Fast and backward stable computation of roots of polynomials

David S. Watkins

Department of Mathematics
Washington State University

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Problem

- $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + x^n \qquad \text{(monic)}$

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$$A = \begin{bmatrix} 0 & & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$$

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- Francis's (implicitly-shifted QR) algorithm
- Structure not fully exploited. Can we do better?

Our Paper

- Yes!

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- Jared L. Aurentz, Thomas Mach, Raf Vandebril, and D. S. W.

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- . . . and also some more recent developments.

- Companion matrix is unitary-plus-rank-one:

Unitary-plus-rank-one Structure

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- Francis's algorithm preserves this structure.

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  - several methods proposed

Some of the Competitors

- Chandrasekaran, Gu, Xia, Zhu (2007)
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- unitary-plus-rank-one structure exploited
- evidence of backward stability
- quasiseparable generator representation
- We do something else.
- Our method is faster, and we can prove backward stability.

Storage Scheme, Part I

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Store Hessenberg matrix in QR decomposed form

$$A = QR$$

- Q is unitary, R is upper triangular

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- looks inefficient!

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- looks inefficient!   but it's not!

Storage Scheme, Part I

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

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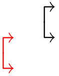

$$
\begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
 & * & * & * & * \\
 & & * & * & * \\
 & & & * & *
\end{bmatrix}
=
\begin{bmatrix}
* & * & * & * & * \\
0 & * & * & * & * \\
 & \color{red}{0} & * & * & * \\
 & & * & * & * \\
 & & & * & *
\end{bmatrix}
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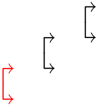

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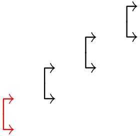

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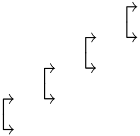

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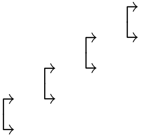

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- **Def:** *Core Transformation*

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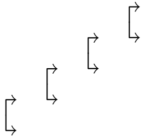

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- **Def:** *Core Transformation*
- Now invert the core transformations to move them to the other side.

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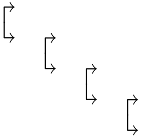

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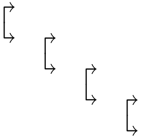

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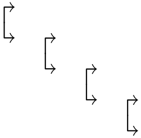

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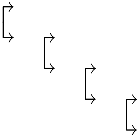

Q requires only $O(n)$ storage space.

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\begin{aligned}
A &= QR \\
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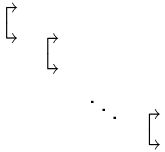
$=$  $\begin{bmatrix} 1 & 0 & \cdots & -a_1 \\ & 1 & & -a_2 \\ & & \ddots & \vdots \\ & & & -a_0 \end{bmatrix}$

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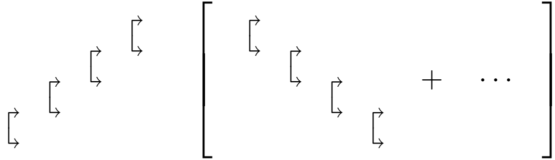
$\underline{R} =$ 

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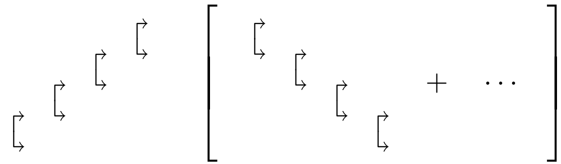
$\underline{R} =$ 

- Bonus: Redundant information (Read our paper.)

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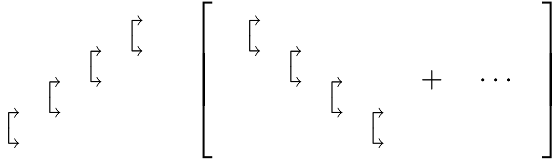

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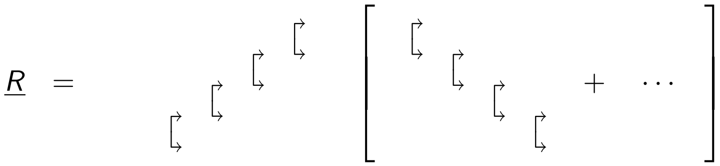

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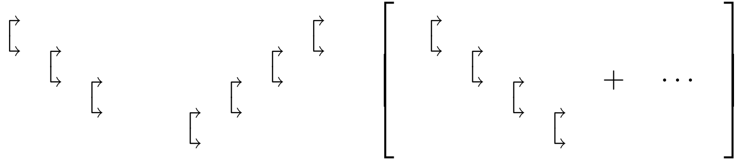
- Storage is $O(n)$.

Representation of A

Altogether we have

$$A = QR$$

$=$



- A is stored entirely in terms of core transformations.

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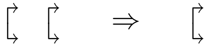

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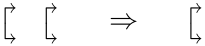
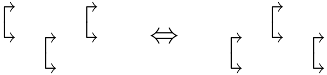
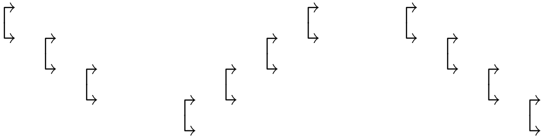


- Turnover (aka shift through, Givens swap, . . .)

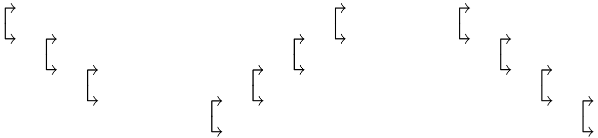


Francis Iteration (the core chase)

- ignoring rank-one part . . .

$A =$ 

The Core Chase



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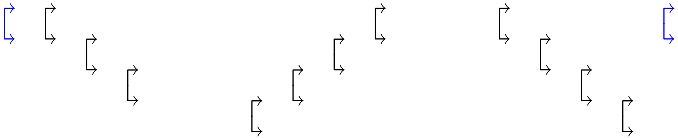

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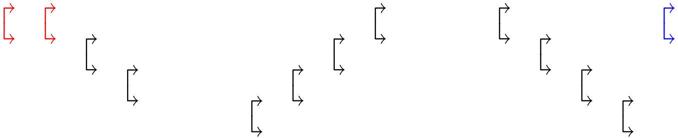

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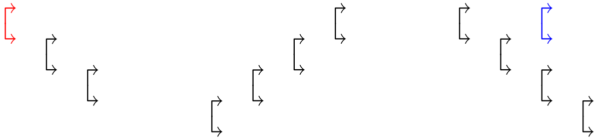

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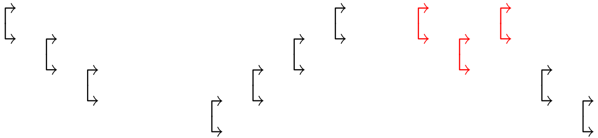

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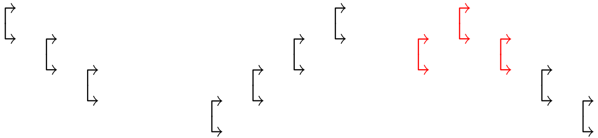

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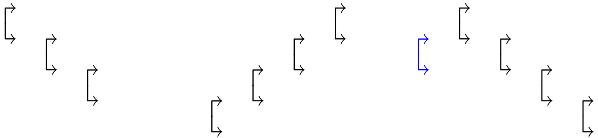

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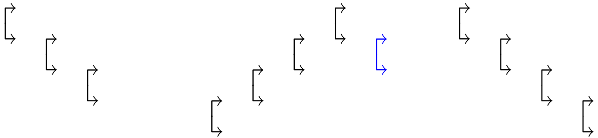

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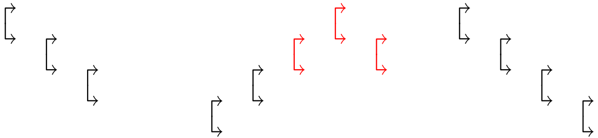

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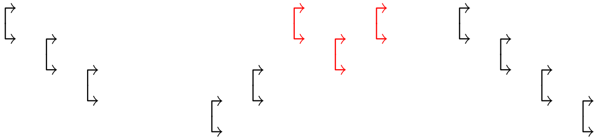

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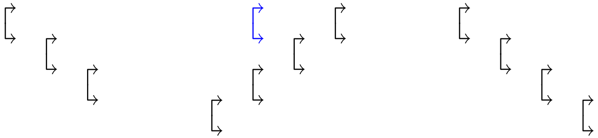

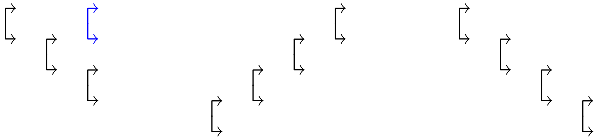

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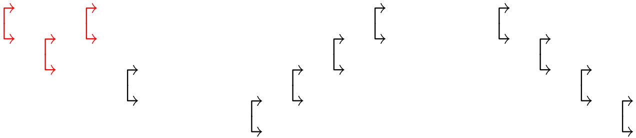

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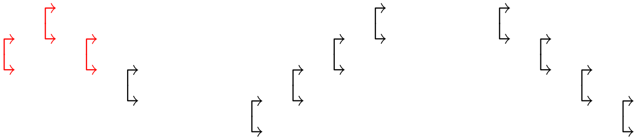

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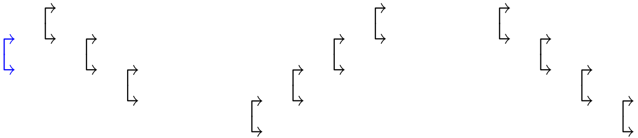

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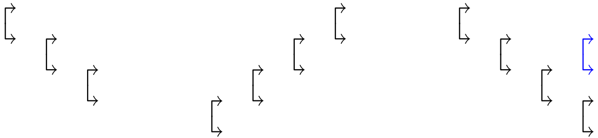

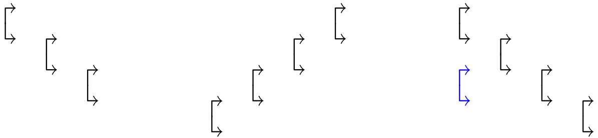

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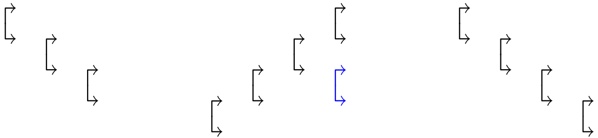

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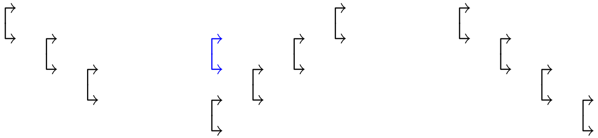

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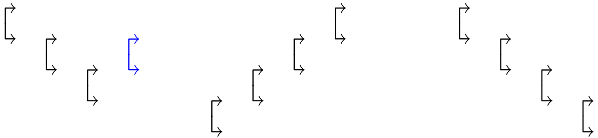

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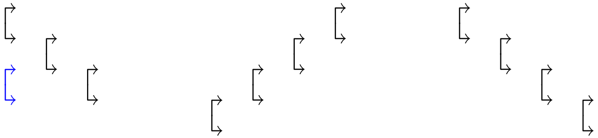

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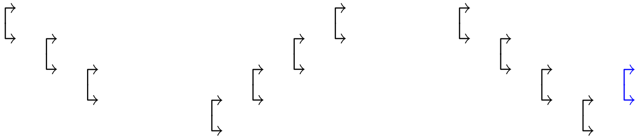

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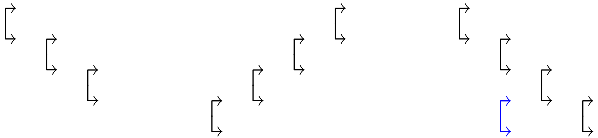

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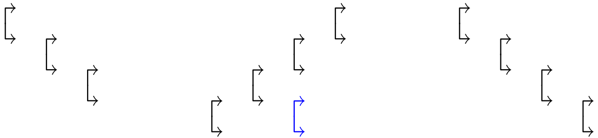

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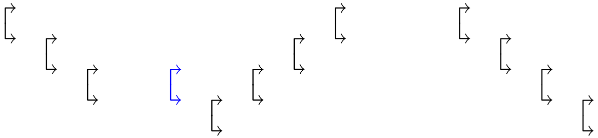

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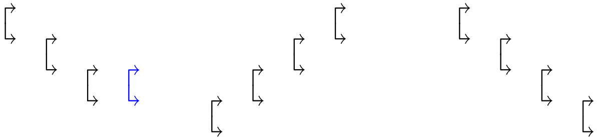

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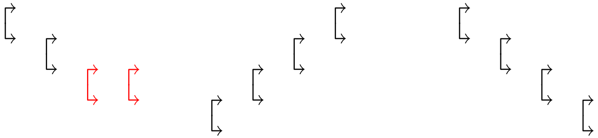

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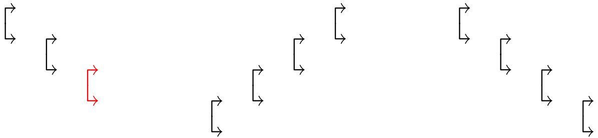

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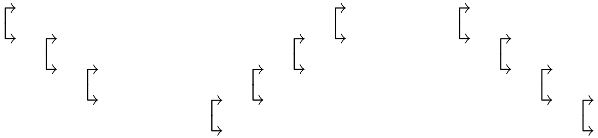

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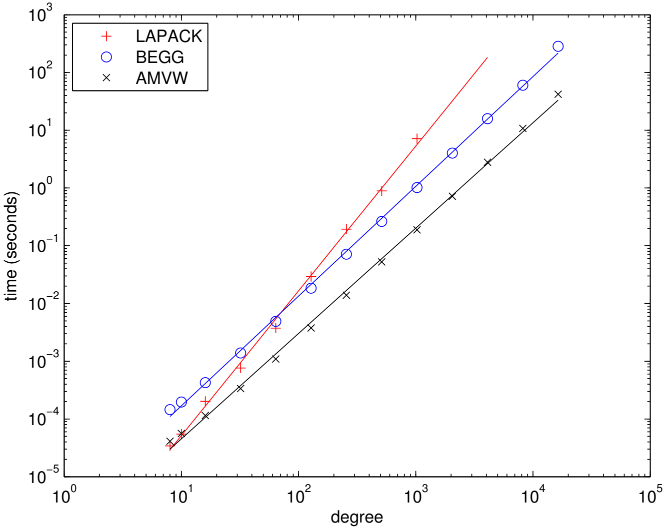

LAPACK
BEGG
AMVW
time (seconds)
degree
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{0}
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Performance

At degree 1000

method	time
LAPACK	7.2
BEGG	1.2
AMVW	0.2

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- We can also handle matrix polynomials.
  (story for another day)

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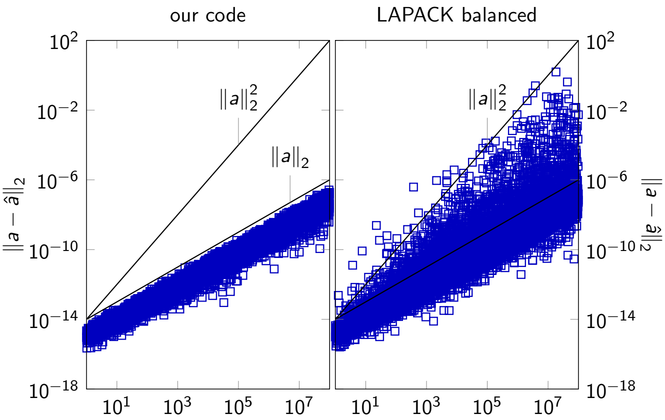
- Meaning: Most of the error is "parallel" to p and is therefore irrelevant.

Nice Picture



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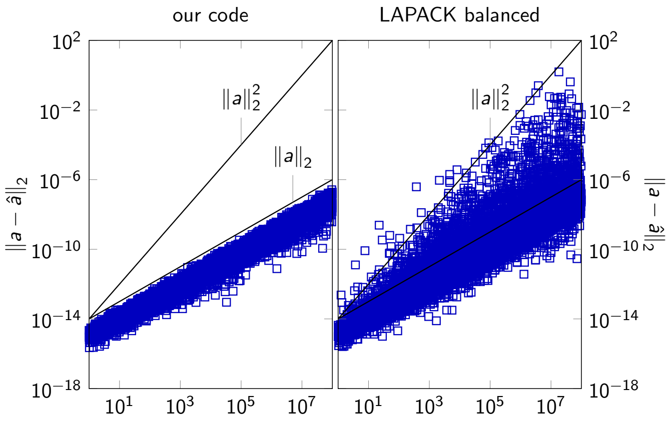

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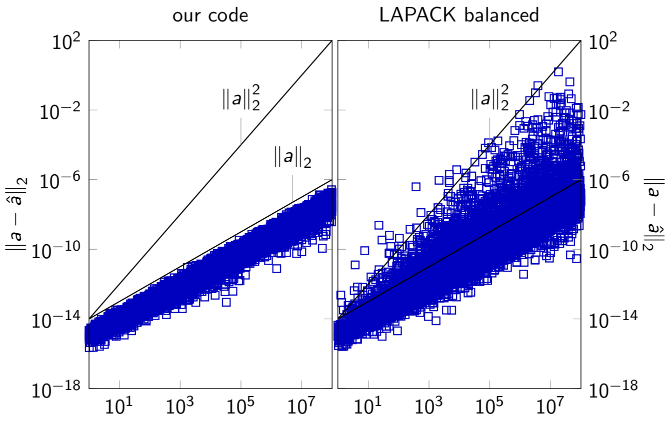

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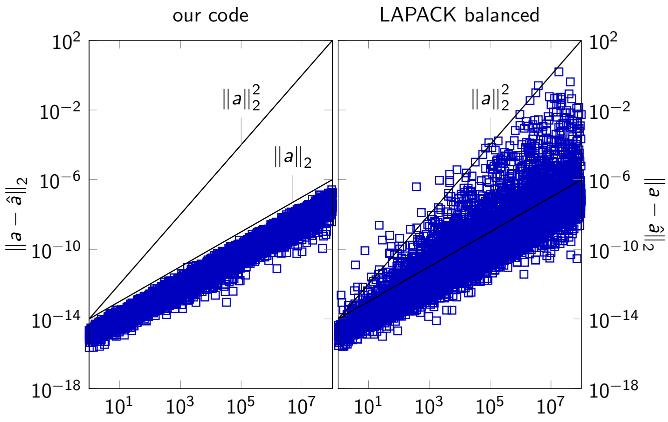

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Thank you for your attention.