Special Values of Euler's Function

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March 18, 2017

Alberta Number Theory Days IX

Euler's Function

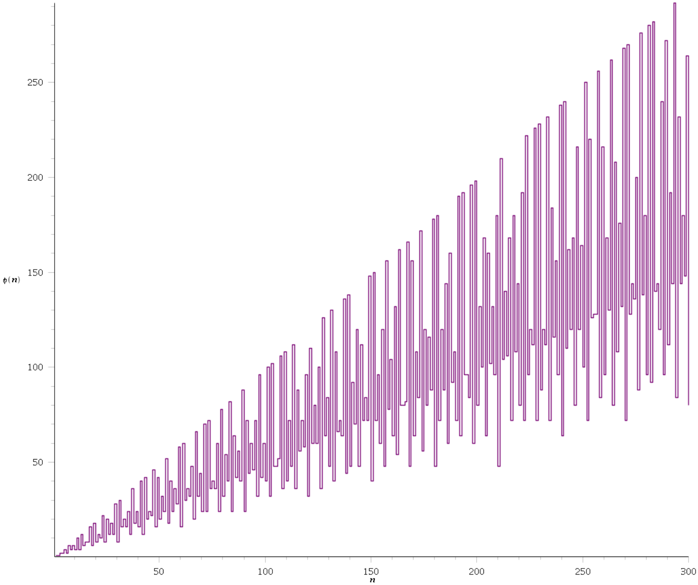
Definition

Let $n \in \mathbb{N}$. *Euler's totient function* is the multiplicative function

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

which counts the natural numbers less than and coprime to n.

Euler's Function



Landau's Theorem

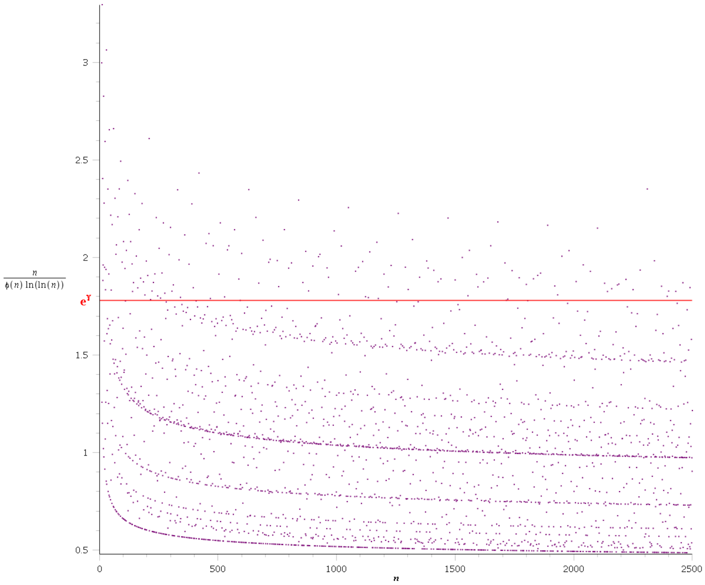
Theorem (Landau, 1909)

$$\limsup_{n\to\infty} \frac{n}{\phi(n)\log\log n} = e^{\gamma},$$

where e is Euler's number and γ is the Euler-Mascheroni constant.

Note: $e^{\gamma} \approx 1.7811$.

Landau's Theorem



Landau's Theorem: Notes

In the proof of Landau's theorem, the relevant sequence is the *primorials*.

Definition

The k-th *primorial*, N_k, is the product of the first k primes. That is,

$$N_k = \prod_{i=1}^{k} p_i,$$

where p_i is the i-th prime.

Landau's Theorem: Notes

Additionally, the proof requires two important theorems, Mertens' (3rd) theorem and the Prime Number Theorem.

Theorem (Mertens, 1874)

$$\prod_{p \leq x} (1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x}$$

as $x \to \infty$.

Theorem (Hadamard and de la Vallée-Poussin, 1896)

Let $x \geq 2$ *and* $\theta(x) = \sum_{p \leq x} \log(p)$. *Then,*

$$\theta(x) \sim x$$

as $x \to \infty$.

A Question of Rosser and Schoenfeld

In their celebrated paper, *Approximate formulas for some functions of prime numbers*, J. B. Rosser and L. Schoenfeld prove that, for $n \geq 3$, $n \neq 223092870$,

$$\frac{n}{\phi(n) \log\log n} \leq e^{\gamma} + \frac{5}{2} \frac{1}{(\log\log n)^2}.$$

Furthermore, the following question is suggested.

Question

Are there infinitely many $n \in \mathbb{N}$ *for which*

$$\frac{n}{\phi(n) \log\log n} > e^{\gamma}?$$

An Answer of Nicolas

Yes!

Theorem (Nicolas, 1983)

There exist infinitely many $n \in \mathbb{N}$ *for which*

$$\frac{n}{\phi(n) \log \log n} > e^{\gamma}.$$

In fact, Nicolas said something much stronger, relating this problem to the Riemann hypothesis.

The Riemann Hypothesis

Definition

The *Riemann zeta function*, $\zeta(s)$, is defined to be the analytic continuation of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

valid where $s = \sigma + it$ is a complex number with $\sigma > 1$. Its *nontrivial zeroes* are those zeroes found in the *critical strip* $0 < \Re(s) < 1$.

Conjecture (The Riemann hypothesis)

If $\rho = \beta + i\gamma$ is a nontrivial zero of the Riemann zeta function, then $\beta = 1/2$.

Nicolas' Answer, Revisted

The theorem of Nicolas already mentioned is an immediate consequence of the following theorem.

Theorem (Nicolas, 1983)

If the Riemann Hypothesis is true, then for all $k \in \mathbb{N}$*,*

$$\frac{N_k}{\phi(N_k)} > e^{\gamma} \log \log N_k.$$

On the other hand, if the Riemann Hypothesis is false, there are infinitely many k *for which the above inequality is true and also infinitely many* k *for which the above inequality is false.*

An Observation

This means that the Riemann Hypothesis is true if and only if there are only finitely many primorials for which

$$\frac{N_k}{\phi(N_k)} \leq e^{\gamma} \log\log N_k.$$

Generalizing Landau

We might recognize the major players in Landau's Theorem have analogues when we replace primes with primes from a fixed arithmetic progression. In particular...

Mertens' (3rd) Theorem for Arithmetic Progressions

Theorem (Languasco and Zaccagnini, 2009)

Let $x \geq 2$ *and* $q, a \in \mathbb{N}$ *such that* $\gcd(q, a) = 1$. *Then,*

$$\prod_{\substack{p \leq x \\ p \equiv a \pmod q}} \left(1 - \frac{1}{p}\right) \sim \frac{C(q, a)}{(\log x)^{\frac{1}{\phi(q)}}},$$

as $x \to \infty$, *where* $C(q, a)^{\phi(q)}$ *is given by*

$$e^{-\gamma} \prod_{p} \left(1 - \frac{1}{p}\right)^{\alpha(p; q, a)}$$

and

$$\alpha(p; q, a) = \begin{cases} \phi(q) - 1 & \text{if } p \equiv a \pmod q, \\ -1 & \text{otherwise.} \end{cases}$$

PNT in Arithmetic Progressions

Moreover,

Theorem (Prime Number Theorem in Arithmetic Progressions)

Let $x \geq 2$ *and* $q, a \in \mathbb{N}$ *such that* $\gcd(q, a) = 1$. *Define*

$$\theta(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p.$$

Then,

$$\theta(x; q, a) \sim \frac{x}{\phi(q)},$$

as $x \to \infty$.

A Notation

Consider the following set

Notation

For $q, a \in \mathbb{N}$ *such that* $\gcd(q, a) = 1$*, we set*

$$S_{q,a} = \{n \in \mathbb{N}; p \mid n \implies p \equiv a \pmod{q}\}.$$

Example

$$S_{5,2} = \{1, 2, 4, 7, 8, 14, 16, 17, 28, 32 \ldots\}$$

A Generalization

Theorem

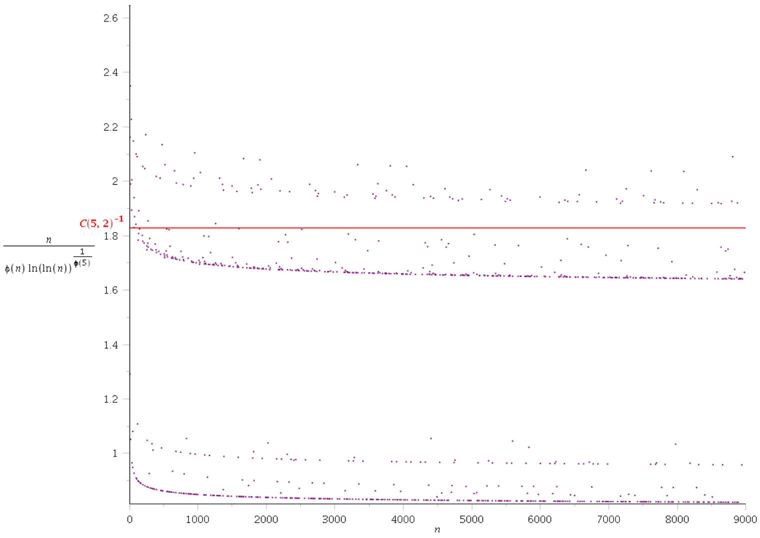
Let $q, a \in \mathbb{N}$ *such that* $\gcd(q, a) = 1$, *then*

$$\limsup_{n \in S_{q,a}} \frac{n}{\phi(n)(\log\log n)^{1/\phi(q)}} = \frac{1}{C(q, a)},$$

where $C(q, a)$ *is the constant arising in Mertens' Theorem for arithmetic progressions.*

Languasco and Zaccagnini have recent work on computing the constants $C(q, a)$. For example, $\frac{1}{C(5,2)} \approx 1.8282$.

A Generalization

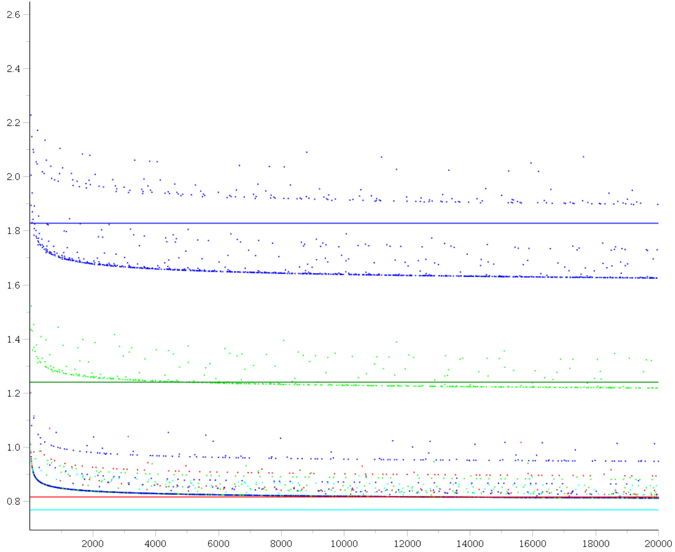


$C(q, a)^{-1}$ for small q

q	a	$C(q, a)^{-1}$
2	1	0.8905...
3	1	0.7125...
3	2	1.6664...
4	1	0.7738...
4	3	1.1508...
5	1	0.8161...
5	2	1.8282...
5	3	1.2407...
5	4	0.7696...

Table: Some values of $C(q, a)^{-1}$ for small q

A Generalization



Rosser and Schoenfeld, Revisted

Given this generalization of Landau, it is natural to ask an analogue of the question of Rosser and Schoenfeld. Namely,

Question

For a fixed q, a *such that* $\gcd(q, a) = 1$*, are there infinitely many* $n \in S_{q,a}$ *for which*

$$\frac{n}{\phi(n)(\log\log n)^{1/\phi(q)}} > \frac{1}{C(q, a)}?$$

Arithmetic Primorials

To discuss what an answer to such a question might look like, we need some analogues of the elements arising in Nicolas' Theorem. Firstly,

Definition

Let $(q, a) = 1$. The k-th (q, a)-*arithmetic primorial*, $\overline{N}_k$, is the product of the first k primes congruent to a modulo q. These will be denoted,

$$\overline{N}_k = \prod_{i=1}^{k} \overline{p}_i,$$

where $\overline{p}_i$ is the i-th prime congruent to $a \pmod q$.

Dirichlet L-functions

Definition

Given a Dirichlet character χ, we define the corresponding *Dirichlet L-function*, $L(s, \chi)$, to be the analytic continuation of the infinite series

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

valid where $s = \sigma + it$ is a complex number with $\sigma > 1$. Its *nontrivial zeroes* are those zeroes found in the *critical strip* $0 < \Re(s) < 1$.

Speculation

We would like to be able to prove:

Statement

For a fixed q, a *such that* $\gcd(q, a) = 1$*, there are infinitely many* $n \in S_{q,a}$ *for which*

$$\frac{n}{\phi(n)(\log\log n)^{1/\phi(q)}} > \frac{1}{C(q, a)}.$$

It seems likely that an infinite set satisfying this inequality will be the (q, a)-arithmetic primorials.

The Main Result

Theorem (F.)

For a fixed q, a *such that* $\gcd(q, a) = 1$*, there are infinitely many* $n \in S_{q,a}$ *for which*

$$\frac{n}{\phi(n)(\log\log n)^{1/\phi(q)}} > \frac{1}{C(q, a)},$$

provided:

- *For* $x \geq x_0$*, there exists* $c \geq 0, m \geq 2$ *s.t.* $\theta(x; q, a) \leq \psi(x; q, a) - cx^{1/m}$.
- *There are no zeroes of* $L(s, \chi)$ *on the real part of the critical strip for any character* $\chi \pmod{q}$.
- *There exists* $\chi \pmod{q}$ *for which* $L(s, \chi)$ *has a zero which is not a zero for any* $L(s, \chi^{'})$*, where* $\chi^{'} \neq \chi$ *is a character modulo* q.

The Function $f(x)$

Let's define the following function:

$$f(x) = \frac{(\log \theta(x; q, a))^{\frac{1}{\phi(q)}}}{C(q, a)} \cdot \prod_{\substack{p \leq x \\ p \equiv a \pmod q}} \left(1 - \frac{1}{p}\right).$$

Then observe that, for $\overline{p}_k \leq x < \overline{p}_{k+1}$,

$$f(x) = \frac{(\log \log \overline{N}_k)^{\frac{1}{\phi(q)}}}{C(q, a)} \cdot \frac{\phi(\overline{N}_k)}{\overline{N}_k}.$$

Hence, if we could show $f(x) < 1$ for infinitely many x, we would obtain our result.

The Function $\log f(x)$

Actually, we'll show that

$$\log f(x) = \frac{\log\log\theta(x;q,a)}{\phi(q)} + \sum_{\substack{p\leq x \\ p\equiv a \pmod q}} \log\left(1-\frac{1}{p}\right) - \log C(q,a)$$

is less than 0 for infinitely many x.

An Upper Bound

We can show that

Proposition

For all x sufficiently large,

$$\log f(x) \leq A(x) + J(x)$$

where $A(x)$ is negative and

$$J(x) = \int_x^\infty \frac{(\psi(t,q,a) - \frac{t}{\phi(q)})(\log t + 1)}{t^2 \log^2 t}\, \mathrm{d}t.$$

From here, one way to answer the analogue of the Rosser and Schoenfeld question would be to show that $J(x)$ changes sign infinitely often.

An Oscillation Theorem of Landau

Theorem (Landau, 1905)

Suppose $h(x)$ is of constant sign for all sufficiently large x. Then the real point $s = \sigma_0$ on the line of convergence of the integral

$$G(s) = \int_1^\infty \frac{h(x)}{x^s}\,\mathrm{d}x$$

is a singularity of the function represented by the integral.

$G(s)$

Hence, to show $J(x)$ changes sign infinitely often, we can show that

$$G(s) = \int_{\overline{p}_1}^{\infty} \frac{J(x)}{x^s} \, \mathrm{d}x$$

(defined initially for $\Re(s) > 1$) extends to a function with no singularities on the real line in the critical strip.

Conclusion

If we assume that $J(x)$ is of constant sign for all sufficiently large x, the oscillation theorem tells us that $G(s)$ must have an abscissa of convergence, σ_0, satisfying $\sigma_0 \leq 0$, and therefore $G(s)$ extends to a holomorphic function for $\Re(s) > 0$. The proof is completed by showing that $G(s)$ has a pole corresponding to a zero of a Dirichlet L-function inside the critical strip.

Conclusion

Hence, $J(x)$ has infinitely many sign changes, and therefore $\log f(x) \leq 0$ for infinitely many x, i.e., there *are* infinitely many $k \in \mathbb{N}$ for which

$$\frac{\overline{N}_k}{\phi(\overline{N}_k)(\log\log \overline{N}_k)^{1/\phi(q)}} > \frac{1}{C(q, a)}.$$

Thank you!