Power Series with Coefficients from a Finite Set

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joint work with Jason P. Bell

Hadamard's problem on power series

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Hadamard then considered the following problem:

What relationships are there between the coefficients of a power series and the singularities of the function it represents?

Two special cases of the problem have been studied:

- Power series with rational or integral coefficients;
- Power series with finitely distinct coefficients.

Power series with rational coefficients

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad \text{where } a_n \in \mathbb{Q}.$$

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G. Eisenstein, Uber eine allgemeine Eigenschaft der Reihenentwicklungen aller algebraischen Funcktionen, Belin, Sitzber, 441-443, 1852

On the general properties of the series expansions of algebraic functions

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On the general properties of the series expansions of algebraic functions

Theorem (Eisenstein 1852, Heine 1853). If $f(x)$ represents an algebraic function over $\mathbb{Q}(x)$, then $\exists\ T \in \mathbb{Z}$, s.t.

$$\sum_{n \geq 0} a_n T^n x^n \in \mathbb{Z}[[x]].$$

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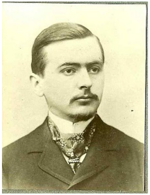

Pierre Fatou (1878-1929)

Pierre Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), no. 1, 335–400.

Fatou's Lemma. If $f(x)$ represents a rational function, then

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P, Q \in \mathbb{Z}[x] \text{ and } Q(0) = 1.$$

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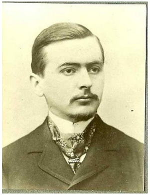

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Fatou's Theorem. If $f(x)$ converges inside the unit disk, then it is either rational or transcendental over $\mathbb{Q}(x)$.

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George Pólya, *Uber Potenzreihen mit ganzzahligen Koeffizienten*, Math. Ann. **77** (1916), no. 4, 497–513.

Fritz Carlson, *Über Potenzreihen mit ganzzahligen Koeffizienten*, Math. Z. **9** (1921), no. 1-2, 1–13.

Pólya-Carlson Theorem. If $f(x)$ converges inside the unit disk, then either it is rational or has the unit circle as natural boundary.

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Pólya-Carlson Theorem. If $f(x)$ converges inside the unit disk, then either it is rational or has the unit circle as natural boundary.

Corollary. If $f(x)$ is algebraic, then it is rational.

Power series with finitely distinct coefficients

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Gábor Szegő (1895-1985)

From 1917 to 1922, there are four papers with the same title:

Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten.

Power Series with Finitely Distinct Coefficients

1. G. Polya in 1917, Math. Ann.
2. R. Jentzsch in 1918, Math. Ann.
3. F. Carlson in 1919, Math. Ann.
4. G. Szego in 1922, Math Ann.

Szegö's Theorem (1922)
A power series with finitely distinct coefficients in $\mathbb{C}$ is either rational or has the unit circle as its natural boundary.

Arithmetical aspects of power series

Problem. Decide whether a given power series is rational, algebraic, transcendental, or hyper-transcendental?

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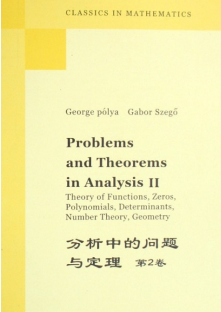

Chapter 3. Arithmetical Aspects of Power Series

Arithmetical aspects of power series

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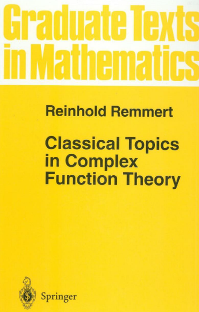

D-finite power series

Throughout this talk, $\mathbb{K}$ is a field of characteristic zero.

Definition. A power series $f(x_1, \ldots, x_d) \in \mathbb{K}[[x_1, \ldots, x_d]]$ is D-finite if

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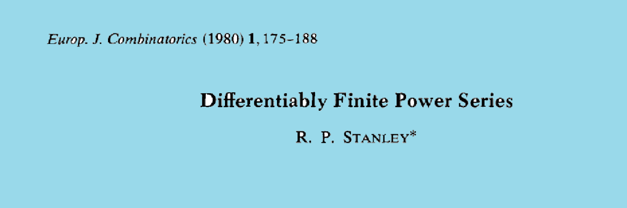
Europ. J. Combinatorics (1980) **1**, 175–188

Differentiably Finite Power Series

R. P. STANLEY*

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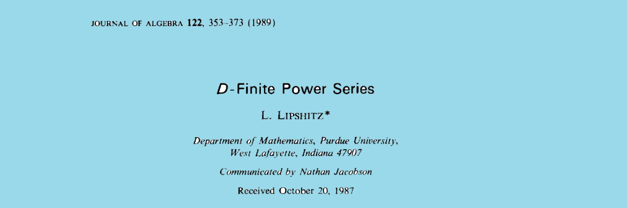
JOURNAL OF ALGEBRA **122**, 353-373 (1989)

D-Finite Power Series

L. LIPSHITZ*

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Communicated by Nathan Jacobson

Received October 20, 1987

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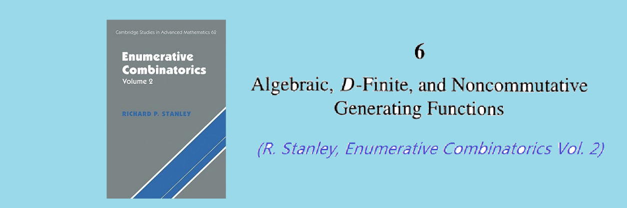

(R. Stanley, Enumerative Combinatorics Vol. 2)

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Definition. A sequence $a : \mathbb{N}^d \to \mathbb{K}$ is P-recursive if for each $i \in \{1,\ldots,d\}$, a satisfies a LPRE:

$$p_{i,r_i} S_{n_i}^{r_i}(a) + p_{i,r_i-1} S_{n_i}^{r_i-1}(a) + \cdots + p_{i,0} a = 0.$$

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$$p_{i,r_i} D_{x_i}^{r_i}(f) + p_{i,r_i-1} D_{x_i}^{r_i-1}(f) + \cdots + p_{i,0} f = 0.$$

Theorem. A sequence $a : \mathbb{N} \to \mathbb{K}$ is P-recursive iff its generating function $f(x) = \sum a(n)x^n$ is D-finite.

Remark. This is not true in the multivariate case.

Closure properties of D-finite power series

Let $\mathbf{n} = n_1, \ldots, n_d$, $\mathbf{x} = x_1, \ldots, x_d$, and $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$.

Definition. Let $f = \sum a(\mathbf{n})\mathbf{x}^{\mathbf{n}}$ and $g = \sum b(\mathbf{n})\mathbf{x}^{\mathbf{n}}$ be in $\mathbb{K}[[\mathbf{x}]]$. The Hadamard product of f and g is

$$f \odot g = \sum a(\mathbf{n})b(\mathbf{n})\mathbf{x}^{\mathbf{n}}.$$

The diagonal of f is defined as $\operatorname{diag}(f) = \sum a(n, \ldots, n)x^n \in \mathbb{K}[[x]]$.

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Theorem (Lipshitz1989). Let $\mathscr{D} := \{f \in \mathbb{K}[[\mathbf{x}]] \mid f \text{ is D-finite}\}$. Then

(i) if $f, g \in \mathscr{D}$, then $f + f$, $f \cdot g$, and $f \odot g$ are in $\mathscr{D}$;

(ii) if $f \in \mathscr{D}$, $\mathrm{diag}(f)$ is D-finite in $\mathbb{K}[[x]]$;

(iii) if $f \in cD$, and $\alpha_1, \ldots, \alpha_d \in K[[\mathbf{y}]]$ are algebraic over $K(\mathbf{y})$ and the substitution makes sense, then $f(\alpha_1, \ldots, \alpha_d)$ is D-finite.

Syndetic sets

Definition. A subset $S \subseteq \mathbb{N}$ is syndetic if there is some positive integer C such that if $n \in S$ then $n+i \in S$ for some $i \in \{1, \ldots, C\}$.

Example. The subset of all even numbers in $\mathbb{N}$ is syndetic, but the subset $S := \{p_1^{m_1} \cdots p_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{N}\}$ with $p_1, \ldots, p_n$ being prime numbers is not syndetic.

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Lemma. Let $f := \sum a(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in \mathbb{K}[[\mathbf{x}]]$ be D-finite. Then the set

$$\{n \in \mathbb{N} \mid \exists (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1} \text{ such that } a(n_1, \ldots, n_{d-1}, n) \neq 0\}$$

is either finite or syndetic.

Power series with integral coefficients
(the multivariate case)

Multivariate extensions of the Pólya-Carlson Theorem:

Power series with integral coefficients (the multivariate case)

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- V. P. Šeĭnov, *Transfinite diameter and certain theorems of Pólya in the case of several complex variables*, Sibirsk. Mat. Ž. **12** (1971), 1382–1389.
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Theorem (BellChen, 2016) If the multivariate power series

$$F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Power series with finitely distinct coefficients (the multivariate case)

Theorem (van der Poorten & Shparlinsky, 1994).

Let $a_n : \mathbb{N} \to \Delta$, where $|\Delta|$ is a finite subset of $\mathbb{Q}$. If the generating function $f(x) = \sum_n a_n x^n$ is D-finite, then it is rational.

Remark. This follows from Szegö's theorem by the fact that a D-finite power series can only have finitely many singularities.

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Theorem (BellChen, 2016). Let $a_{n_1,\ldots,n_d} : \mathbb{N}^d \to \Delta$, where $|\Delta|$ is a finite subset of $\mathbb{Q}$. If the generating function

$$f(x_1,\ldots,x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

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Nonnegative integer points on algebraic varieties

Let V be an algebraic variety over an algebraically closed field K of characteristic zero. We define the listing generating function

$$F_V(x_1,\ldots,x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

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We may ask the following questions:

When F_V is zero?

Remark. This is Hilbert Tenth Problem when K is $\mathbb{Q}$. In 1970, Matiyasevich proved that this problem is undecidable.

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We may ask the following questions:

When F_V is a polynomial?

Remark. In 1929, Siegel proved that a smooth algebraic curve C of genus $g \geq 1$ has only finitely many integer points over a number field K.

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We may ask the following questions:

When F_V is a rational function?

Remark. If V is defined by linear polynomials over $\mathbb{Q}$, then F_V is rational.

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When F_V is a D-finite function?

Corollary.

F_V is D-finite $\Leftrightarrow$ F_V is rational.

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Theorem.

The problem of testing whether F_V is rational is undecidable!

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We may ask the following questions:

When F_V is a differentially algebraic function?

Definition. $F \in K[[x_1,\ldots,x_d]]$ is differentially algebraic if the transcendence degree of the filed generated by the derivatives $D_{x_1}^{i_1}\cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ over $K(x_1,\ldots,x_d)$ is finite.

Nonnegative integer points on algebraic curves

Theorem. Let $p(x,y) \in \mathbb{C}[x,y]$. If the generating function

$$F_p(x,y) := \sum_{(n,m)\in V(p)\cap\mathbb{N}^2} x^n y^m$$

is rational. Then $p = f \cdot g$, where $f, g \in \mathbb{C}[x,y]$ s.t.

$$f = \prod_i (s_i \cdot x + t_i \cdot y + c_i) \quad \text{with } s_i, t_i \in \mathbb{Z} \text{ and } c_i \in \mathbb{C}$$

and g has only finite zeros in $\mathbb{N}^2$.

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and g has only finite zeros in $\mathbb{N}^2$.

Example. Let $p = x^2 - y$. Since p is not a product of integer-linear polynomials, the power series $F_p(x,y)$ is not D-finite.

Open problems

Conjecture. Let V be an algebraic variety over $\mathbb{C}$. Then the power series

$$\sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is differentially algebraic if and only if it is rational.

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Example. Let $p = x^2 - y$. Then the power series

$$F_p(x,y) := \sum_{m\geq 0} x^m y^{m^2}$$

is not differentially algebraic, otherwise, $F_p(x,2) = \sum 2^{m^2} x^m$ is differentially algebraic. By Mahler's lemma, we get a contradiction

$$2^{m^2} \ll (m!)^c \quad \text{for any positive constant } c.$$

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Conjecture (Chowla-Chowla-Lipshitz-Rubel). The power series

$$f := \sum_{n\in\mathbb{N}} x^{n^3} \in \mathbb{C}[[x]]$$

is not differentially algebraical, i.e., satisfies no ADE.

Remark. The power series $\sum x^{n^2}$ is differentially algebraic.

Summary

Theorem 1. If the power series

$$F = \sum f(n_1,\ldots,n_d)x_1^{n_1}\cdots x_d^{n_d} \in \mathbb{Z}[[x_1,\ldots,x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Theorem 2. If the power series

$$f(x_1,\ldots,x_d) = \sum a_{n_1,\ldots,n_d}x_1^{n_1}\cdots x_d^{n_d}, \quad a_{n_1,\ldots,a_d} \in \Delta \text{ with } |\Delta| < +\infty$$

is D-finite, then it is rational.

J. P. Bell, S. Chen. Power Series with Coefficients from a Finite Set. Journal of Combinatorial Theory, Series A, 151, pp. 241–253, 2017.

Summary

Theorem 1. If the power series

$$F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Theorem 2. If the power series

$$f(x_1, \ldots, x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}, \quad a_{n_1,\ldots,a_d} \in \Delta \text{ with } |\Delta| < +\infty$$

is D-finite, then it is rational.

J. P. Bell, S. Chen. Power Series with Coefficients from a Finite Set. Journal of Combinatorial Theory, Series A, 151, pp. 241–253, 2017.

Thank you!