An Iterative Method for Generated Jacobian Equations

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Outline

Joint work with C. E. Gutiérrez.

1. Motivating Example: the Parallel Reflector Problem.
2. Weak Solutions of Generated Jacobian Equations.
3. An Iterative Method for Constructing Approximate Solutions.
4. Finite Step Convergence.

The Parallel Reflector Problem

- $\Omega, \Omega^* \subset \mathbb{R}^n$ bounded domains;
  $\Omega =$ source domain, $\Omega^* =$ target domain.
- μ and ν Radon measures on Ω and Ω^* respectively;
  $\mu =$ source intensity, $\nu =$ target intensity
- Conservation of energy: $\mu(\Omega) = \nu(\Omega^*)$.
- Light beams emanate from Ω in the e_{n+1} direction, strike a surface $\Sigma \subset \mathbb{R}^{n+1}$ and are reflected onto Ω^*.
- Σ determines the *reflector map*, $\Phi_\Sigma : \Omega \to \Omega^*$, which takes points from the source to the target according to the law of reflection.
- **Parallel Reflector Problem:** Given domains Ω, Ω^* and measures μ, ν s.t. $\mu(\Omega) = \nu(\Omega^*)$, find the reflecting surface Σ whose reflector map Φ_Σ conserves energy locally; i.e.

$$\mu(\Phi_\Sigma^{-1}(F)) = \nu(F) \quad \forall F \subset \Omega^* \text{ Borel}.$$

The Parallel Reflector Problem: the Semi-Discrete Case

1. Ω^* consists of a finite number of distinct points $y_1, \dots, y_N$.
2. Source intensity μ assumed to be an absolutely continuous measure with density $g \in L^1(\Omega)$, $g > 0$ a.e.
3. Target measure ν assumed to be a Dirac measure; $\nu = \sum_{i=1}^{N} f_i \delta_{y_i}$.
4. Conservation of Energy implies

$$\int_\Omega g(x)\, dx = \sum_{i=1}^{N} f_i.$$

The Parallel Reflector Problem: the Semi-Discrete Case

- Law of reflection determines underlying geometry.
- The reflecting surface Σ consists of pieces of downward facing paraboloids P_i with focus at $y_i \in \Omega^*$, given by the equation

$$P_i(x) = P(x, y_i, b_i) = \frac{1}{b_i} - b_i|x - y_i|^2, \;\; b_i > 0$$

- $b_i =$ opening of the paraboloid; determines how much light is reflected onto y_i.
- Knowledge of the numbers $b_1, \ldots, b_N$ allows reconstruction of the reflector surface Σ.

Statement of the Parallel Reflector Problem

Determine the numbers $b_1, \ldots, b_N$ so that the graph of the function $u(x) = \max_{1 \le i \le N} P(x, y_i, b_i)$ reflects f_i amount of radiation onto the point y_i for each $i = 1, \ldots, N$.

Aim of this Talk

- Given the source intensity g and the target intensities $f_1, \dots, f_N$ for each target point $y_1, \dots, y_N$, is there an iterative method to solve for the coefficients $b_1, \dots, b_N$ up to a prescribed error?
- The method we will consider first appeared in work of Caffarelli-Kochengin-Oliker on the far-field reflector problem; subsequently generalized by Kitagawa to the semi-discrete optimal mass transport problem.
- Our contribution: generalize this method to the setting of generated Jacobian equations (GJEs) and provide a simpler proof of finite-step convergence under minimal assumptions on the data.
- Previous works used smoothness of source density g and the Ma-Trudinger-Wang Condition on the cost function; the idea behind the simplified proof originates in work of DeLeo-Gutierrez-Mawi on the far-field refractor problem.

From the Parallel Reflector Problem to GJEs

- The parallel reflector problem provides the prototypical example of a generated Jacobian equation.
- $\Omega, \Omega^* \subset \mathbb{R}^n$ bounded domains.
- μ an absolutely continuous measure on Ω with density $g \in L^1(\Omega)$, $g > 0$ Lebesgue a.e..
- $\nu = \sum_{i=1}^{N} f_i \delta_{y_i}$ for $y_1, \ldots, y_N \in \Omega^*$ distinct and $f_1, \ldots, f_N > 0$.
- μ and ν satisfy the mass-balance condition $\mu(\Omega) = \nu(\Omega^*)$; that is

$$\int_{\Omega} g(x)\, dx = \sum_{i=1}^{N} f_i.$$

Generating Functions for GJEs

- Let $G : \overline{\Omega} \times \overline{\Omega^*} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a given *generating function*.
- Assume $G = G(x, y, v)$ satisfies the following structural conditions:
  1. (Regularity) $G(x, y, v)$ continuously differentiable in v, twice continuously differentiable in x for $x \in \Omega$, and, for any $\alpha > 0$,
  $$\sup_{\Omega \times \Omega^* \times (0,\alpha)} |G_x(x, y, v)| < \infty.$$
  2. (Monotonicity) $G_v(x, y, v) < 0$ for all $(x, y) \in \Omega \times \Omega^*$.
  3. (Twist) The map $(y, v) \mapsto (G(x, y, v), G_x(x, y, v))$ is injective for each $x \in \Omega$.
  4. (Uniform Convergence Property) For each $y \in \Omega^*$, we have $G(x, y, v) \to \infty$ uniformly in $x \in \overline{\Omega}$ as $v \to 0^+$.

Notions of Convexity for GJEs

G-Convexity

A function $\phi : \Omega \to \mathbb{R}$ is said to be G-convex if for all $x_0 \in \Omega$, there exists $y_0 \in \Omega^*$ and $v_0 \in \mathbb{R}$ such that $\phi(x) \geq G(x, y_0, v_0)$ with equality at $x = x_0$. The function $G(\cdot, y_0, v_0)$ is said to be a G-support to ϕ at x_0.

G-Normal Map

Given a G-convex function ϕ, we define the G-normal map of ϕ to be the set-valued function

$$\partial_G \phi(x_0) = \{y \in \Omega^* : \exists v_0 \in \mathbb{R} \text{ s.t. } G(\cdot, y, v_0) \text{ supports } \phi \text{ at } x_0\}.$$

- (Regularity) $\Rightarrow$ each G-convex function ϕ is uniformly Lipschitz.
- (Twist) $\Rightarrow$ $\partial_G \phi(x)$ is single-valued for Lebesgue a.e. $x \in \Omega$.

Weak Solutions of GJEs

Tracing Map

The tracing map of ϕ is defined as

$$\tau_G\phi(y_0) := (\partial_G\phi)^{-1}(y_0) = \{x \in \Omega : y_0 \in \partial_G\phi(x)\}\,.$$

For each $F \subset \Omega^*$, we define $\tau_G\phi(F) := \bigcup_{y\in F} \tau_G\phi(y)$.

Weak (Brenier) Solutions

The G-convex function ϕ is said to be a weak (Brenier) solution of the generated Jacobian equation if $(\partial_G\phi)_{\#}\,\mu = \nu$; that is, for each Borel set $F \subset \Omega^*$, we have

$$\mu[\tau_G\phi(F)] = \nu(F).$$

Back to the Parallel Reflector Problem

- The generating function for the parallel reflector problem is $G(x, y, v) = \frac{1}{2v} - \frac{v}{2}|x - y|^2$. It satisfies all the structural conditions outlined above under certain restrictions on the configuration of the target points $y_1, \ldots, y_N$ (more later).
- The reflector surface Σ is the graph of a G-convex function ϕ.
- The G-normal map for the reflector problem is the set of target points $y_1, \ldots, y_N$.
- The tracing map $\tau_\phi(y_i)$ for a point $y_i \in \Omega^*$ is the set of points $x \in \Omega$ which are reflected by Σ onto y_i.
- The solution to the parallel reflector problem for discrete targets is a weak (Brenier) solution of the GJE associated to the above generating function.

Setup for the Iterative Method

- We use the short-hand $\mathbf{b} > 0$ to denote a vector $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{R}^N$ with $b_i > 0$ for $1 \le i \le N$.
- Given $\mathbf{b} > 0$, define the envelope

$$\phi_{\mathbf{b}}(x) := \max_{1 \le i \le N} G(x, y_i, b_i).$$

- Intensity functions:

$$H_i(\mathbf{b}) := \mu[\tau_G \phi_{\mathbf{b}}(y_i)], \ 1 \le i \le N.$$

- Voronoi Cells:

$$V_{i,j}^{\mathbf{b}} := \{x \in \Omega : G(x, y_i, b_i) \ge G(x, y_j, b_j)\},$$

$$V_i^{\mathbf{b}} := \Omega \cap \bigcap_{j \ne i} V_{i,j}^{\mathbf{b}} = \{x \in \Omega : \phi_{\mathbf{b}}(x) = G(x, y_i, b_i)\}.$$

- By (Twist), the sets $V_i^{\mathbf{b}}$ form a partition of Ω.

An Important Lemma

Lemma

Fix $i \in \{1, \ldots, N\}$.

1. If $V_i^{\mathbf{b}} \neq \emptyset$, then $V_i^{\mathbf{b}} = \tau_G \phi_{\mathbf{b}}(y_i)$.
2. If $V_i^{\mathbf{b}} = \emptyset$, then $H_i(\mathbf{b}) = 0$.

Corollary

Let $1 \leq i \leq N$ and $b_j > 0$ for all $j \neq i$. Then $H_i(\mathbf{b})$ is increasing in b_i and $H_j(\mathbf{b})$ is decreasing in b_i if $j \neq i$. Furthermore,

$$\lim_{b_i \to 0^+} H_i(\mathbf{b}) = \mu(\Omega) \text{ and } \lim_{b_i \to 0^+} H_j(\mathbf{b}) = 0 \text{ for all } j \neq i.$$

Initializing the Iterative Method

- Let $\epsilon > 0$ be a given tolerance.
- We wish to find a vector $\mathbf{b}_\epsilon > 0$ such that $|H_i(\mathbf{b}_\epsilon) - f_i| < \epsilon$ for each $i = 1, \ldots, N$.
- Fix $\delta := \min\left\{\frac{\epsilon}{N-1}, \frac{f_1}{N}\right\}$ and initialize $b_2 = \cdots = b_N = 1$.
- By the uniform convergence property, there exists $\beta > 0$ such that if $b_1 = \beta$, then $G(x, y_1, b_1) > G(x, y_i, 1)$ for each $i = 2, \ldots, N$ and $x \in \Omega$.
- The vector $\mathbf{b}_{\text{initial}} := (\beta, 1, \ldots, 1)$ thus satisfies $H_1(\mathbf{b}) = \mu(\Omega)$ and $H_i(\mathbf{b}) = 0$ for each $i = 2, \ldots, N$.
- Define the set

$$W_\delta := \{\mathbf{b} > 0 : b_1 = \beta \text{ and } H_i(\mathbf{b}) \leq f_i + \delta \text{ for all } i = 2, \ldots, N\}.$$

- Clearly $\mathbf{b}_{\text{initial}} \in W_\delta$, and so $W_\delta \neq \emptyset$.

Description of the Iterative Method

Choose any $\mathbf{b}^0 \in W_\delta$ and construct the sequence $\mathbf{b}^M \in W_\delta$ as follows:

1. Given $\mathbf{b}^M \in W_\delta$, $M \geq 0$, construct N intermediate vectors $\mathbf{b}^{M,1}, \ldots, \mathbf{b}^{M,N} \in W_\delta$ (recall, $N =$ number of target points).
2. Start by letting $\mathbf{b}^{M,1} = \mathbf{b}^M$. Since $\mathbf{b}^M \in W_\delta$, we know $H_2(\mathbf{b}^{M,1}) \leq f_2 + \delta$.
   - Case 1: $H_2(\mathbf{b}^{M,1}) \geq f_2 - \delta$. Then $|H_2(\mathbf{b}^{M,1}) - f_2| \leq \delta$, so set $\mathbf{b}^{M,2} = \mathbf{b}^{M,1}$.
   - Case 2: $H_2(\mathbf{b}^{M,1}) < f_2 - \delta$. Since $f_2 < \mu(\Omega)$, $\exists \bar{b} \in (0, b_2^{M,1})$ s.t. $\mathbf{b}^{M,2} := (b_1^{M,1}, \bar{b}, b_3^{M,1}, \ldots, b_N^{M,1})$ satisfies $H_2(\mathbf{b}^{M,2}) \in (f_2, f_2 + \delta)$.
3. The inequalities $H_i(\mathbf{b}^{M,2}) \leq f_i + \delta$ for $i = 3, \ldots, N$ follow due to the Corollary. Hence, $\mathbf{b}^{M,2} \in W_\delta$.
4. Continue in this manner for each $\mathbf{b}^{M,k}$, $k = 2, \ldots, N$ and set $\mathbf{b}^{M+1} := \mathbf{b}^{M,N}$.

Stopping Criteria

- If at some step M we have $\mathbf{b}^M := \mathbf{b}^{M,1} = \mathbf{b}^{M,2} = \cdots = \mathbf{b}^{M,N}$, then $|H_i(\mathbf{b}^M) - f_i| \leq \delta < \epsilon$ for each $i = 2, \ldots, N$.
- By the choice of δ, and the mass-balance condition $\mu(\Omega) = \nu(\Omega^*)$

$$\begin{aligned}\left|H_1(\mathbf{b}^M) - f_1\right| &= \left|\mu(\Omega) - \sum_{i=2}^{N} H_i(\mathbf{b}^M) - \nu(\Omega^*) + \sum_{i=2}^{N} f_i\right| \\ &\leq \sum_{i=2}^{N} \left|H_i(\mathbf{b}^M) - f_i\right| \\ &\leq (N-1)\delta < \epsilon.\end{aligned}$$

- Thus, $\mathbf{b}^M$ is the desired vector.

Finite Step Convergence

- Suppose we are at the (M, i)-th step of the iterative procedure. Then we either decrease $b_{i+1}^{M,i}$ to $b_{i+1}^{M,i+1}$ or leave it unchanged.
- In the first scenario, we have

$$H_{i+1}(\mathbf{b}^{M,i+1}) - H_{i+1}(\mathbf{b}^{M,i}) > f_{i+1} - (f_{i+1} - \delta) = \delta$$

- Assume $H_i(\mathbf{b})$ is Lipschitz on W_δ for each $i = 2, \ldots, N$, with Lipschitz constant L; then

$$\delta < H_{i+1}(\mathbf{b}^{M,i+1}) - H_{i+1}(\mathbf{b}^{M,i}) \leq L(b_{i+1}^{M,i} - b_{i+1}^{M,i+1}).$$

- Since only positive vectors $\mathbf{b}$ are admissible, we conclude that each b_i can only be decreased a finite number of times.
- **Conclusion:** If $H_i(\mathbf{b})$ satisfies a Lipschitz estimate on W_δ for each $i = 1, \ldots, N$, then the method terminates in a finite number of steps.

Main Result

Let $j \in \{1, \dots, N\}$, $j \neq i$, and let $\mathcal{G}_{ij}(x) := G(x, y_j, b_j) - G(x, y_i, b_i)$.
Assume $\exists \lambda > 0$ s.t.

$$\inf_{x \in \Omega,\ \Lambda \leq b_i, b_j \leq 1} |D_x \mathcal{G}_{ij}(x)| \geq \lambda > 0. \tag{1}$$

Lipschitz Estimate for H_i

Let G be a generating function satisfying the structural conditions and (1). Then for $\mathbf{b} \in W_\delta$ and $0 < t \leq b_i - \Lambda$, we have the one-sided Lipschitz estimate

$$0 \leq H_i(\mathbf{b}^t) - H_i(\mathbf{b}) \leq \frac{C}{\lambda}(N-1)||g||_{L^\infty(\Omega)} \left(\mathcal{H}^{n-1}(\partial\Omega) + K\mathcal{L}^n(\Omega)\right) t,$$

where $K = K\left(\lambda, \|D_x G\|_{L^\infty(\Omega)}, \|D_x^2 G\|_{L^\infty(\Omega)}\right)$ is a positive constant, λ is the constant in (1), and $C = \sup_{x \in \Omega, \Lambda \leq b \leq 1} |G_v(x, y_i, b)|$.

Parallel Reflectors Once Again

Let us check the condition (1) for the parallel reflector. Recall that $G(x, y, v) = \frac{1}{2v} - \frac{v}{2}|x - y|^2$. An easy calculation shows

$$D_x \mathcal{G}_{ij}(x) := b_j(y_j - x) - b_i(y_i - x).$$

This vanishes if and only if the points x, y_i, y_j are colinear.

Conclusion: Suppose the target Ω^* is arranged in such a way that for any distinct pair of points $y_i, y_j \in \Omega^*$, the line containing y_i and y_j does not intersect Ω. Then by compactness of Ω and the fact that $b_i, b_j \neq 0$, we obtain (1) for the parallel reflector problem.

Lipschitz Estimate for H_i (Idea of Proof)

W_δ stays away from zero

There exists a positive number $\Lambda = \Lambda(\beta, \Omega, \Omega^*)$ such that for all $\delta > 0$, $W_\delta \subset \mathcal{B}_\Lambda$, where $\mathcal{B}_\Lambda := \{\mathbf{b} > 0 : b_1 = \beta,\ b_k \geq \Lambda \text{ for } k = 2, \ldots, N\}$.

- **Proof:** By the assumption $\delta \leq \dfrac{f_1}{N}$ and the mass-balance condition, it follows that for any $\mathbf{b} \in W_\delta$,

$$
\begin{aligned}
0 &\leq f_1 - N\delta < f_1 - (N-1)\delta \\
&= \nu(\Omega^*) - \sum_{i=2}^{N} (f_i + \delta) \leq \mu(\Omega) - \sum_{i=2}^{N} H_i(\mathbf{b}) = H_1(\mathbf{b}).
\end{aligned}
$$

- On the other hand, by the uniform convergence property, there exists a positive number $\Lambda = \Lambda(\beta, \Omega, \Omega^*) < \beta$ such that if $0 < b_i < \Lambda$ for any $i \neq 1$, then $G(x, y_i, b_i) > G(x, y_1, \beta)$ for all $x \in \Omega$.
- Hence, $V_1^{\mathbf{b}} = \emptyset$ and so $H_1(\mathbf{b}) = 0$, which is a contradiction. $\square$

Lipschitz Estimate for H_i (Idea of Proof)

Fix $i, j = 1, \ldots, N$, $i \neq j$. Let $0 < t < b_i$ and $\mathbf{b}^t := b - t\mathbf{e}_i$.

Shorthand: $V_{i,j} = V_{i,j}^{\mathbf{b}}$, $V_{i,j}^t = V_{i,j}^{\mathbf{b}_t}$, $V_i = V_i^{\mathbf{b}}$, $V_i^t = V_i^{\mathbf{b}_t}$.

We have

$$0 \leq H_i(\mathbf{b}^t) - H_i(\mathbf{b}) = \mu(V_i^t) - \mu(V_i) = \mu(V_i^t \backslash V_i) = \int\limits_{V_i^t \backslash V_i} g(x)\ dx.$$

It can be shown that

$$V_i^t \backslash V_i \subset \bigcup_{j \neq i} \left(V_{i,j}^t \backslash V_{i,j} \right).$$

Therefore,

$$0 \leq H_i(\mathbf{b}^t) - H_i(\mathbf{b}) = \int\limits_{V_i^t \backslash V_i} g(x)\ dx \leq ||g||_{L^\infty(\Omega)} \sum_{j \neq i} \mathcal{L}^n \left(V_{i,j}^t \backslash V_{i,j} \right).$$

Lipschitz Estimate for H_i (Idea of Proof)

By definition of $V_{i,j}$,

$$V_{i,j}^t \backslash V_{i,j} = \{x \in \Omega : G(x, y_i, b_i) < G(x, y_j, b_j) \leq G(x, y_i, b_i - t)\}$$
$$= \{x \in \Omega : 0 < \mathcal{G}_{ij}(x) \leq G(x, y_i, b_i - t) - G(x, y_i, b_i)\}.$$

By the mean value theorem,

$$G(x, y_i, b_i - t) - G(x, y_i, b_i) \leq \sup_{x \in \Omega, \Lambda \leq v \leq 1} |G_v(x, y_i, v)| \cdot t \leq Ct.$$

Thus, $V_{i,j}^t \backslash V_{i,j} \subset \{x \in \Omega : 0 < \mathcal{G}_{ij}(x) \leq Ct\}$. Under the assumption (1), it can be shown using the divergence theorem and co-area formula that

$$\mathcal{L}^n \left(\{x \in \Omega : 0 < \mathcal{G}_{ij}(x) \leq Ct\}\right) \simeq t.$$

Thank You.