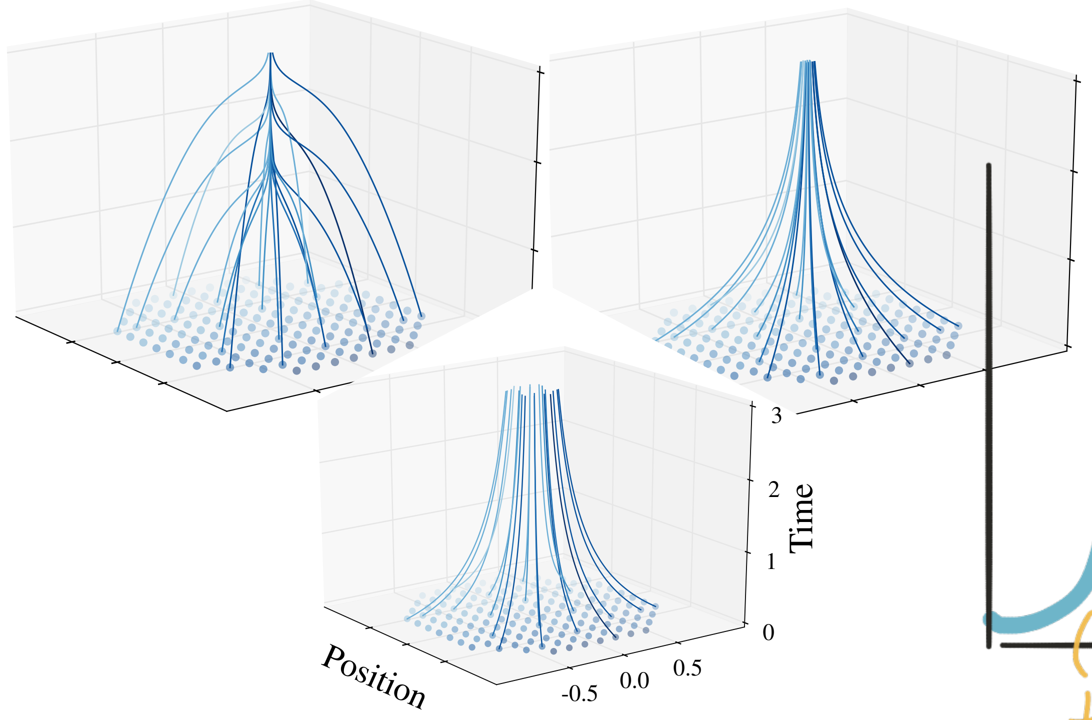

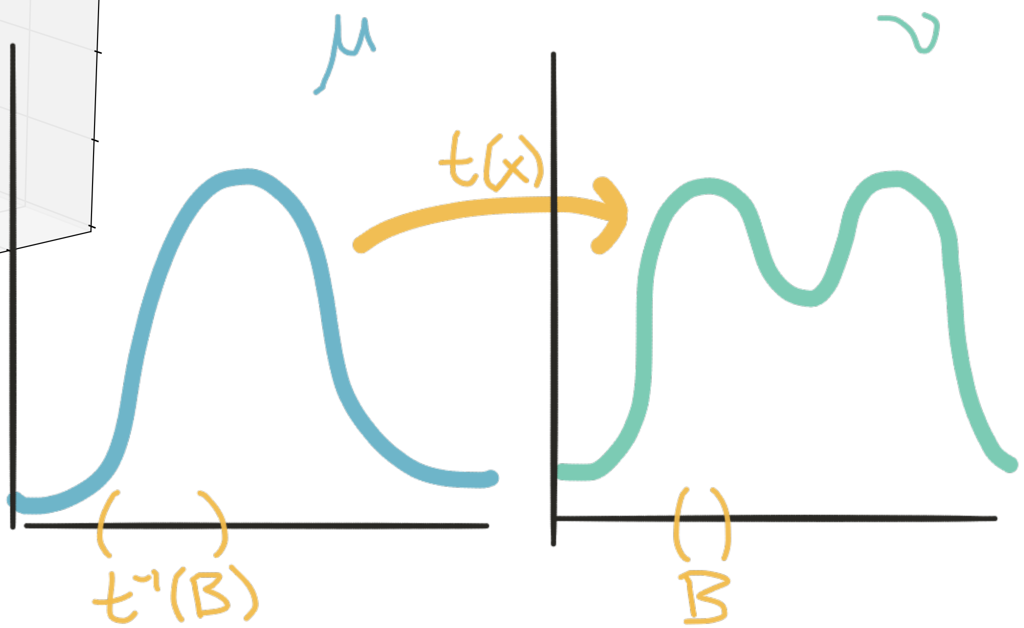

From slow diffusion to a hard height constraint: characterizing congested aggregation

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BIRS
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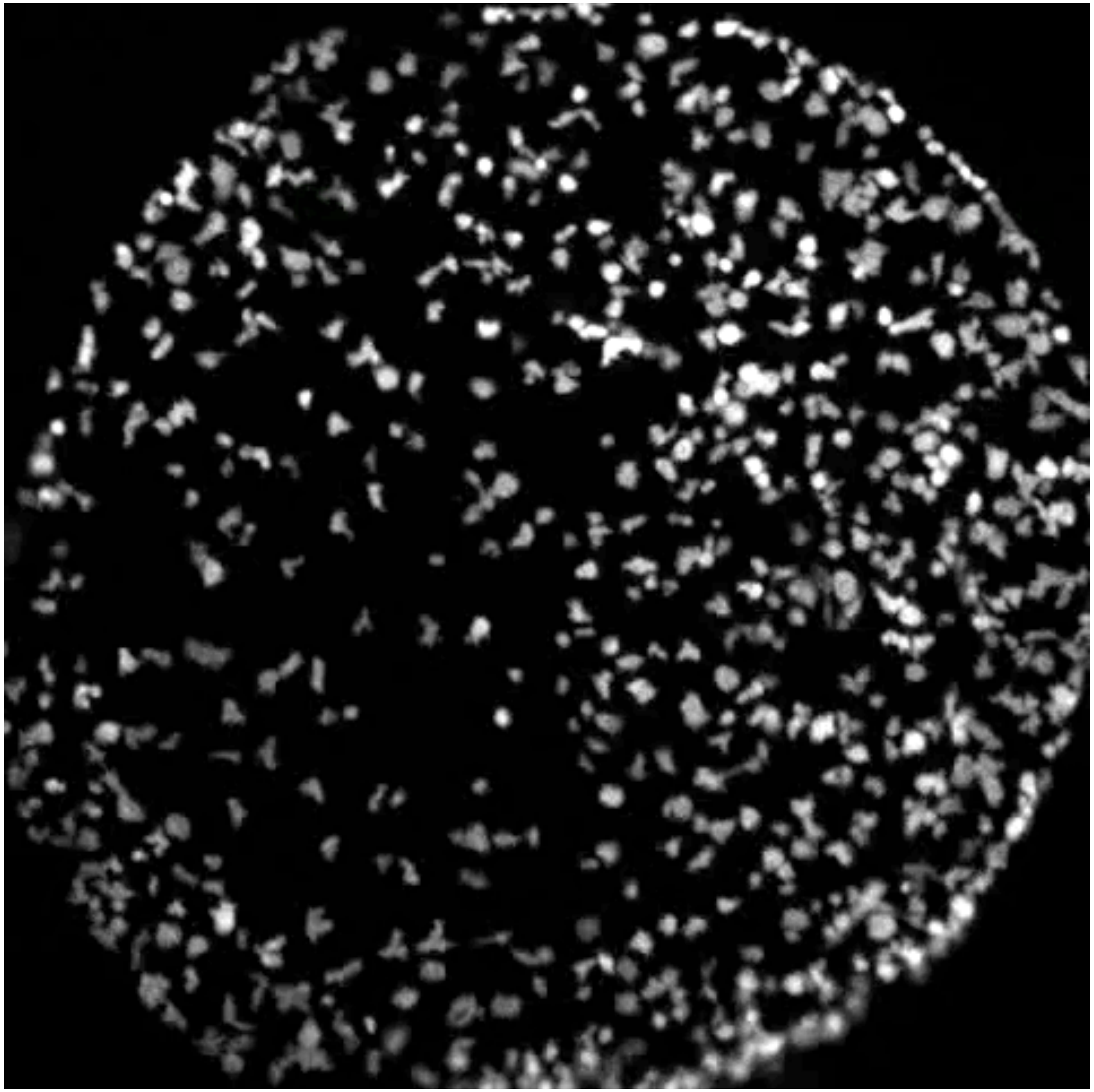
collective dynamics

biological chemotaxis (a colony of slime mold)



plan

- collective dynamics
- optimal transport and Wasserstein gradient flow
- ω-convexity and height constrained aggregation
- future work

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motivation

- ρ(x,t): $\mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ nonnegative density
- mass is conserved ⇒ $\int \rho(x)\, dx = 1$

aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho = \nabla \cdot ((\underbrace{\nabla K * \rho)}_{\text{self attraction}}\rho) + \underbrace{\Delta\rho^m}_{\text{degenerate diffusion}}} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

interaction kernels:

- granular media: K(x) = $|x|^3$
- swarming: K(x) = $|x|^a/a - |x|^b/b$, $-d<b<a$
- chemotaxis: $K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2, \\ C_d|x|^{2-d} & \text{otherwise.} \end{cases}$

degenerate diffusion:

- $\Delta\rho^m = \nabla \cdot (\underbrace{m\rho^{m-1}}_{D}\nabla\rho)$

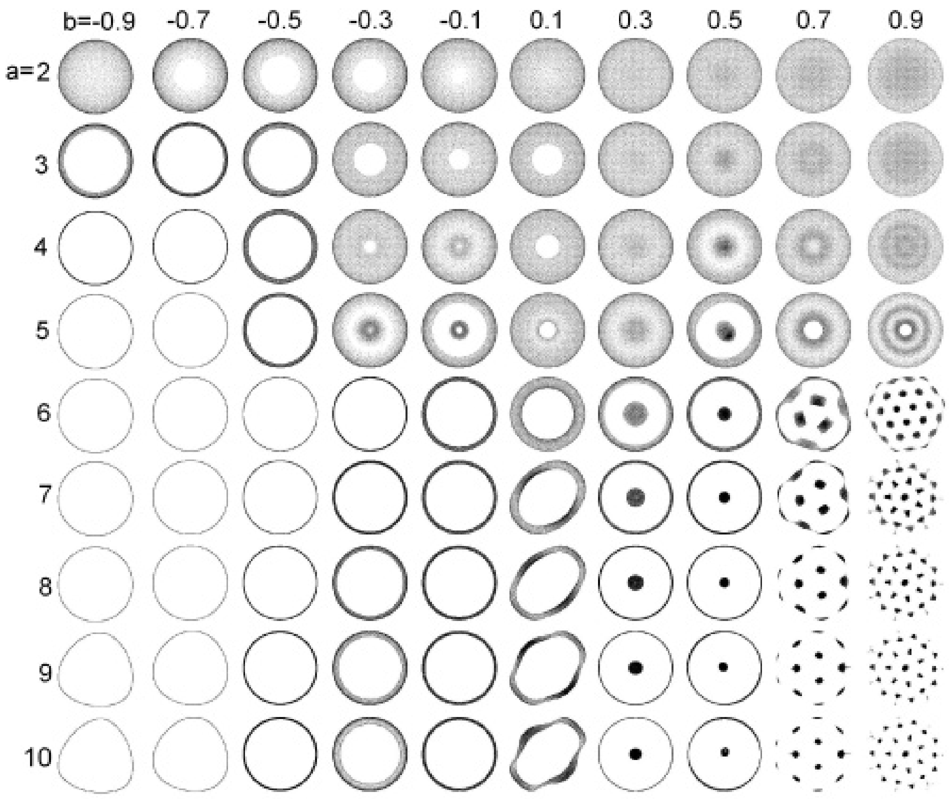
collective dynamics: mathematics

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

Mathematical interest:

- Nonlinear
- Nonlocal
- Competing effects of attraction/repulsion
- Rich structure of equilibria



[Kolokolnikov, Sun, Uminsky, Bertozzi, 2011]

collective dynamics: main questions

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

Main questions:

1. Do solutions exist?
2. Are they unique? stable?
3. How do they behave in the long time limit?
4. How can we simulate them numerically?

Key tool: optimal transport

plan

- collective dynamics
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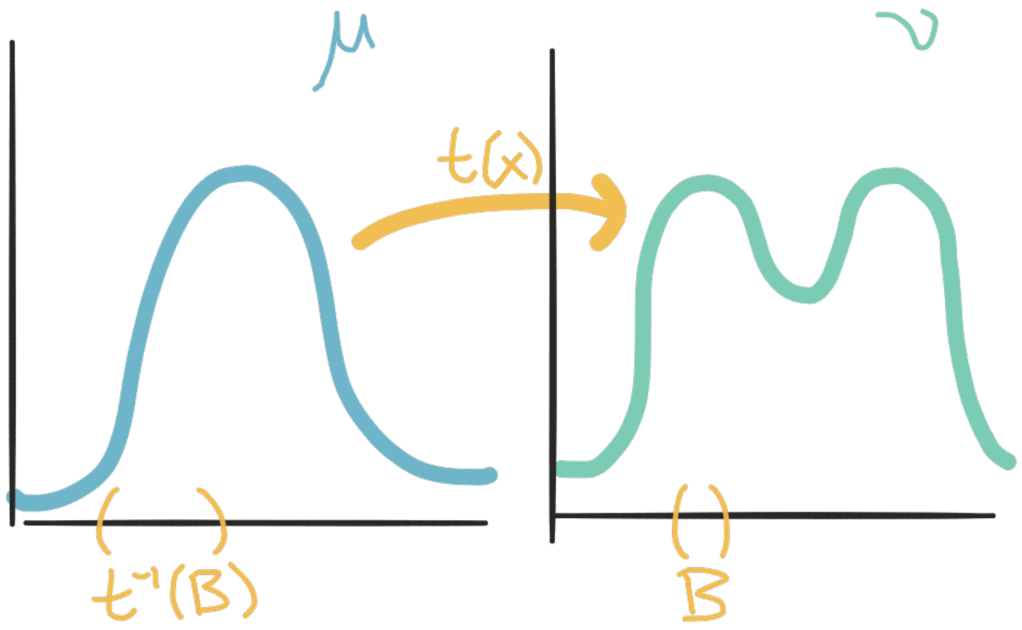
Wasserstein metric

- Given two probability measures μ and ν on $\mathbb{R}^d$, $\mathbf{t} : \mathbb{R}^d \to \mathbb{R}^d$ *transports* μ *onto* ν if $\nu(B) = \mu(\mathbf{t}^{-1}(B))$. Write this as $\mathbf{t}\#\mu = \nu$.



- The *Wasserstein distance* between μ and ν ∈ $P_2(\mathbb{R}^d)$ is

$$W_2(\mu,\nu) := \inf\left\{ \underbrace{\left(\int |t(x)-x|^2 d\mu(x)\right)^{1/2}}_{\text{effort to rearrange μ to look like ν, using t(x)}} : \underbrace{t\#\mu = \nu}_{\text{t sends μ to ν}} \right\}$$

For simplicity of notation,
μ, ν ≪ $\mathscr{L}^d$

geodesics

Not just a metric space… a geodesic metric space: there is a constant speed geodesic $\sigma : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ connecting any μ and ν.

$$\sigma(0) = \mu,\ \sigma(1) = \nu,\ W_2(\sigma(t), \sigma(s)) = |t-s| W_2(\mu, \nu)$$

Wasserstein geodesic $\sigma(t)$

linear interpolation $(1-t)\mu + t\nu$

[Payré, Papadakis, Oudet 2013]

convexity

Since the Wasserstein metric has geodesics, it has a notion of convexity.

Recall: in **Euclidean space**, E: $\mathbb{R}^d \to \mathbb{R}$ is…

convex

endpoints

$$D^2E \geq 0 \iff E(\overbrace{(1-t)x+ty}^{\text{Euclidean geodesic}}) \leq (1-t)E(x)+tE(y)$$

λ-convex

$$D^2E \geq \lambda I_{d\times d} \iff E((1-t)x+ty) \leq (1-t)E(x)+tE(y) - t(1-t)\frac{\lambda}{2}|x-y|^2$$

Likewise, in the **Wasserstein metric**, E: $P_2(\mathbb{R}^d) \to \mathbb{R}$ is λ-convex if

$$E(\sigma(t)) \leq (1-t)E(\mu)+tE(\nu) - t(1-t)\frac{\lambda}{2}W_2^2(\mu,\nu)$$

Wasserstein geodesic

endpoints

gradient flow

How does this relate to PDE? Wasserstein gradient flow.

- Informally, a curve x(t):$\mathbb{R}$ → X is the gradient flow of an energy E: X → $\mathbb{R}$ if

“ $$\boxed{\frac{d}{dt}x(t) = -\nabla_X E(x(t))}$$ ”

- “x(t) evolves in the direction of steepest descent of E”

Examples:

metric	energy functional	gradient flow
$(L^2(\mathbb{R}^d), \|\cdot\|_{L^2})$	$E(f) = \frac{1}{2}\int \|\nabla f\|^2$	$\frac{d}{dt}f = \Delta f$
$(\mathcal{P}_2(\mathbb{R}^d), W_2)$	$E(\rho) = \int \rho \log \rho$	$\frac{d}{dt}\rho = \Delta \rho$
	$E(\rho) = \frac{1}{m-1}\int \rho^m$	$\frac{d}{dt}\rho = \Delta \rho^m$

gradient flow

ρ(t): $\mathbb{R}$ → $P_2(\mathbb{R}^d)$ is the *Wasserstein gradient flow* of energy E: $P_2(\mathbb{R}^d)$ → $\mathbb{R}$ if

$$\text{“}\ \frac{d}{dt}\rho(t) = -\nabla_{W_2} E(\rho(t))\ \text{”}$$

More precisely, ρ(t) is the gradient flow of E if…

- there exists $v(t) \in L^2_{\mathrm{loc}}((0,+\infty), L^2(\rho(t)))$ so that

$$\boxed{\frac{d}{dt}\rho(x,t) + \nabla \cdot (v(x,t)\rho(x,t)) = 0}$$

- for a.e. t>0, $-v(t) \in \partial E(\rho(t))$

$$\xi \in \partial E(\rho) \text{ if as } \nu \to \mu,\ E(\nu) - E(\rho) \geq \int \underbrace{\langle \xi, \mathbf{t}_\rho^\nu - \mathrm{id} \rangle \mathrm{d}\mu}_{\xi\,(\nu\,-\,\rho)} + \mathrm{o}(\mathrm{W}_2(\rho,\nu))$$

- If E and ρ are nice, $\partial E(\rho) = \left\{ \nabla \frac{\partial E}{\partial \rho} \right\}$, and solutions of the gradient flow can be characterized as solutions to a PDE.

collective dynamics: main questions

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

$$E(\mu) = \iint K(x-y) d\mu(x) d\mu(y) + \frac{1}{m-1} \int \mu(x)^m dx$$

Main questions:

1. Do solutions exist?
2. Are they unique? stable?
3. How do they behave in the long time limit?
4. How can we simulate them numerically?

collective dyr

If K(x) is λ-convex, λ ≤ 0, so is E(μ) [CDFLS, 2011].
But what about when K(x) isn't λ-convex?

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

$$E(\mu) = \iint K(x-y)d\mu(x)d\mu(y) + \frac{1}{m-1}\int \mu(x)^m dx$$

Theorem (Ambrosio, Gigli, Savaré 2005): If the energy is λ-convex,

1. Do solutions exist? Yes (JKO)
2. Are they unique? Yes stable? contract (λ>0)/expand (λ≤0) exponentially
3. How do they behave in the long time limit? For λ>0, there is a unique steady state, which solutions approach exponentially quickly.
4. How can we simulate them numerically?

collective dynamics: applications

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

Applied interest:

- Slime mold (chemotaxis): $K(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } d = 2, \\ C_d |x|^{2-d} & \text{otherwise.} \end{cases}$
- Swarming: $K(x) = |x|^a/a - |x|^b/b, \quad -d < b < a$
- Granular media: $K(x) = |x|^3$

not λ-convex

“merely” 0-convex

plan

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- future work

height constrained aggregation

a new model (C., Kim, Yao 2016):

inspired by the aggregation equation with degenerate diffusion, we consider a height constrained aggregation equation, for $K = \Delta^{-1}$

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta\rho^m \quad \Rightarrow \quad \text{“}\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}\text{”}$$

- Both models have self-attraction from $\nabla K * \rho$.

- The role of repulsion is played by hard height constraint instead of degenerate diffusion.

- Heuristically, hard height constraint is singular limit of degenerate diffusion:

  Idea: $\Delta\rho^m = \nabla \cdot (\underbrace{m\rho^{m-1}}_{D} \nabla\rho)$, so as m→+∞, $D \to \begin{cases} +\infty & \text{if } \rho > 1 \\ 0 & \text{if } \rho < 1 \end{cases}$

height constrained aggregation

$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

- Hard height constraint appeared in previous work by [Maury, Roudneff-Chupin, Santambrogio 2010]—instead of K*ρ(x) had V(x).
- Has a (formal) Wasserstein gradient flow structure:

equation	energy
$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta\rho^m$	$E(\mu) = \iint K(x-y)d\mu(x)d\mu(y) + \frac{1}{m-1}\int \mu(x)^m dx$
$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$	$E_\infty(\mu) = \begin{cases} \iint K(x-y)d\mu(x)d\mu(y) & \text{if } \|\mu\|_{L^\infty} \leq 1 \\ +\infty & \text{otherwise} \end{cases}$

Since K(x) is not λ-convex, E_∞ falls outside the scope of the existing theory.

ω-convexity

Even though we don't have

$$E_\infty(\sigma(t)) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{\lambda}{2}t(1-t)W_2^2(\mu,\nu)$$

λ-convexity

E∞ does satisfy a similar inequality for a modulus of convexity ω(x) = x |log(x)|.

$$E_\infty(\sigma(t)) \leq (1-t)E_\infty(\mu) + tE_\infty(\nu) - \frac{\lambda}{2}\left[(1-t)\omega\left(t^2W_2^2(\mu,\nu)\right) + t\omega\left((1-t)^2W_2^2(\mu,\nu)\right)\right]$$

ω-convexity

[Carrillo, McCann, Villani, 2006] [Ambrosio, Serfaty, 2008]
[Carrillo, Lisini, Mainini, 2014]

- Inequalities coincide for ω(x) = x; ω-convexity generalizes λ-convexity.
- Sufficient condition: above the tangent line inequality

$$E(\mu_1) - E(\mu_0) - \frac{d}{d\alpha}E(\mu_\alpha)|_{\alpha=0} \geq \frac{\lambda}{2}\omega(W_2^2(\mu_0,\mu_1))$$

collective dynamics: main questions

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m} \quad \text{for } K(x) : \mathbb{R}^d \to \mathbb{R} \text{ and } m \geq 1$$

$$E(\mu) = \iint K(x-y)d\mu(x)d\mu(y) + \frac{1}{m-1}\int \mu(x)^m dx$$

Main questions:

1. Do solutions exist?
2. Are they unique? stable?
3. How do they behave in the long time limit?
4. How can we simulate them numerically?

collective d

Aggregation equation

$$\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K$$

$$E(\mu) = \iint$$

In general, for ω(x) satisfying Osgood's condition, i.e.

$$\int_0^1 \frac{dx}{\omega(x)} = +\infty$$

we obtain the stability estimate

$$F_{2t}(W_2^2(\rho_1(t), \rho_2(t))) \leq W_2^2(\rho_1(0), \rho_2(0))$$

$$\frac{d}{dt}F_t(x) = \lambda\, \omega(F_t(x))$$

from which we recover [AGS, 2005] & [CMV, 2006].

Theorem (C. 2016): If the energy is ω-convex, ω(x) = x |log(x)|,

1. Do solutions exist? Yes (JKO)
2. Are they unique? Yes stable? expand at most double-exponentially

$$W_2^2(\rho_1(t), \rho_2(t)) \leq W_2^2(\rho_1(0), \rho_2(0))^{e^{2\lambda t}}$$

ω-convexity: applications

$$\begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

- Slime mold singular limit: $K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ C_d|x|^{2-d} & \text{otherwise} \end{cases}$

ω-convex

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta\rho^m$$

- Slime mold (chemotaxis): $K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ C_d|x|^{2-d} & \text{otherwise} \end{cases}$

ω-convex on bounded measures

- Swarming: $K(x) = |x|^a/a - |x|^b/b, \quad -d < b < a$

ω-convex on L^p measures for $2-d \leq b < a$

- Granular media: $K(x) = |x|^3$

ω-convex on measures with fixed center of mass and $\omega(x) = x^{3/2}$

collective dynamics: main questions

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m}$$ for $K(x) : \mathbb{R}^d \to \mathbb{R}$ and $m \geq 1$

<u>Main questions:</u>

☑ 1. Do solutions exist?

☑ 2. Are they unique? stable?

☐ 3. How do they behave in the long time limit? depends on choice of K(x)

4. How can we simulate them numerically?

long time behavior: $K = \Delta^{-1}$

For $K = \Delta^{-1}$ and $1 \leq m \leq +\infty$, long time behavior of Keller-Segel equation has been the subject of recent interest.

- **Supercritical power** ($m \leq 2-2/d$):
  Profiles of steady states known for certain of m; solutions can "blow up" to a Dirac mass in finite time or remain bounded.

  [Sugiyama 2006, 2007], [Luckhaus and Sugiyama 2006, 2007], [Blanchet, Carlen, Carrillo 2012], [Chen, Liu, Wang 2012]

- **Subcritical power** ($m > 2-2/d$):
  All steady states are radially symmetric and decreasing; still, convergence to equilibrium is only known in d=1, 2 and for radial solutions in higher dimensions.

  [Carrillo, Hitter, Volzone, Yao 2016], [Kim, Yao 2012]

long time behavior: $K = \Delta^{-1}$, $m = +\infty$

In the case of the height constrained aggregation equation, we obtained quantitative rates of convergence to equilibrium for patch solutions:

Theorem (C., Kim, Yao 2016):

- Suppose $\rho(x,t)$ solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
- Then, in two dimensions,

$$\rho(x,t) \xrightarrow{L^p} 1_B(x) \text{ for all } 1 \leq p < +\infty$$

and

$$|E_\infty(\rho(\cdot,t)) - E_\infty(1_B)| \leq C_{\Omega(0)} t^{-1/6}$$

- In any dimension, the Riesz Rearrangement Inequality guarantees that the unique minimizer of E_∞ is $1_B(x)$.
- The tricky part is showing mass of $\rho(x,t)$ doesn't escape to $+\infty$. To do this, we characterize the dynamics of patch solutions in terms of a free boundary problem and control $M_2(\rho(t))$ by Talenti inequality (d=2).

collective dynamics: main questions

Aggregation equation with degenerate diffusion:

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta \rho^m}$$ for $K(x) : \mathbb{R}^d \to \mathbb{R}$ and $m \geq 1$

<u>Main questions:</u>

☑ 1. Do solutions exist?

☑ 2. Are they unique? stable?

☑ 3. How do they behave in the long time limit?

☐ 4. How can we simulate them numerically?

numerics

- For nice velocity fields and $\rho = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}$,

$$\boxed{\frac{d}{dt}\rho(x,t) + \nabla\cdot(v(x,t)\rho(x,t)) = 0} \leftrightarrow \boxed{\frac{d}{dt}x_i(t) = v(x_i(t),t), \quad \forall i = 1,\dots,N}$$

- For any ρ(x), there exist x_1, …, x_N so that $W_2\left(\rho, \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\right) \xrightarrow{N\to+\infty} 0$

<u>General Numerical Strategy:</u> to approximate a solution ρ(x,t) of a PDE…

1) Approximate ρ(x,0) by $\rho_N(x,0) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$.

2) Compute the solution with initial data ρ_N by numerically solving the corresponding system of ODEs.

3) Use stability of PDE to conclude that the numerical solution $\rho_N(x,t)$ must be close to ρ(x,t) on bounded time intervals.

What about when v(x,t) is not "nice"?

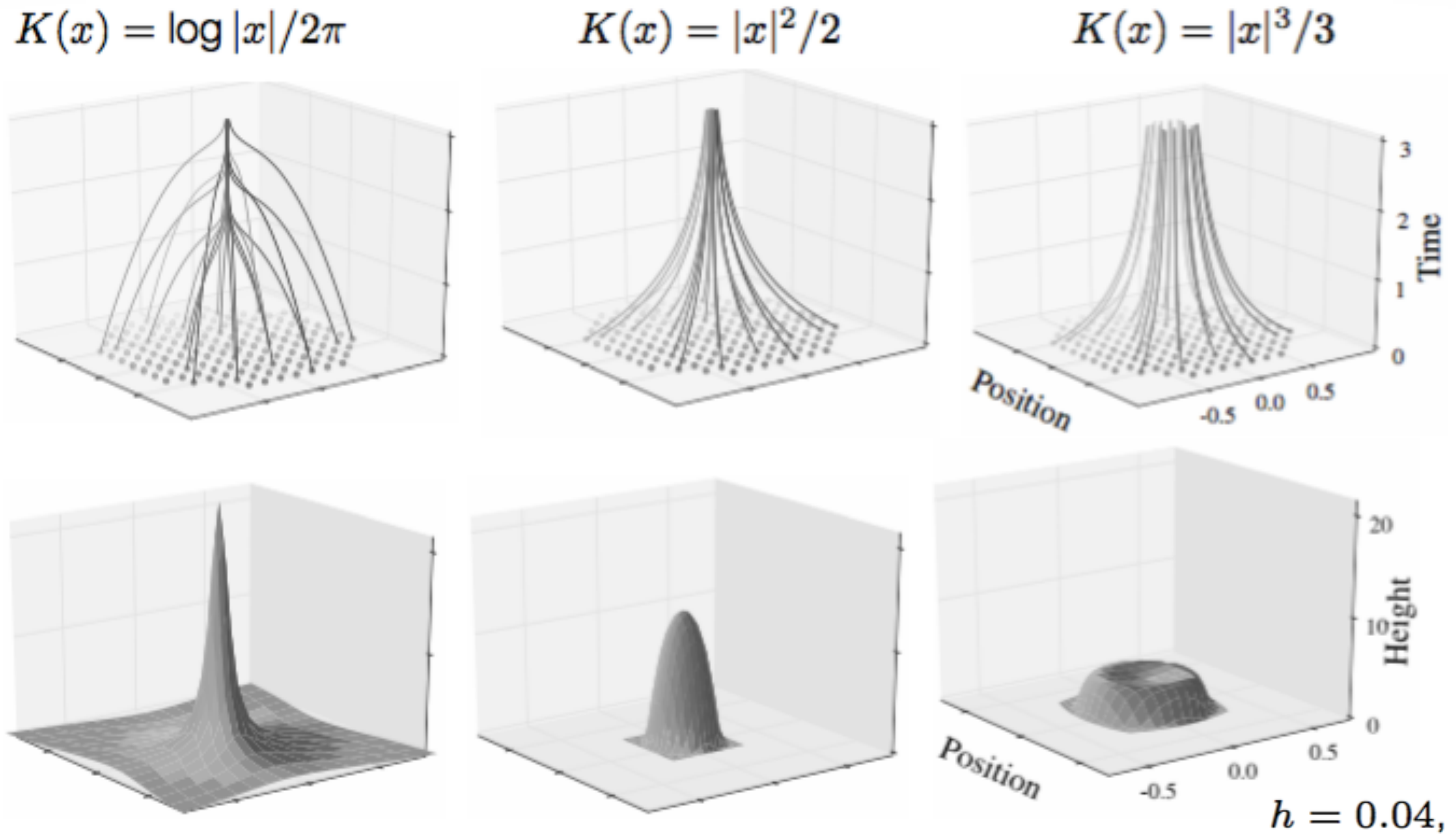
numerics

None of the v(x,t) mentioned so far are nice! We need to make them nice.

Aggregation equation without diffusion:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho)$$

- Regularize K by convolution with a mollifier ("blob")
- **Theorem** [C., Bertozzi 2014]: If you remove the mollification as you add particles, the particle "blob" method converges.



$K(x) = \log|x|/2\pi$

$K(x) = |x|^2/2$

$K(x) = |x|^3/3$

$h = 0.04,\ q = 0.9,\ m = 4$

numerics

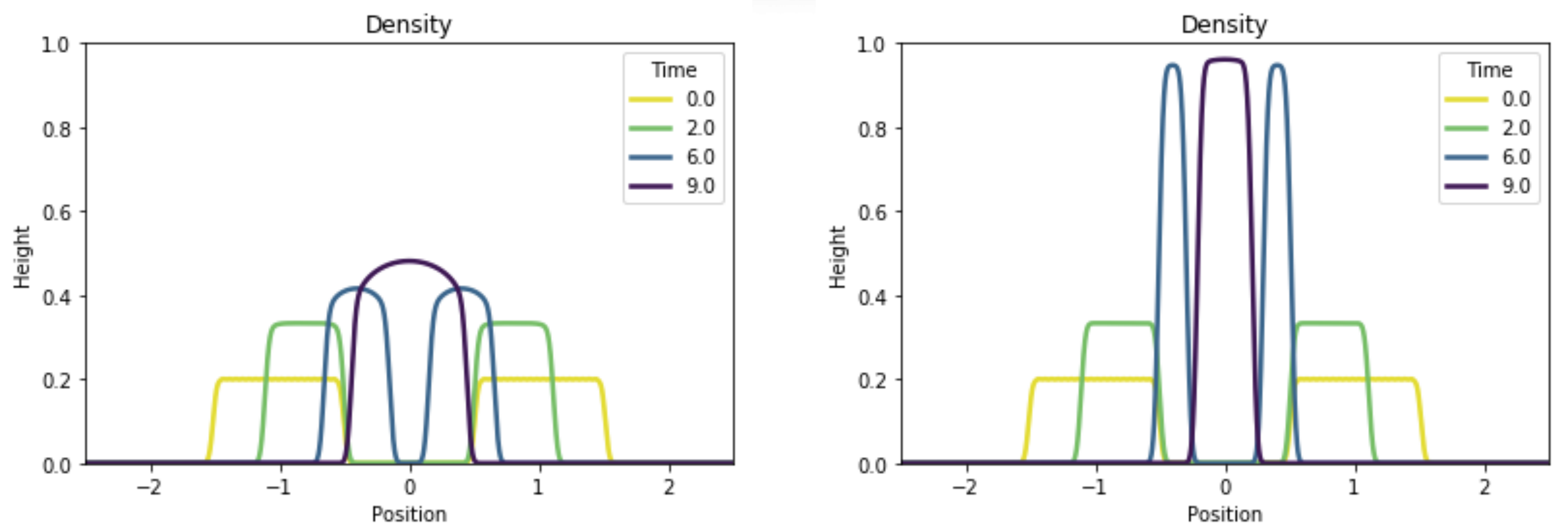
Aggregation equation w/ deg. diffusion:

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m$$

- Regularize both K and v by convolution

$$\Delta \rho^m = \nabla \cdot (m\rho^{m-1}\nabla\rho) = \nabla \cdot (\underbrace{(m\rho^{m-2}\nabla\rho)}_{v}\rho)$$

- **Theorem** [Carrillo, C., Patacchini (in progress)]: If you remove the mollification as you add particles, the particle "blob" method Γ-converges.



Newtonian attraction ($K = \Delta^{-1}$) and m=2 and m=100 diffusion

future work:

Does Keller-Segel converge to congested aggregation?

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla K * \rho)\rho) + \Delta \rho^m \xrightarrow{m\to+\infty} \begin{cases} \frac{d}{dt}\rho = \nabla \cdot (\nabla(K * \rho)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

- For V(x) convex, [Alexander, Kim, Yao 2014] showed

$$\frac{d}{dt}\rho = \nabla \cdot ((\nabla V)\rho) + \Delta \rho^m \xrightarrow{m\to+\infty} \begin{cases} \frac{d}{dt}\rho = \nabla \cdot ((\nabla V)\rho) \text{ if } \rho < 1 \\ \rho \leq 1 \text{ always} \end{cases}$$

- Connecting Keller-Segel and the congested aggregation eqn would lead to greater insight in long-time behavior of supercritical (m>2-2/d) Keller-Segel.

Further examples of ω-convex energies?

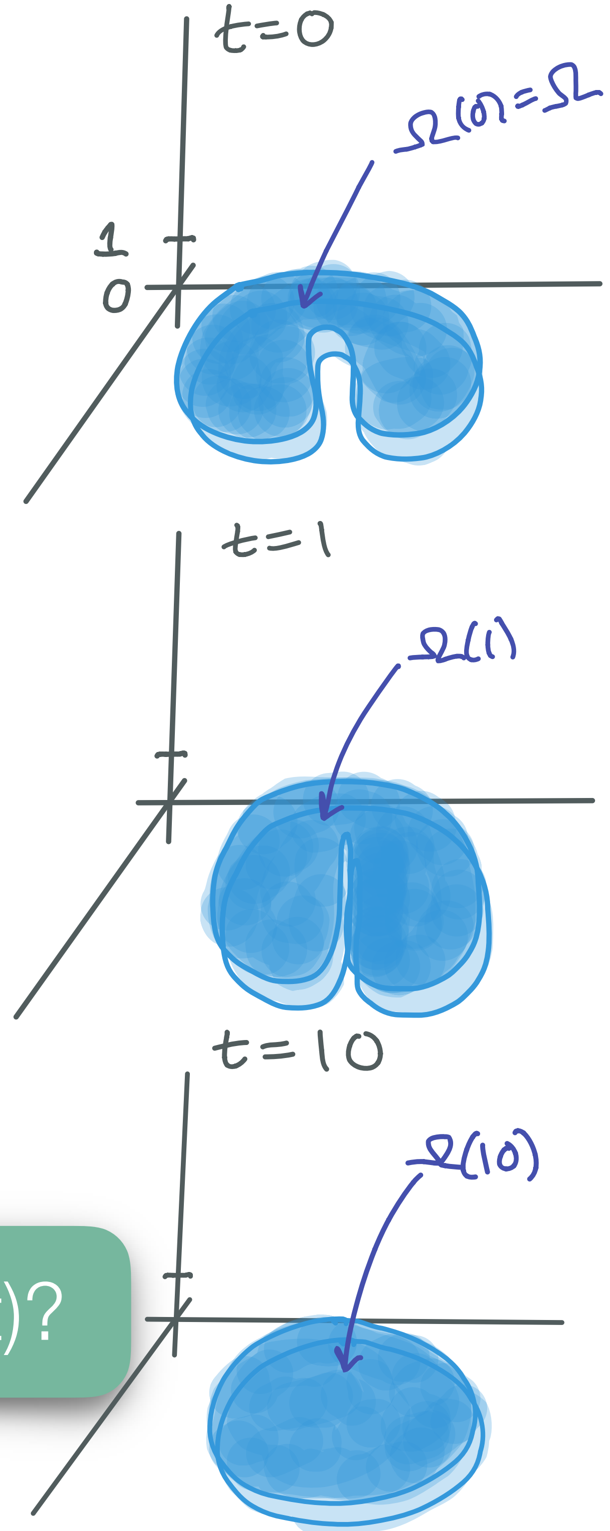
More applications with a height constraint?

Thank you!

motivation for free boundary problem

How does congested aggregation equation relate to free boundary problem?

“
$$\begin{cases}\frac{d}{dt}\rho = \nabla\cdot(\nabla(K*\rho)\rho) \text{ if } \rho<1\\ \rho\le 1 \text{ always}\end{cases}$$
”



- Consider patch solutions. For a domain Ω, suppose that ρ(x,t) is a solution with initial data

$$\rho(x,0) = \begin{cases}1 & \text{if } x\in\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

- Since $K = \Delta^{-1}$, $\nabla K * \rho$ causes self-attraction. Thus, we expect ρ(x,t) to remain a characteristic function.

- Let Ω(t)={ρ=1} be *congested region*, so $\rho(x,t)=1_{\Omega(t)}(x)$.

What free boundary problem describes evolution of Ω(t)?

formal derivation

- Here is a **formal** derivation of the related free boundary problem.
- Suppose ρ(x,t) solves

$$“\boxed{\begin{cases}\frac{d}{dt}\rho = \nabla\cdot(\nabla(K*\rho)\rho) \text{ if } \rho<1\\ \rho\le 1 \text{ always}\end{cases}}”$$

- Since mass is conserved, we expect ρ(x,t) satisfies a continuity equation

$$\boxed{\frac{d}{dt}\rho = \nabla\cdot\Big(\underbrace{(\nabla K*\rho+\nabla\mathbf{p})}_{v}\,\rho\Big)}$$

where **∇p(x,t)** is the pressure arising from the height constraint.

Height constraint is active on the congested region {**p**>0} = Ω(t).

Height constraint is inactive outside the congested region {**p**=0}= Ω(t)^c.

formal derivation

Given $\boxed{\frac{d}{dt}\rho = \nabla \cdot \Big(\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\,\rho\Big)}$ what happens on congested region?

- Because of hard height constraint, on the congested region Ω(t)={ρ=1}, the velocity field is incompressible, **∇**·v=0.

- Since K= Δ^{-1}, $\nabla \cdot v = \Delta K * \rho + \Delta \mathbf{p} = \rho + \Delta \mathbf{p}$, so incompressibility means

$$\boxed{-\Delta \mathbf{p} = \rho \text{ on } \Omega(t) = \{\rho = 1\}}$$

- Using that the height constraint is active on the congested region, Ω(t)={**p**>0}, we obtain the following equation for the pressure:

$$\boxed{-\Delta \mathbf{p} = 1 \text{ on } \{\mathbf{p} > 0\}}$$

formal derivation

Given $\boxed{\frac{d}{dt}\rho = \nabla \cdot (\underbrace{(\nabla K * \rho + \nabla \mathbf{p})}_{v}\rho)}$ what about bdy of congested region?

outward normal velocity of ∂Ω(t)

- By conservation of mass,

$$\boxed{0 = \frac{d}{dt}\int_{\Omega(t)} \rho = \int_{\Omega(t)} \frac{d}{dt}\rho + \int_{\partial\Omega(t)} V\rho}$$

- Using that ρ(x,t) solves the above continuity equation, this equals

$$\boxed{= \int_{\Omega(t)} \nabla \cdot ((\nabla K * \rho + \nabla \mathbf{p})\rho) + \int_{\partial\Omega(t)} V\rho = \int_{\partial\Omega(t)} (\partial_\nu K * \rho + \partial_\nu \mathbf{p} + V)\rho}$$

- Using that $\rho(x,t)=1_{\Omega(t)}(x)$, for $\Omega(t)=\{\mathbf{p}>0\}$, we again obtain an equation for **p**,

$$\boxed{\partial_\nu K * 1_{\{\mathbf{p}>0\}} + \partial_\nu \mathbf{p} + V = 0 \text{ on } \partial\{\mathbf{p} > 0\}}$$

free boundary problem

Combining the observations that…

- on the congested region,

$$\boxed{-\Delta \mathbf{p} = 1 \text{ on } \{\mathbf{p} > 0\}}$$

- and on the boundary of the congested region,

outward normal velocity of ∂Ω(t)

$$\boxed{\partial_\nu K * 1_{\{\mathbf{p}>0\}} + \partial_\nu \mathbf{p} + V = 0 \text{ on } \partial\{\mathbf{p} > 0\}}$$

Theorem (C., Kim, Yao 2016):

- Suppose ρ(x,t) solves congested aggregation eqn with $\rho(x,0) = 1_{\Omega(0)}(x)$.
- Then $\rho(x,t)=1_{\Omega(t)}(x)$, for Ω(t) = {**p(x,t)**>0}, where **p** a viscosity solution of

$$\boxed{\begin{cases} -\Delta \mathbf{p} = 1 & \text{on} \quad \{\mathbf{p} > 0\} \\ \quad V = -\partial_\nu K * 1_{\{\mathbf{p}>0\}} - \partial_\nu \mathbf{p} & \text{on } \partial\{\mathbf{p} > 0\}. \end{cases}}$$

collective dynamics

- $\rho(x,t) : \mathbb{R}^d \times \mathbb{R} \to [0,+\infty)$ nonnegative density
- Mass is conserved (assume ∫ ρ(x) dx = 1), and ρ(x,t) evolves according to a continuity equation:

$$\boxed{\frac{d}{dt}\rho(x,t) + \nabla \cdot (v(x,t)\rho(x,t)) = 0}$$

- Particle approximation:
  - Suppose $\rho = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}$
  - For "nice" velocity fields, ρ(x,t) solves the continuity equation iff

$$\boxed{\frac{d}{dt}x_i(t) = v(x_i(t),t), \quad \forall i = 1,\ldots,N}$$

collective dynamics: slime mold

In the case of the slime mold, we have 1) self-attraction and 2) diffusion.

1) Self-Attraction

- At the particle level, we may formulate self-attraction as

$$\boxed{\frac{d}{dt}x_i(t) = -\frac{1}{N}\sum_{j=1}^{N}\nabla K(x_i(t) - x_j(t))}$$

$$K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2, \\ C_d|x|^{2-d} & \text{otherwise.} \end{cases}$$

- Since $\rho = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}$, we write the resulting velocity field as

$$\boxed{-\frac{1}{N}\sum_{j=1}^{N}\nabla K(x - x_j(t)) = -\int \nabla K(x-y)d\rho(y) = -\nabla K * \rho(x)}$$

collective dynamics: slime mold

In the case of the slime mold, we have 1) self-attraction and 2) diffusion.

2) Diffusion

- Combining self-attraction with diffusion gives the Keller-Segel equation

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta\rho}$$

- More generally, we can consider degenerate diffusion for m ≥ 1

$$\boxed{\frac{d}{dt}\rho + \nabla \cdot ((-\nabla K * \rho)\rho) = \Delta\rho^m}$$

$$\Delta\rho^m = \nabla \cdot (\underbrace{m\rho^{m-1}}_{D} \nabla\rho)$$