A rapid course on Generated Jacobian Equations

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BIRS Workshop on Generated Jacobian Equations:
from Geometric Optics to Economics

Outline

Prescribed Jacobian Equations

A function of the form

$$T : \Omega \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d,\ \Omega \subset \mathbb{R}^d$$

determines an operator that assigns to any scalar function $u : \Omega \mapsto \mathbb{R}$ a map $T_u : \Omega \mapsto \mathbb{R}^d$ via

$$T_u(x) := T(x, \nabla u(x), u(x))$$

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$$T_u(x) := T(x, \nabla u(x), u(x))$$

In this context, given $\psi : \Omega \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$, the equation

$$\det(DT_u(x)) = \psi(x, \nabla u(x), u(x))$$

is called a **Prescribed Jacobian equation**.

Prescribed Jacobian Equations

Differentiating $T_u(x) = T(x, \nabla u(x), u(x))$ yields

$$DT_u(x)) = D_x T + D_{\bar{p}} T D^2 u + T_u \otimes \nabla u$$

and the Prescribed Jacobian Equation can be written as

$$\det(D^2 u + (D_{\bar{p}} T)^{-1}(D_x T + T_u \otimes \nabla u))$$
$$= \det(D_{\bar{p}} T)^{-1} \psi(x, \nabla u(x), u(x))$$

Prescribed Jacobian Equations

The most basic example of such an equation is given by

$$T(x, \bar{p}, u) = \bar{p}$$

and the resulting equation is the real Monge-Ampère equation

$$\det(D^2 u) = \psi(x, \nabla u(x), u(x)) \tag{MA}$$

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Yes, as long as we have a little more **structure** than this!.

Prescribed Jacobian Equations

A **generating function** is a real valued function

$$G : \Omega \times \bar{\Omega} \times \mathbb{R} \mapsto \mathbb{R},$$
$$G(x, \bar{x}, z) \text{ is monotone decreasing in } z$$

Provided certain assumptions on G hold, one can associate to G a map $T : \Omega \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d \ldots$ (the "G-exponential map")

Essentially, one defines functions $T(x, \bar{p}, u)$ and $Z(x, \bar{p}, u)$ by

$$\bar{p} = (D_x G)(x, T, Z))$$
$$u = G(x, T, Z)$$

Prescribed Jacobian Equations

Implicit differentiation of both equations yields

$$\det(D^2u + (D_x^2G)(x, T_u(x), Z_u(x))) = \psi_G(x, \nabla u, u)$$

where

$$\psi_G(x, \bar{p}, u) = \det(E(x, T, Z))\psi(x, \bar{p}, u)$$

$$E(x, \bar{x}, z) = D_{x\bar{x}}^2 G(x, \bar{x}, z) - \frac{D_x G_z}{G_z} \otimes D_{\bar{x}} G(x, \bar{x}, z)$$

Generated Jacobian Equations

Motivated in part by problems in geometric optics, Trudinger (2014) made the considerations above, setting up a framework to study a large class of scalar PDE we call **Generated Jacobian Equations**, which are given by

$$\det(D^2 u + A_G(x, \nabla u, u)) = \psi_G(x, \nabla u, u) \tag{GJE}$$

with A_G and ψ_G given by a generating function G.

Generated Jacobian Equations

Degenerate ellipticity?

The linearization of this equation at a given function u is degenerate elliptic as soon as

$$D^2u + A_G(x, \nabla u, u) \geq 0, \ \ \forall\, x,$$

which, as in optimal transport, leads to a notion of "convex function" that is natural for the PDE.

Generated Jacobian Equations

We are not limited to working in $\mathbb{R}^d$, in fact, we may consider

- Domains in Riemannian manifolds $\Omega \subset M^n$, $\bar{\Omega} \subset \overline{M}^n$ –or even compact metric spaces X and $\overline{X}$.
- Generating function defined for some set of $(x, \bar{x}, z)$: $G : (x, \bar{x}, z) \in \mathrm{dom}(G) \subset \Omega \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$
- (If we are in a manifold) G is C^2 in $(x, \bar{x})$.
- Last but not least, $G_z < 0$ everywhere (or $G_z > 0$).

Since $G_z < 0$, for $(x, \bar{x}, u) \in \mathbb{R}$ there is a unique real number $H = H(x, \bar{x}, u)$ solving

$$G(x, \bar{x}, H) = u$$

this defines a function $H : (x, \bar{x}, u) \in \mathrm{dom}(H) \subset \Omega \times \bar{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$. Plus: $H(x, \bar{x}, G(x, \bar{x}, z)) = z$, H is C^2 in $(x, \bar{x})$, and $H_u < 0$.

This is called the **Dual Generating Function** (sometimes the Inverse Generating Function).

Following Trudinger, $u : \Omega \to \mathbb{R}$ is said to be G-convex if

$$u(x) = \sup_{(\bar{x},z)\in\mathcal{A}} G(x,\bar{x},z) \ \forall\, x,$$

for some set $\mathcal{A} \subset \bar{\Omega} \times \mathbb{R}$.

Generated Jacobian Equations

G-transform

Given $u : \Omega \to \mathbb{R}$ and $v : \bar{\Omega} \mapsto \mathbb{R}$, we define, respectively

$$u^G(\bar{x}) = \sup_x H(x, \bar{x}, u(x)), \quad \text{for } \bar{x} \in \bar{\Omega},$$
$$v^H(x) = \sup_{\bar{x}} G(x, \bar{x}, v(x)), \quad \text{for } x \in \Omega,$$

known as the G- and H-transform of the function, respectively.

Generated Jacobian Equations

G-transform

For u, being G-convex amounts to $u = v^H$ for some $v(\bar{x})$, and for v, being H-convex amounts to $v = u^G$ for some v.

In analogy with the Legendre transform, if (u, v) are functions such that $u = v^H$ and $v = u^G$ then we say they are **conjugate**.

In particular, if u is G-convex, it is not hard to see* that

$$u = (u^G)^H$$

and analogously if v is H-convex.

*Under some natural some assumptions on G –see "Twist" condition in OT.

Generated Jacobian Equations

G-gradient map

If u is G convex, we set

$$\partial_G u(x) = \{\bar{x} \mid u(\cdot) \geq G(\cdot, \bar{x}, H(x, \bar{x}, u(x))\}$$

Note that, by the definition of H, we have

$$u(x) = G(x, \bar{x}, H(x, \bar{x}, u(x)))$$

so u is being touched from below at x by $G(\cdot, \bar{x}, H(x, \bar{x}, u(x)))$.

This is called the **G-subdifferential** of u at x.

Generated Jacobian Equations

G-gradient map

If further, u differentiable at x, then $\partial_G u(x)$ is a singleton

$$\partial_G u(x) = \{T_G(x, \nabla u(x), u(x))\}$$

where $T_G = T_G(x, \bar{p}, u)$ is determined by solving

$$\begin{aligned} \bar{p} &= (D_x G)(x, T_G, Z) \\ u &= G(x, T_G, Z) \end{aligned}$$

Accordingly, $T_G(x, \nabla u(x), u(x))$ is called the G**-gradient map**.

In general with G-convex functions

- Can define weak solutions for the GJE (“A-type” or “B-type”), allows for discontinuous RHS.

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- Can define weak solutions for the GJE ("A-type" or "B-type"), allows for discontinuous RHS.
- Makes (GJE) degenerate elliptic.
- Strong / uniform G-convexity $\leftrightarrow$ strong / uniform ellipticity

Examples:

Examples:
The “Trivial” Ones

Examples

The “Trivial” Ones

- Certainly the simplest interesting example is given by

$$G(x, \bar{x}, z) = -x \cdot \bar{x} - z,$$

  which corresponds to (MA) and convex functions.

- Given a cost function $c(x, \bar{x})$, we have the generating function

$$G(x, \bar{x}, z) = -c(x, \bar{x}) - z,$$

  which corresponds to Optimal Transport and c-convex functions.

Examples

The "Trivial" Ones

The fundamentally new phenomenon for GJE:

There is a possibly nonlinear dependence in z!

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There is a possibly nonlinear dependence in z!

Thus, naïvely, one may say GJE is what you get when you look at c-affine functions

$$-c(x, \bar{x}) - z$$

and their associated Monge-Ampère equation, and stop assuming that they depend linearly on the **height** parameter z.

Examples

The "Trivial" Ones

The fundamentally new phenomenon for GJE:

There is a possibly nonlinear dependence in z!
(*what's next? nonlocal dependence?!*)

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Examples

Beyond the "Trivial" Ones?

Affine functions (i.e. hyperplanes!)

$$\ell(x) = -x \cdot \bar{x} - z$$

c-Affine functions (e.g. $d_g(x, \bar{x})^2$!)

$$f(x) = -c(x, \bar{x}) - z$$

G-Affine functions (e.g. **???**)

$$f(x) = G(x, \bar{x}, z)$$

Examples

Beyond the "Trivial" Ones?

Affine functions (i.e. hyperplanes!)

$$\ell(x) = -x \cdot \bar{x} - z$$

c-Affine functions (e.g. $d_g(x, \bar{x})^2$!)

$$f(x) = -c(x, \bar{x}) - z$$

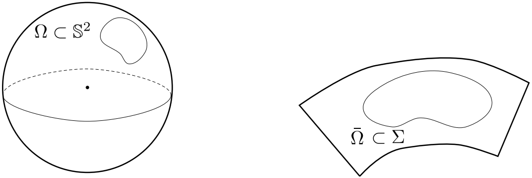
G-Affine functions (e.g. **ellipsoids!**)

$$f(x) = G(x, \bar{x}, z)$$

Examples:
Near-field reflectors
and ellipsoids of revolution

Examples

Near-field reflecting surfaces

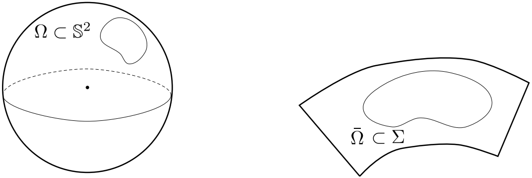


The design/reconstruction of reflective surfaces leads naturally to a GJE (V. Oliker, late 1980's-today).

(Physics/engineering literature: Norris-Westcott 1970's, Brickell-Marder-Westcott 1970's , J.B. Keller 1950's).

Examples

Near-field reflecting surfaces

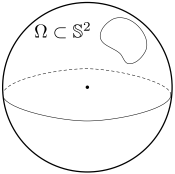


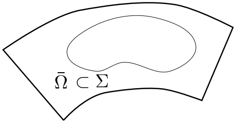
Already for the far field regime, geometric optics provided examples crucial to development of Optimal Transport (OT).

Works by
Gutierrez, Huang, Karakhanyan, Kochengin, Liu, Oliker, Tournier, X-J. Wang…

Examples

Near-field reflecting surfaces



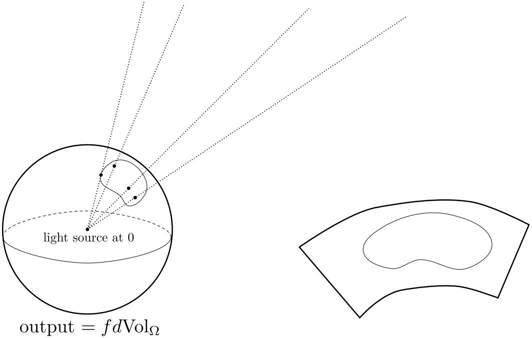


Domains:

- $\Omega \subset \mathbb{S}^2$
- $\bar{\Omega} \subset \Sigma$, a surface in $\mathbb{R}^3$

Examples

Near-field reflecting surfaces

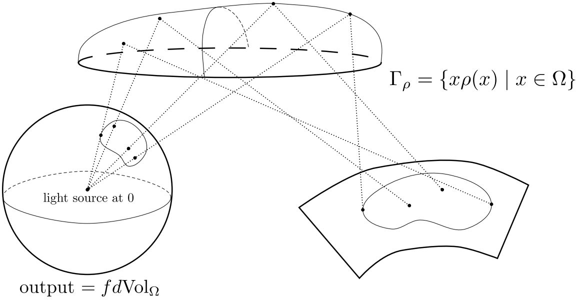


Light source:

- point source at origin
- emits energy $f\,\mathrm{dVol}_\Omega$

Examples

Near-field reflecting surfaces

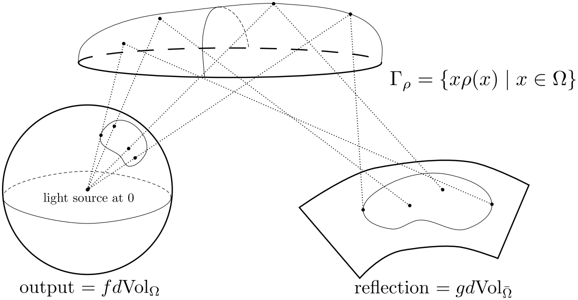


Reflector:

- radial graph of ρ over Ω
- perfectly reflective surface

Examples

Near-field reflecting surfaces

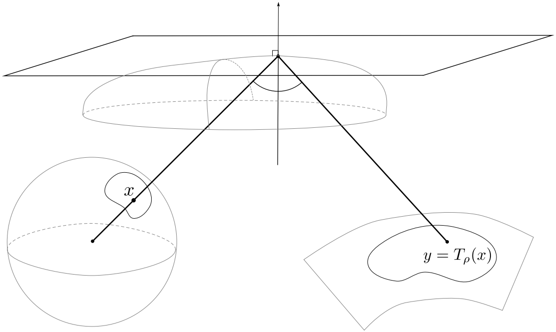


Goal:

- reflected pattern $g d\mathrm{Vol}_{\bar{\Omega}}$
- $\int_\Omega f d\mathrm{Vol}_\Omega = \int_{\bar{\Omega}} g d\mathrm{Vol}_{\bar{\Omega}}$

Examples

Near-field reflecting surfaces

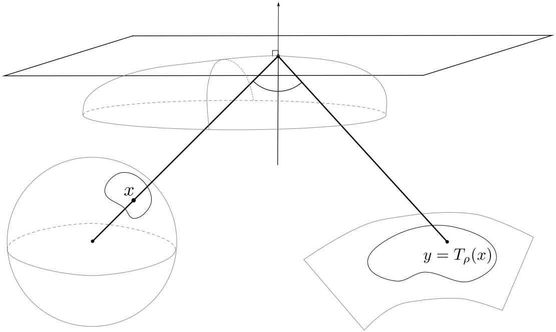


Ray tracing map:

$$T_\rho : \Omega \to \bar{\Omega} \text{ (Snell's law: } \angle \text{ incidence} = \angle \text{ reflection)}$$

Examples

Near-field reflecting surfaces

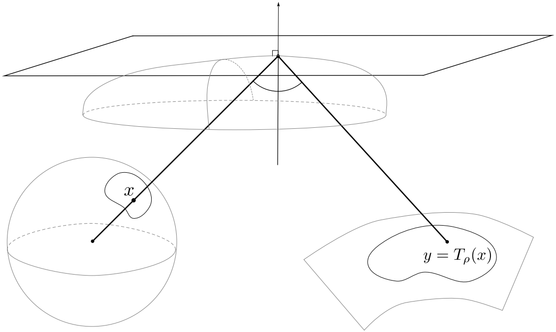


Prescribed Jacobian equation:

$$g(T_\rho(x))\,\det DT_\rho(x) = f(x)$$

Examples

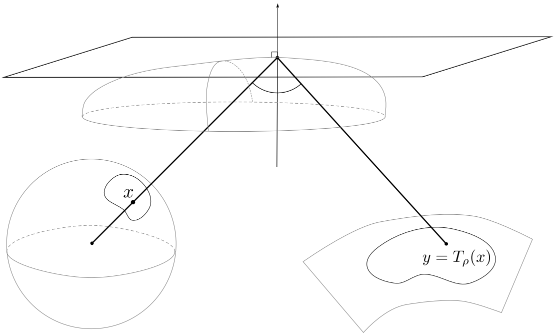
Near-field reflecting surfaces



Prescribed Jacobian equation:

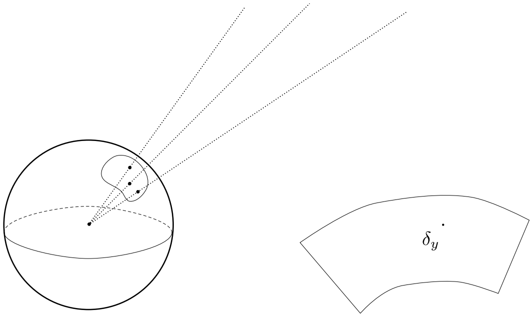
$$\det(D^2\rho + A_\Sigma(x, \rho, D\rho)) = \psi_\rho(x, \rho, D\rho)\frac{f(x)}{g(T_\rho(x))}$$

Near-field reflecting surfaces



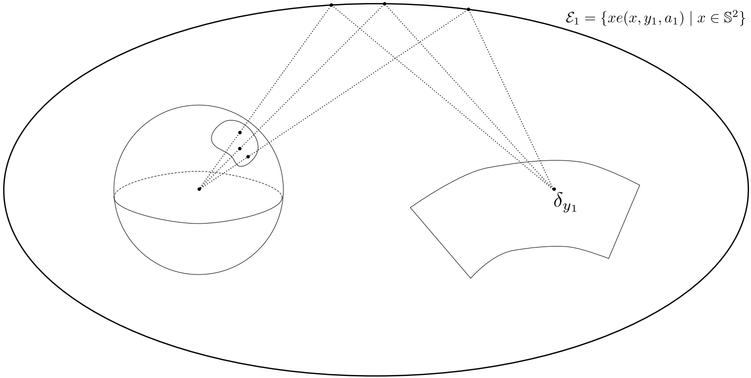
Now, what is the **generating function** for this example?

Near-field reflecting surfaces



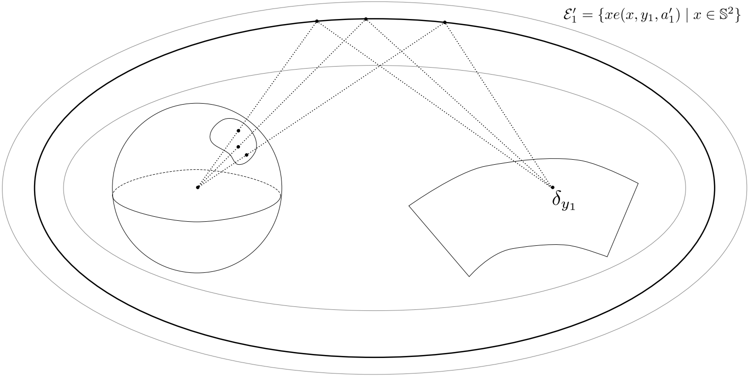
Consider what happens if $g\mathrm{dVol}_{\bar{\Omega}}$ becomes a **single point mass**.

Near-field reflecting surfaces



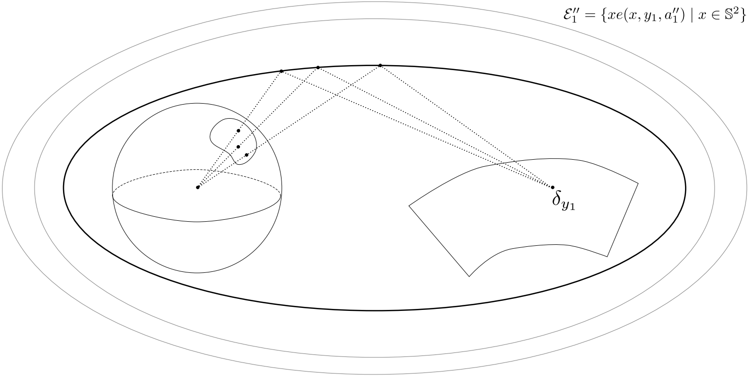
$\rho(\cdot) = e(\cdot, y_1, a_1)$, an ellipsoid

Near-field reflecting surfaces



$$\rho(\cdot) = e(\cdot, y_1, a_1')$$

Near-field reflecting surfaces



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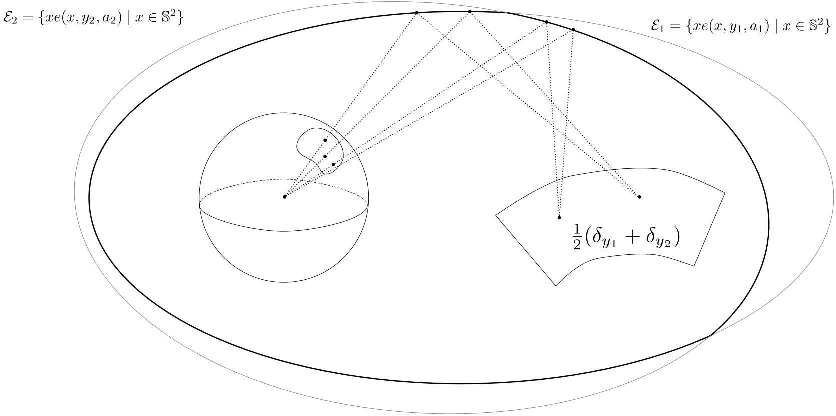
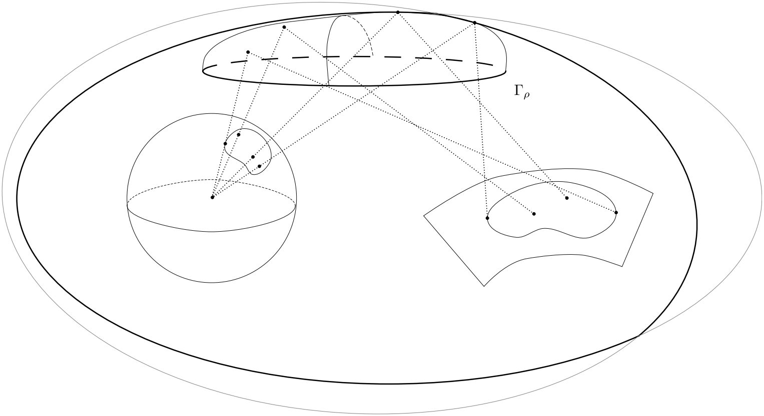


$$\rho(\cdot) = \min\{e(\cdot, y_1, a_1), e(\cdot, y_2, a_2)\}$$

Near-field reflecting surfaces



General: reflector = boundary (intersection of ellipsoids)

Near-field reflecting surfaces

The ellipsoids then give us the generating function.

We define, for $(x, \bar{x}, z) \in \mathbb{S}^2 \times \Sigma \times \mathbb{R}$ such that $\frac{1}{2} z |\bar{x}| < 1$,

$$G(x, \bar{x}, z) = \frac{1}{e(x, \bar{x}, z^{-1})}, \quad \left(e(x, \bar{x}, a) = \frac{a^2 - \frac{1}{4}|\bar{x}|^2}{a - \frac{1}{2}(x, \bar{x})} \right)$$

$$\psi_G(x, \bar{x}, z) = |\det(D^2_{x\bar{x}} G - \tfrac{D_x G_z \otimes D_{\bar{x}} G}{G_z})| \frac{f(x)}{g(T_u(x))}.$$

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Then, for a G-convex function u

- The reflector $\rho = 1/u$ is an envelope of ellipsoids.
- The ray tracing map for ρ plays the role of T_u.

A priori estimates for the near field reflector

Theorem (Karakhanyan and Wang, JDG 2010)

1. Ω, $V \subset \mathbb{S}^{n-1}$, $\Omega \cap V = \emptyset$, Ω *has Lipschitz boundary.*
2. Σ *is given by a radial graph of some smooth ($C^{1,1}$ function over* V*.*
3. f, g *are* $C^{2,\alpha}$*.*
4. $\partial\bar{\Omega}$ *is "convex."*

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Then, there is a $U \subset \mathbb{R}^n$ *such that*

1. *If* $O \in U$*, then any reflector going through* U *has a priori* C^2 *estimates in a neighborhood of* U*.*
2. *If* $O \notin U$*, there are examples of smooth* f, g *and a weak reflector going through* O *which is not* C^1*.*

Examples:
Optimal matchings (& more. . .)
with non-quasilinear utility functions

Optimal matchings & more

Non quasilinear utility functions

An important context for generating functions is economics.

Consider, first, the following interpretation

$$X = \text{ a set of buyers}$$
$$Y = \text{ a set of sellers}$$

A utility function $G(x, y, v)$ is given, representing the following

The maximal utility buyer x may obtain when matched with seller y after paying a utility v to the seller.

Optimal matchings & more

Non quasilinear utility functions

In the economic literature, when

$$G(x, y, z) = -c(x, y) + z$$

it is said that the utility function is **quasilinear**.

Optimal matchings & more

Principal-agent problems

We consider a situation where a *principal* is dealing with a set of *agents*, and set

$$X = \text{ a set of agents}$$
$$Y = \text{ a set of decisions taken by agents}$$

A utility function for agents $G(x, y, v)$ is given, representing the following

The utility of the agent x upon taking decision y wihile providing a transfer of v to the principal

Optimal matchings & more

Principal-agent problems

The Principal has a utility function,

$$\pi : X \times Y \times \mathbb{R} \mapsto \mathbb{R}$$

and she wishes to maximize

$$\int_{x \in X} \pi(x, y(x), v(y(x)) \; d\mu(x)$$

Here: μ represents the distribution of agents, and…

... following Nöldeke-Samuelson (2015), the supremum is taken over certain admissible pairs $(v(x), y(x))$ where

$$y : X \mapsto Y \text{ represents an assignment to the agents}$$
$$v : X \mapsto \mathbb{R} \text{ represents a tariff}$$

and, y is *implemented* by the tariff v.

In our notation, this means that y is given the G-gradient map associated to the dual of $v(y)$.

Regularity Theory For Weak Solutions

Regularity

Optimal Transport

(Only a very few highlights)

Let us recall

- $G(x, y, z) = -c(x, y) - z$, $(\mathrm{dom}(G) = \Omega \times \bar{\Omega} \times \mathbb{R})$
- $\det(D^2 u(x) + D_x^2 c(x, T_u(x))) = |\det D_{x,y}^2 c(x, T_u(x))| \frac{f(x)}{g(T_u(x))}$ (c-MA)
- u is *c-convex*
- Mapping $T = T_u$ solves the above.

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Existence of weak solutions, smooth solutions, c-convexity:

... Brenier (1991), Gangbo-McCann (1996), McCann (2001), Ma-Trudinger-Wang (2005)

Regularity

Optimal Transport
(Only a very few highlights)

- Cheng-Yau (1970's) and Urbas (1990's) existence of smooth solutions for real Monge-Ampère.
- Caffarelli (early 1990's) regularity for weak solutions of real Monge-Ampère.
- Ma, Trudinger, and Wang (2005): condition $(\mathrm{MTW})_+$ on c for smoothness of solutions (f, g smooth, need $c \in C^4$)
- Loeper (2009): equivalent geometric formulation of $(\mathrm{MTW})_0$, necessary for regularity when $c \in C^4$
- Figalli, Kim, and McCann (2013): local $C^{1,\alpha}$ regularity of weak solutions under the minimal sharp assumptions: i.e. $(\mathrm{MTW})_0$ ("A3-weak") and densities bounded and bounded away from zero.

Regularity for GJE

Theorem (Trudinger 2014)

Let u be a weak solution of (GJE) with $\partial_G u(\Omega) = \bar{\Omega}$ and where the right hand side is given by

$$\frac{f}{g \circ T_u}$$

where f, g are $C^{1,1}$, bounded, and bounded away from zero. If the generating function G satisfies ***natural structural conditions****, and $\bar{\Omega}$ is G-convex with respect to Ω, then the solution u is of class $C^3(\Omega)$.*

Regularity of weak solutions of (GJE)

Theorem (with J. Kitagawa, 2016)

*If G, Ω, and Ω^* satisfy **natural** structural conditions and F_G, F_G^{-1} are bounded, then any "nice" weak solution u of* (GJE) *is C^1 in Ω.*

If G is also locally C^{1,α_0} in the x variable for some $\alpha_0 \in (0,1)$, then any "nice" weak solution u is locally $C^{1,\alpha}$ in Ω for some $\alpha > 0$.

Note: By "nice", we mean the following: there is a region $U \subset \Omega \times \mathbb{R}$ determined by G, u **is nice** means $graph(u) \subset U$.

Aims of this workshop

- Disseminate GJE as a new framework tha encompases many fields.
- Bring together experts in different fields that touch on aspects of GJE.
- Identify new lines of investigation –GJE is a fertile ground for research: many open questions, many basic important examples not fully understood.

Thank You!

(BONUS SLIDES)

Regularity

The Real Monge-Ampère equation

- $\Omega \subset \mathbb{R}^n$ bounded

The Monge-Ampère equation

$$\det D^2 u(x) = \psi(x, u(x), \nabla u(x)), \;\; x \in \Omega. \tag{1}$$

- $\Omega \subset \mathbb{R}^n$ bounded

The Monge-Ampère equation

$$\det D^2 u(x) = \psi(x, u(x), \nabla u(x)),\ x \in \Omega. \tag{1}$$

- $G(x, y, z) = -\langle x, y \rangle - z$, $(\mathrm{dom}(G) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$
- G-convex functions $\to$ convex functions

Regularity

The Real Monge-Ampère equation

Theorem (Caffarelli (1990's))

If u is a weak solution of (MA), $\psi, \frac{1}{\psi}$ *are bounded, and* $\bar{\Omega}$ *is convex, then it is strictly convex and loc.* $C^{1,\alpha}$ *for some* $\alpha > 0$.

Regularity

The Real Monge-Ampère equation

Theorem (Caffarelli (1990's))

If u is a weak solution of (MA), $\psi, \frac{1}{\psi}$ are bounded, and $\bar{\Omega}$ is convex, then it is strictly convex and loc. $C^{1,\alpha}$ for some $\alpha > 0$.

The key to this result: using barrier arguments, one proves opposing **pointwise** inequalities for a solution u in any "normal"' convex domain. Using affine invariance one obtains proper pointwise bounds for general domains.

The two estimates, applied to proper rescalings of u, rule out "corners" and "flat" pieces in the graph of u.

Aleksandrov estimate

There exists $C_n > 0$ s.t. if u is convex and $x_0 \in S := \{u \leq 0\}$ with $B_1(0) \subset S \subset B_n(0)$, then

$$|u(x_0)|^n \leq C_n d(x_0, \partial S) \mathrm{Vol}\left(\nabla u(S)\right)$$

Regularity

Key ingredients for regularity

Aleksandrov estimate

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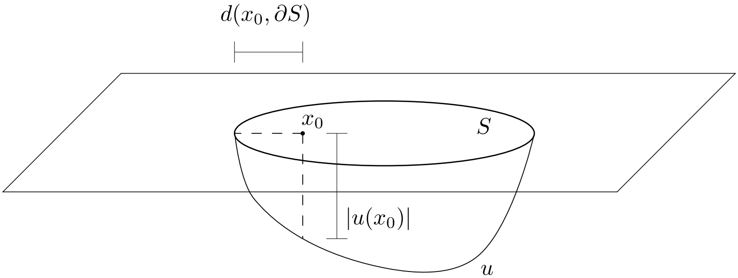


Regularity

Key ingredients for regularity

Sharp growth estimate

There exists $C_n > 0$ s.t. if u is convex and $x_0 \in S := \{u \leq 0\}$ with $B_1(0) \subset S \subset B_n(0)$, then

$$\sup_S |u|^n \geq C_n \mathrm{Vol}\left(\nabla u(\frac{1}{2}S)\right)$$

These two inequalities allows us to prove that u is strictly convex and $C^{1,\alpha}$.

To illustrate the method used in the theory for GJE, let us review it in the special case of the MA, where we have the following important result.

Lemma

Let $u : \Omega \to \mathbb{R}$ be a convex function such that

$$\Lambda^{-1} \leq \det(D^2 u) \leq \Lambda \text{ in } \Omega,$$

*the above understood in the **sense of Aleksandrov**, then, if ℓ is supporting to u at some point in Ω, then*

$$\{u = \ell\} \text{ is either a single point or it intersects } \partial\Omega.$$

Unnormalized estimates:

By affine invariance, a scale-invariant or unnormalized version of the Aleksandrov estimate can be written (this is one way of seeing Caffarelli's result)

New proofs of these estimates -covering c-convex functions in optimal transport- have been obtained by Figalli, Kim, and McCann (2013), and Guillen and Kitagawa (2014). **They don't rely on affine invariance**.

In the latter work, it is shown that these estimates follow from a **quantitative quasiconvexity** property of cost functions, which itself is equivalent to the $(\mathrm{MTW})_0$ condition introduced by Ma, Trudinger, and Wang when the cost is C^4.

Quantitative Quasiconvexity: motivation

Dilemma:

- Need $c \in C^4$ for $(\mathrm{MTW})_0$
- Only need $c \in C^2$ for Loeper's maximum principle.

Related Question: If c_k is a sequence of costs all satisfying $(\mathrm{MTW})_0$, and $c_k \to c$ in C^2 norm, can we prove regularity for OT problem associated to c?

Quantitative Quasiconvexity: motivation

Loeper's maximum principle refers to an important property of costs satisfying the $(\mathrm{MTW})_0$:

if c_t is a tilting family of c-functions, that is

$$c_t(x) = -c(x, y(t)) + c(x_0, y_0) + \alpha$$

Then, for every $t \in [0, 1]$ we have

$$c_t(x) \leq \max\{c_0(x), c_1(x)\}$$

Quantitative Quasiconvexity: motivation

Loeper's maximum principle refers to an important property of costs satisfying the $(\mathrm{MTW})_0$:

if f_t is a "tilting family" of c-functions, then, for every $t \in [0,1]$ we have

$$f_t(x) \leq \max\{f_0(x), f_1(x)\}$$

Tilting family refers to the fact that

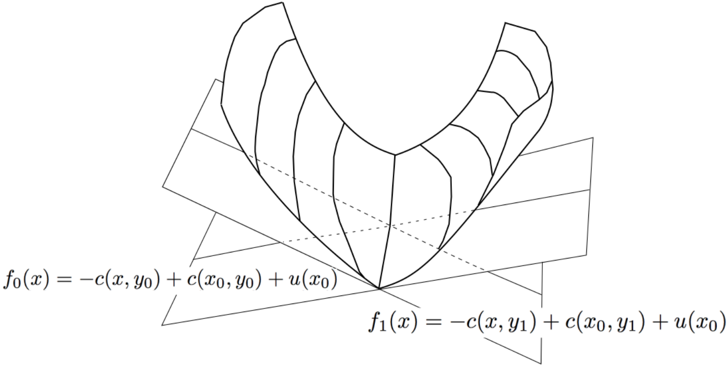
$$f_t(x) = -c(x, y(t)) + c(x_0, y_0) + \alpha$$

where $y(t)$ is what is known as a "c-segment with respect to x_0"

Quantitative Quasiconvexity: motivation

When $c(x, y) = x \cdot y$, this corresponds to "tilting hyperplanes"

$$f_t(x) \leq \max\{c_0(x), c_1(x)\}$$



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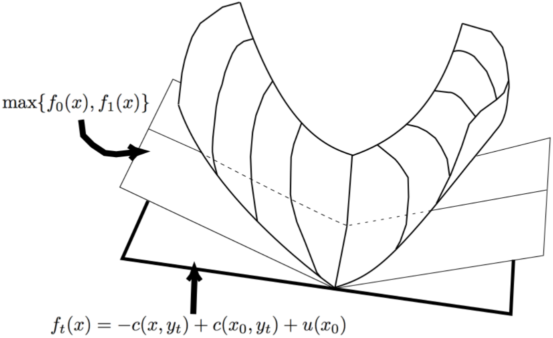

Quantitative Quasiconvexity: motivation

The cost c is said to satisfy (QQConv), if $\exists\, M \geq 1$ such that

$$f_t(x) - f_0(x) \leq M(f_1(x) - f_0(x))_+ \;\; \forall\, t \in [0,1].$$

This notion also extends to G-functions, however, the exact definition in this general setting would take us too far adrift.

- (QQConv) implies **quasiconvexity**.
- For $M = 1$, it implies to **convexity**.

Quantitative Quasiconvexity and optimal transport

Theorem (with J. Kitagawa, 2014)

- *If* $c \in C^4$ *satisfies* $(\mathrm{MTW})_0$ *(and conditions on domains), then* c *satisfies* (QQConv).
- *If* $c \in C^3$ *and satisfies* (QQConv), *(+ conditions on domains), then* c*-convex functions have Aleksandrov / sharp growth estimates.*
- *These estimates lead to strict* c*-convexity and* $C^{1,\alpha}$ *regularity of weak solutions.*

Quantitative Quasiconvexity and GJE

Theorem (with J. Kitagawa, 2016)

- *If* $G \in C^4$ *satisfies analogue of* $(\mathrm{MTW})_0$ *(and conditions on domains), then* G *satisfies* (QQConv).
- *If* $G \in C^2$ *and satisfies* (QQConv), *(+ conditions on domains), then* G*-convex functions have Aleksandrov / sharp growth estimates.*
- *These estimates lead to strict* G*-convexity and* $C^{1,\alpha}$ *regularity of weak solutions.*

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- *These estimates lead to strict* G*-convexity and* $C^{1,\alpha}$ *regularity of weak solutions.*

In particular: the previous question has a positive answer, the class of cost functions for which the OT problem enjoys $C^{1,\alpha}$ regularity is closed under C^2 limits.