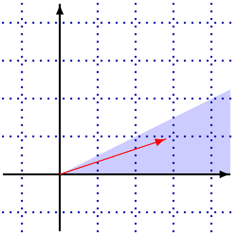

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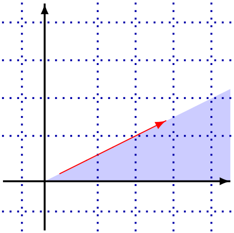

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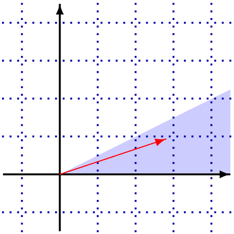

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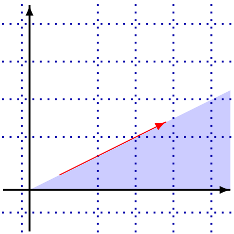

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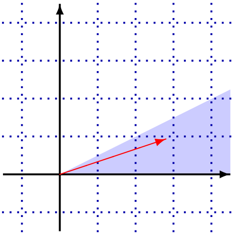

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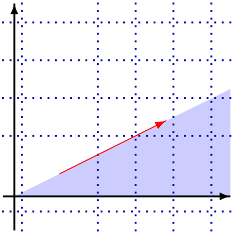

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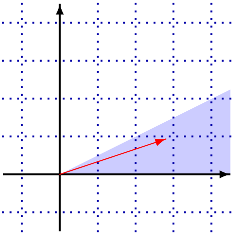

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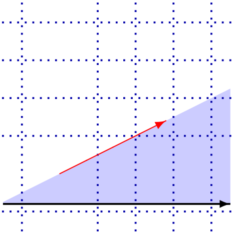

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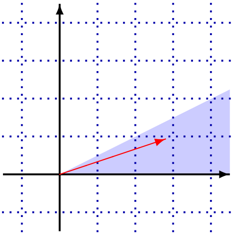

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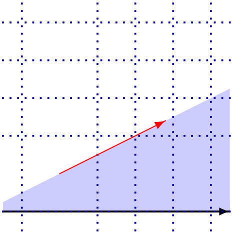

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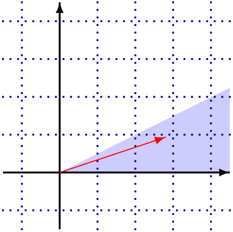

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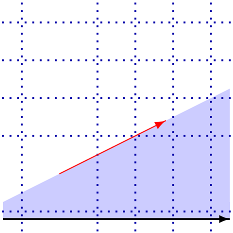

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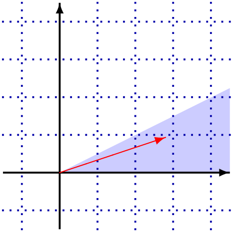

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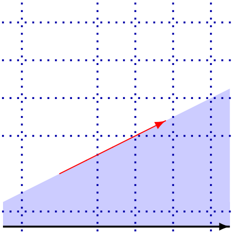

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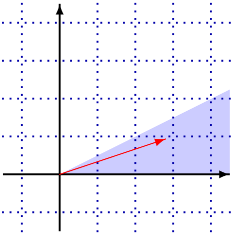

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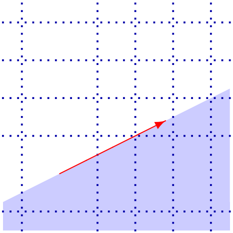

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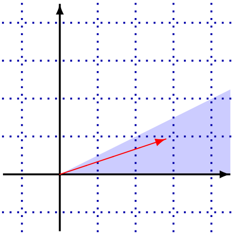

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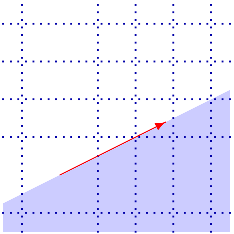
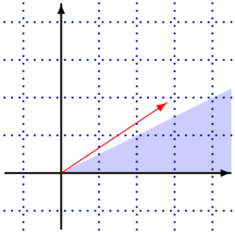


- Half-plane case
- Exponent $\alpha = \frac{1}{2}$
- *Cannot* be used as a filter to detect non-D-finiteness
- Exponent $\alpha = \frac{i}{2}$ for non-smooth boundary

Non-universal exponents: six cases

Case #3: directed drift



- Half-plane case
- Exponent $\alpha = \frac{3}{2}$
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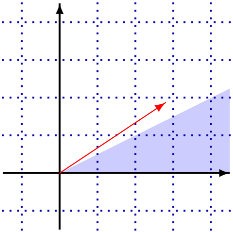
Non-universal exponents: six cases

Case #3: directed drift



- Half-plane case
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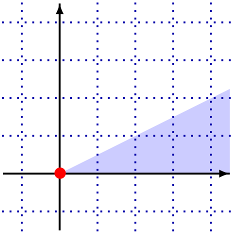
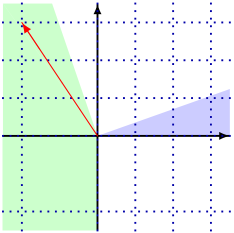
Case #4: zero drift



- See ✎ Varopoulos '99; Denisov & Wachtel '15
- Exponent $\alpha_1 = \frac{1}{2}\left(\sqrt{\lambda_1 + (\frac{d}{2} - 1)^2} - (\frac{d}{2} - 1)\right)$
- *Can* be used as a filter to detect non-D-finiteness

Non-universal exponents: six cases

Case #5: polar interior drift



- See ✎ Duraj '14
- Exponent $2\alpha_1 + 1$
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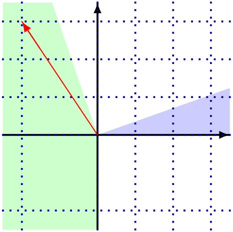
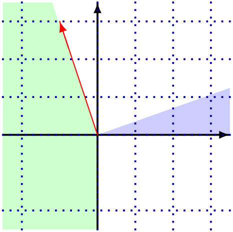
Non-universal exponents: six cases

Case #5: polar interior drift



- See ✎ Duraj '14
- Exponent $2\alpha_1 + 1$
- *Can* be used as a filter to detect non-D-finiteness

Case #6: polar boundary drift



- Exponent $\alpha_1 + 1$
- *Can* be used as a filter to detect non-D-finiteness

Non-universal exponents: six cases

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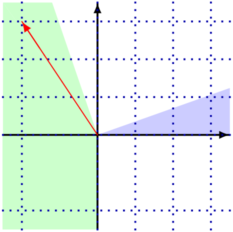

- See ✎ Duraj '14
- Exponent $2\alpha_1 + 1$
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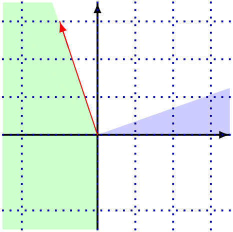
Case #6: polar boundary drift

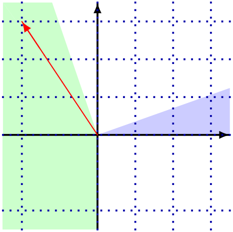


- Exponent $\alpha_1 + 1$
- *Can* be used as a filter to detect non-D-finiteness

Weighted GB model: with J. Courtiel, S. Melczer & M. Mishna

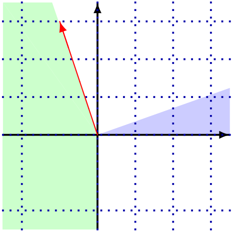
Non-universal exponents: six cases

Case #5: polar interior drift



- See ✎ Duraj '14
- Exponent $2\alpha_1 + 1$
- *Can* be used as a filter to detect non-D-finiteness

Case #6: polar boundary drift



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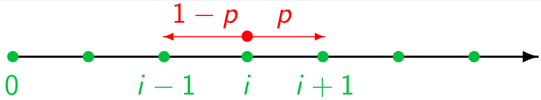
Six-exponents-result: joint with R. Garbit & S. Mustapha

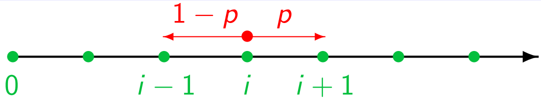
Introductory example & definition

Absorption probabilities for the SRW on **N**



Introductory example & definition

Absorption probabilities for the SRW on $\mathbf{N}$



Probability $a(i) := \mathbf{P}_i[\exists n \geqslant 0 : \text{SRW } S(n) = 0]$ satisfies

- ▷ $a(0) = 1 \rightsquigarrow$ *initial condition*
- ▷ $a(i) = p \cdot a(i+1) + (1-p) \cdot a(i-1) \rightsquigarrow$ *recurrence*

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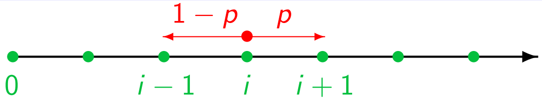

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Solution $a(i) = \begin{cases} 1 & \text{if } p \leqslant \frac{1}{2} \\ \left(\frac{1-p}{p}\right)^i & \text{if } p > \frac{1}{2} \end{cases}$

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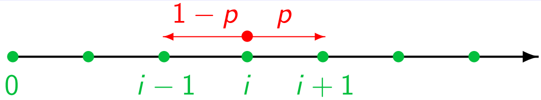

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Definition: f harmonic if $L[f](x) = 0$ for all x in a region $\subset \mathbf{Z}^d$

$$L[f](x) = \sum_{y \in N_x} p(y)\{f(x+y) - f(x)\},$$

with *set of neighbors* $N_x \subset \mathbf{Z}^d$ and *weights* $p = \{p(y)\}_{y \in \mathbf{Z}^d}$

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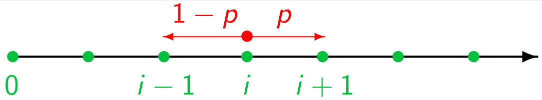

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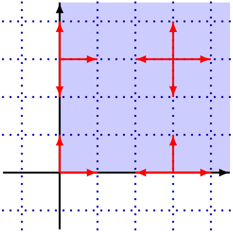
- ▷ Multivariate linear recurrences with constant coefficients

✎ Bousquet-Mélou & Petkovšek '00

Warning: lattice walk enum. vs. preharmonic functions

Multivariate recurrence relations in both cases



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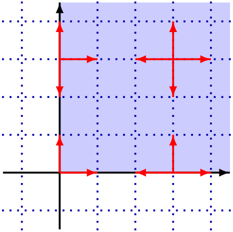

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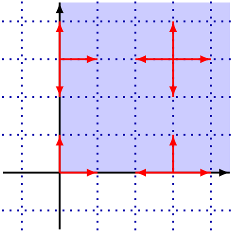

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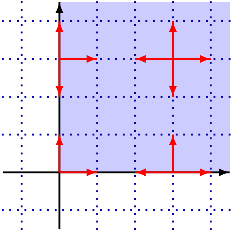

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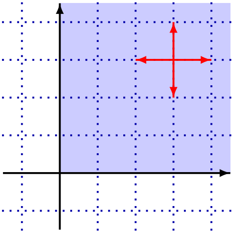
- Preharmonic functions $\rightsquigarrow$ counting numbers asymptotics

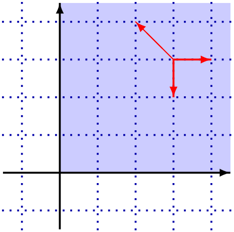
Two examples of rational harmonic functions

The simple walk



- Uniform weights $\frac{1}{4}$
- $f(i,j) = i \cdot j$
- *Unique preharmonic function* (up to multiplicative factors)
- Product form ✎ Picardello & Woess '92

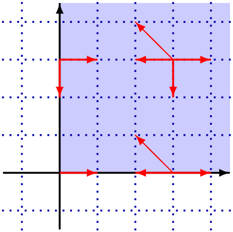
The Tandem walk



- Uniform weights $\frac{1}{3}$
- $f(i,j) = i \cdot j \cdot (i+j)$
- *Unique preharmonic function* (up to multiplicative factors) ✎ Biane '92

Asymptotics of some numbers of walks

Asymptotic statements



- *Total number of walks* starting at (k, ℓ):

$$q(n; k, \ell; \mathbf{N}^2) = \#_{\mathbf{N}^2}\{(k,\ell) \xrightarrow{n} \mathbf{N}^2\}$$
$$\sim f_1(k,\ell) \cdot \rho_1^n \cdot n^{\alpha_1}$$

✎ Not proved yet!

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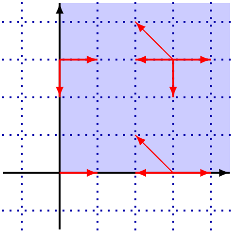

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✎ Denisov & Wachtel '15

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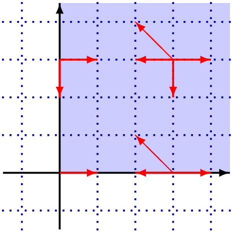

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▷ f_1 is ρ_1-harmonic & f_2 is ρ_2-harmonic

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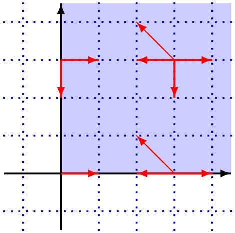

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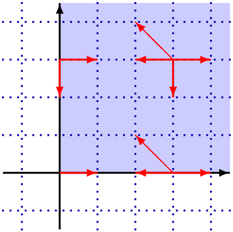

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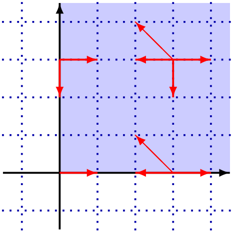

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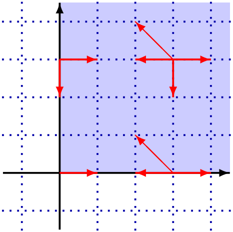
- Drift zero: unique harmonic function $\Longrightarrow$ f_1, f_2 and f_2'

Functional equation & Tutte's invariants

A functional equation reminiscent of the enumeration



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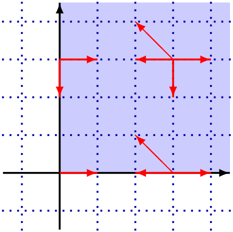

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Definition of Tutte's invariants

- Introduced to count q-colored triangulations & planar maps

  ✎ Tutte '73; Bernardi & Bousquet-Mélou '11
- Define X_0 & X_1 by $K'(X_0,y) = K'(X_1,y) = 0$
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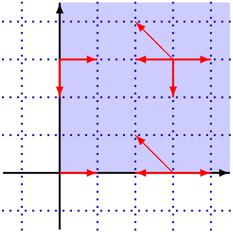

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The sections $K'(x,0)F(x,0)$ & $K'(0,y)F(0,y)$ are invariants

- Evaluate the functional equation at X_0 & X_1
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Functional equation & Tutte's invariants

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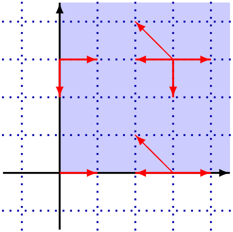

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Does this characterize the sections?

Example: the SRW

A product-form generating function

$$f(i,j) = i \cdot j \Longrightarrow \boxed{F(x,y) = \sum_{i,j\geqslant 1} i \cdot j \cdot x^{i-1}y^{j-1} = \frac{1}{(1-x)^2(1-y)^2}}$$

Kernel: $K'(x,y) = xy\{\frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4}$

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Verification of the functional equation

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Verification of the functional equation

$$\begin{aligned} K'(x,y)F(x,y) &= K'(x,0)F(x,0) + K'(0,y)F(0,y) - K'(0,0)F(0,0) \\ &= \frac{x}{4} \times \frac{1}{(1-x)^2} + \frac{y}{4} \times \frac{1}{(1-y)^2} - 0 \times 1 \end{aligned}$$

Example: the SRW

A product-form generating function

$$f(i,j) = i \cdot j \implies \boxed{F(x,y) = \sum_{i,j\geqslant 1} i \cdot j \cdot x^{i-1}y^{j-1} = \frac{1}{(1-x)^2(1-y)^2}}$$

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Tutte's invariants

- ▷ $I(X_0) = I(X_1) \overset{X_0X_1=1}{\implies} I(x) = I(\frac{1}{x}) \implies I$ function of $x + \frac{1}{x}$
- ▷ $\boxed{K'(x,0)F(x,0) = \frac{x}{4}\frac{1}{(1-x)^2} = \frac{1}{4}\frac{1}{x+\frac{1}{x}-2}}$ is an invariant

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Why *this* function of $x + \frac{1}{x}$?

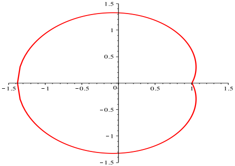
- Of order 1 in $x + \frac{1}{x} \rightsquigarrow$ *Minimality* (conformal mappings)
- $F(1,0) = \infty \rightsquigarrow$ *Liouville's theorem*

Tutte's invariants & conformal mappings

A general theorem

$K'(x,0)F(x,0) = w(x)$, *characterized by*

- ▷ Conformal mapping of a quartic
- ▷ $w(x) = w(\overline{x})$
- ▷ $w(x) = \frac{c+o(1)}{(1-x)^{\alpha-1}}$, $\alpha =$ crit. exponent
- ▷ Same for $K'(0,y)F(0,y)$



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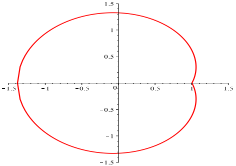

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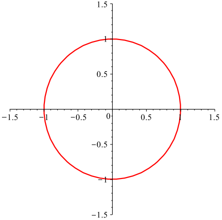
Going back to the SRW

$K'(x,0)F(x,0) = \frac{x}{4(1-x)^2}$, *characterized by*



- ▷ Conformal mapping of the unit disc
- ▷ $w(e^{i\theta}) = w(e^{-i\theta})$
- ▷ $w(1) = \infty$
- ▷ Same for $K'(0,y)F(0,y) = \frac{y}{4(1-y)^2}$

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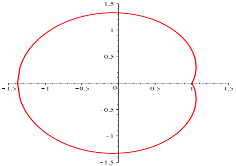

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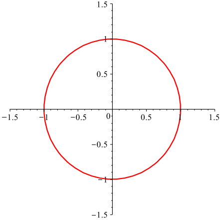
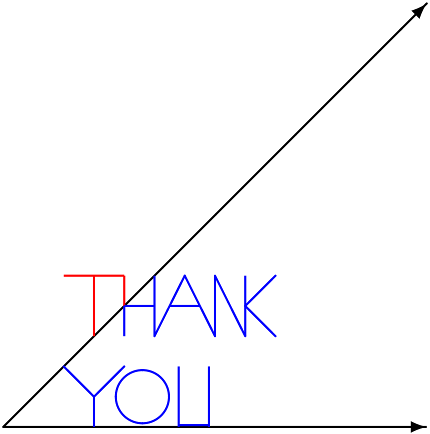


Question

How deep is this *connection conformal maps/harmonic functions*?



cnrs

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