OPEN PROBLEMS

1. Introduction

We would like to study weighted walks with small steps in the quarter plane $\mathbb{Z}_{\geq 0}^2$. More explicitly, we let $(d_{i,j})_{(i,j)\in\{0,\pm1\}^2}$ be a family of elements of $\mathbb{Q}\cap[0,1]$ such that $\sum_{i,j} d_{i,j}=1$ and we consider the walk in the quarter plane $\mathbb{Z}_{\geq 0}$ satisfying the following properties :

- it starts at $(0,0)$,
- it has steps in $\{\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow,\searrow,\downarrow,\swarrow\}$ – these steps will be identified with pairs $(i,j)\in\{0,\pm1\}^2\backslash\{(0,0)\}$,
- it goes to the direction $(i,j)\in\{0,\pm1\}^2\backslash\{(0,0)\}$ (resp. stays at the same position) with probability $d_{i,j}$ (resp. $d_{0,0}$).

The $d_{i,j}$ are called the weights of the walk. This walk is unweighted if $d_{0,0}=0$ and if the nonzero $d_{i,j}$ all have the same value.

For any $(i,j)\in\mathbb{Z}_{\geq 0}^2$ and any $k\in\mathbb{Z}_{\geq 0}$, we let $q_{i,j,k}$ be the probability for the walk to be reach the position (i,j) from the initial position $(0,0)$ after k steps.We introduce the corresponding trivariate generating series*

$$Q(x,y,t):=\sum_{i,j,k\geq 0} q_{i,j,k}x^iy^jt^k.$$

It is easily seen that, for any $k\in\mathbb{Z}_{\geq 0}$, $|q_{i,j,k}|\leq\sum_{i,j\geq 0}|q_{i,j,k}|\leq(\sum_{i,j\geq 0}|d_{i,j}|)^k=1$. From this, one sees that $Q(x,y,t)$ converges for all $(x,y,t)\in\mathbb{C}^3$ such that $|x|<1$, $|y|<1$ and $|t|\leq 1$.

The *Kernel* of the walk is defined by

$$K(x,y,t):=xy(1-tS(x,y))$$

where

$$\begin{aligned} S(x,y) &= \textstyle\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j}x^iy^j \\ &= A_{-1}(x)\frac{1}{y}+A_0(x)+A_1(x)y \\ &= B_{-1}(y)\frac{1}{x}+B_0(y)+B_1(y)x, \end{aligned}$$

and $A_i(x)\in x^{-1}\mathbb{Q}[x]$, $B_i(y)\in y^{-1}\mathbb{Q}[y]$. Similarly to [FIM99, Section 1], we may prove that the generating series $Q(x,y,t)$ satisfies the following functional equation:

$$K(x,y,t)Q(x,y,t)=xy-F^1(x,t)-F^2(y,t)+td_{-1,-1}Q(0,0,t) \tag{1.1}$$

where

$$F^1(x,t):=K(x,0,t)Q(x,0,t),\quad F^2(y,t):=K(0,y,t)Q(0,y,t).$$

From now on, let us fix $0<t<1$ with $t\notin\overline{\mathbb{Q}}$. We assume that the algebraic curve $\overline{E_t}$ defined as the zero set in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ of the following homogeneous polynomial

$$\overline{K}(x_0,x_1,y_0,y_1,t)=x_0x_1y_0y_1-t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^ix_1^{2-i}y_0^jy_1^{2-j}=x_1^2y_1^2K(\frac{x_0}{x_1},\frac{y_0}{y_1},t). \tag{1.2}$$

is an elliptic curve. This is the situation studied in [BMM10, KR12], and from a Galoisian point of view, see [DHRS17].

*In several papers as [BMM10], it is not assumed that $\sum_{i,j} d_{i,j}=1$. But after a rescaling of the t variable, we may always reduce to the case $\sum_{i,j} d_{i,j}=1$.

Thanks to uniformization, we can identify E_t with $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ via a map

$$\begin{array}{ccc} \mathbb{C} & \to & \overline{E}_t \\ \omega & \mapsto & (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{array}$$

where $\mathfrak{q}_1, \mathfrak{q}_2$ are rational functions of $\mathfrak{p}$ and its derivative $d\mathfrak{p}/d\omega$, $\mathfrak{p}$ the Weirestrass function associated with the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (cf. [KR12, Section 3.2]). Therefore one can lift the functions $F^1(x,t)$ and $F^2(y,t)$ to functions $r_x(\omega) = F^1(\mathfrak{q}_1(\omega), t)$ and $r_y(\omega) = F^2(\mathfrak{q}_2(\omega), t)$.

One can deduce from [KR12, Theorems 3 and 4], that the functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically as univalent functions on the universal cover $\mathbb{C}$. Furthermore, for any $\omega \in \mathbb{C}$, we have

$$\tau(r_x(\omega)) - r_x(\omega) = b_1, \text{ where } b_1 = \iota_1(\mathfrak{q}_2(\omega))(\tau(\mathfrak{q}_1(\omega)) - \mathfrak{q}_1(\omega)) \tag{1.3}$$

$$\tau(r_y(\omega)) - r_y(\omega) = b_2, \text{ where } b_2 = \mathfrak{q}_1(\omega)(\iota_1(\mathfrak{q}_2(\omega)) - \mathfrak{q}_2(\omega)) \tag{1.4}$$

$$r_x(\omega + \omega_1) = r_x(\omega) \tag{1.5}$$

$$r_y(\omega + \omega_1) = r_y(\omega). \tag{1.6}$$

where τ is the automorphism of the field of meromorphic functions sending $f(\omega)$ onto $f(\omega + \omega_3)$ and ω_3 is explicitely given in [KR12, Section 3.2].

One can show that $r_x(\omega)$ is differentially algebraic with respect to $\frac{d}{d\omega}$ over C_E, the field of elliptic functions with respect to $\overline{E_t}$ if and only if $Q(x,0,t)$ is differentially algebraic with respect to $\frac{d}{dx}$ over $\mathbb{C}(x)$.

2. Open problems

2.1. **Order of the differential operator.** In [DHRS17], it is shown that if $r_x(\omega)$ is differentially algebraic then there exists complex numbers $a_1, a_0 \in \mathbb{C}$ ang $g \in C_E$ such that $a_1 \frac{d}{d\omega}(r_x(\omega)) + a_0 r_x(\omega) - g$ is an ω_3 periodic function, that is is left invariant by τ.

For the 51 unweighted walks associated to genus 1 curve and infinite group of the walk , only 9 walks give rise to a differentially algebraic generating function (see [DHRS17]). For these walks, it seems that the results of [BBMR15, BBMR17] show that there exists $g \in C_E$ such that $r_x(\omega) - g(\omega)$ is ω_3-periodic.

One can then ask the following:

For weighted walks associated to genus 1 curve and infinite group of the walk, is it always true that when $r_x(\omega)$ is differentially algebraic then there exists $g \in C_E$ such that $r_x(\omega) - g(\omega)$ is ω_3-periodic?

2.2. **Dependence in t.** What about the differential dependence in t for the 51 unweighted walks associated to genus 1 curve and infinite group of the walk?

2.3. **Finite group of the walk.** Let us consider a walk associated with an elliptic curve $E_t = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (the curve defined by the Kernel equation) and let us assume that this walk has finite group. This is equivalent to the fact that an integer multiple of ω_3 above belongs to $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. One can ask the following questions:

- The prolongation of $r_x(\omega)$ into a meromorphic function over $\mathbb{C}$ as well as the functional equation (1.3) obtained in [KR12] for the 51 unweighted walks associated to genus 1 curve and infinite group of the walk, depend heavily on the fact that the group of the walk is infinite. Can one still obtain analogue results in the finite group case?
- For weighted walks associated to genus 1 curve and finite group of the walk, are there some general closed form for the generating series?

References

[BBMR15] Olivier Bernadi, Mireille Bousquet-Mélou, and Kilian Raschel, *Counting quadrant walks via Tutte's invariant method*, An extended abstract to appeared in *Proceedings of FPSAC 2016*, Discrete Math. Theor. Comput. Sci. Proc., arXiv:1511.04298, 2015.

[BBMR17] ______, *Counting quadrant walks via Tutte's invariant method*, Preprint, arXiv:1708.08215, 2017.

[BMM10] Mireille Bousquet-Mélou and Marni Mishna, *Walks with small steps in the quarter plane*, Algorithmic probability and combinatorics, Contemp. Math., vol. 520, Amer. Math. Soc., Providence, RI, 2010, pp. 1–39.

[DHRS17] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F. Singer, *On the nature of the generating series of walks in the quarter plane*, Preprint (2017).

[FIM99] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev, *Random walks in the quarter-plane*, Applications of Mathematics (New York), vol. 40, Springer-Verlag, Berlin, 1999, Algebraic methods, boundary value problems and applications. MR 1691900

[KR12] Irina Kurkova and Kilian Raschel, *On the functions counting walks with small steps in the quarter plane*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 69–114. MR 3090255